Elastic Shape Analysis of Curves and Surfaces in Euclidean Space

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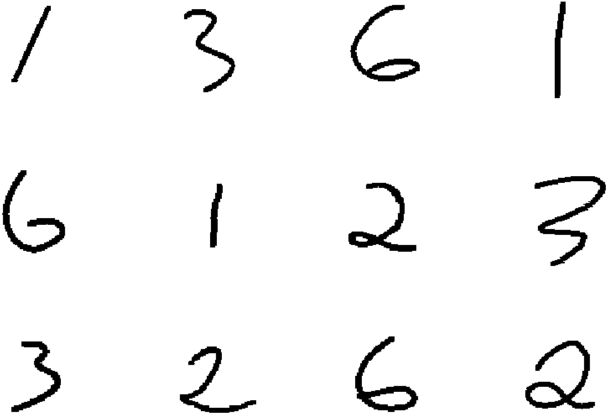
Shapes in $\mathbb{R}^2$



Basic Questions

1. What does it mean for two objects in $\mathbb{R}^n$ to have the "same shape"?

   For us, it means that they differ by isometries (rigid motions) of $\mathbb{R}^n$ and/or dilations.
2. Can we quantify the difference between two given shapes, i.e., put a metric on the space of shapes?
3. Can we find an optimal deformation between two shapes; i.e., a geodesic in the space of all shapes?
4. Statistics: Given a collection of shapes in $\mathbb{R}^n$, can we define the average of these shapes? Can we find the principal modes of variation from this average, i.e., perform principal component analysis (PCA) on sets of shapes?

Shapes of What Kinds of Objects?

1. Ordered sets of points
2. Unordered sets of points
3. Parametrized Curves
4. Parametrized Surfaces

Definition

Two sets of points have the same shape if they differ by a rigid motion and/or a rescaling. Two parametrized objects have the same shape if they differ by a rigid motion and/or a rescaling and/or a reparametrization. A rigid motion of $\mathbb{R}^n$ is a function

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

of the form $f(x) = Ax + b$ where $A \in O(n, \mathbb{R})$ and $b \in \mathbb{R}^n$.

Review of Riemannian Geometry

Our approach to shape analysis will rest heavily on the concepts of Riemannian Geometry, so we begin with a quick review of these ideas!

Definition

A topological space M is called an n-dimensional manifold (or simply n-manifold) if it satisfies these three properties:

1. M is Hausdorff
2. The topology on M has a countable basis
3. Each point $p \in M$ has a neighborhood U in M that is homeomorphic to an open subset of $\mathbb{R}^n$.

A chart on an n-manifold M is a pair (U, ϕ), where $U \subseteq M$ is open, $\phi : U \to \mathbb{R}^n$ and $\phi : U \to \phi(U)$ is a homeomorphism.

Smooth manifolds

To use the tools of calculus on a manifold, we need smooth compatibility between charts. Suppose (U, ϕ) and (V, ψ) are charts on a manifold M. The we have a homeomorphism

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

Definition

We say that (U, ϕ) and (V, ψ) are smoothly compatible if $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are smooth (i.e., all partial derivatives of all orders exist and are continuous).

A smooth atlas $\mathcal{A}$ for M is a collection of charts on M whose domains cover M such that each pair of charts in $\mathcal{A}$ is smoothly compatible.

Smooth structures

Definition

Given an n-manifold M, a smooth structure on M is a maximal smooth atlas $\mathcal{A}$ on M. ("Maximal" means that every chart that is smoothly compatible with all the charts in $\mathcal{A}$ is already in $\mathcal{A}$.)

A smooth n-dimensional manifold is a pair $(M, \mathcal{A})$, where M is an n-manifold and $\mathcal{A}$ is a smooth structure on M.

Examples:

1. $\mathbb{R}^n$ is a smooth n-manifold covered by a single chart $(\mathbb{R}^n, \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n)$.

2. Any open subset N of a smooth manifold M inherits a smooth structure from M. (Just intersect the domain of each chart on M with N to obtain a smooth structure on N.) Important special case: $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, hence it is a smooth manifold.

More Examples

3. If M is a smooth m-manifold and N is a smooth n-manifold, then $M \times N$ is a smooth $(m+n)$-manifold. Charts are obtained as follows: if (U, ϕ) is a chart on M and (V, ψ) is a chart on N, then $(U \times V, \phi \times \psi)$ is a chart on $M \times N$. For example $\mathbb{R}^n \times GL(n, \mathbb{R})$ is a smooth $(n + n^2)$-manifold.

4. **Spheres:** For $p \in \mathbb{R}^{n+1}$, write $p = (p_1, \ldots, p_{n+1})$. Define

$$S^n = \{p \in \mathbb{R}^{n+1} : p_1^2 + \cdots + p_{n+1}^2 = 1\}.$$

S^n, called the n-dimensional sphere can easily be given a smooth structure, as follows. Let $N = (0, \ldots, 0, 1)$ and $U = S^n - \{N\}$.
Define a chart $\phi : U \to \mathbb{R}^n$ by
$\phi(p) =$ the unique point in $\mathbb{R}^n$ such that $(\phi(p), 0)$ is on the line in $\mathbb{R}^{n+1}$ passing through p and N.

It's easy to show that ϕ is a homeomorphism from $U \to \mathbb{R}^n$.

S^n (continued)

Now let $S = (0, \dots, 0, -1)$ and $V = S^n - \{S\}$. Define $\psi : V \to \mathbb{R}^n$ in the same way as ϕ, but replacing N by S. It's clear that U and V cover S^n, and that ϕ and ψ are smoothly compatible. Thus, we can enlarge the atlas $\{(U, \phi), (V, \psi)\}$ to a smooth structure on S^n.

Mappings between manifolds

From here on, assume all manifolds that we talk about are smooth. Recall that a mapping $\mathbb{R}^n \to \mathbb{R}^n$ is smooth (or C^∞), if all partial derivatives exist and are continuous everywhere.

Smooth mappings between manifolds

Definition

Suppose $f : M \to N$, where M and N are manifolds. Let $x \in M$. We say f is smooth at x if we can find a chart (U, ϕ) on M with $x \in U$ and a chart (V, ψ) on N with $f(x) \in V$ such that

$$\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(V)) \to \mathbb{R}^n$$

is smooth on some neighborhood of $\phi(x)$ in $\mathbb{R}^n$. Because of smooth compatibility of charts, it's easy to see this definition does not depend on the charts (U, ϕ) and (V, ψ).

We say that $f : M \to N$ is smooth if it is smooth at every $x \in M$.

Diffeomorphism

If $f : M \to N$ is bijective and f and f^{-1} are both smooth, we say that f is a diffeomorphism from $M \to N$ or that M is diffeomorphic to N.

Examples:

1. If we define $f : (-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \tan x$, then f is a diffeomorphism.

2. If we define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$, then f is smooth and bijective, but it is not a diffeomorphism because f^{-1} is not smooth at 0.

3. Given an $n \times n$ matrix A, we say A is orthogonal if $AA^T = I = A^TA$. Let

$$O(n) = \{\text{all orthogonal } n \times n \text{ matrices}\}$$

It's easy to check that $O(n)$ is a group under matrix multiplication. Given a specific $A \in O(n)$, we can define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = Ax$, where we are thinking of $\mathbb{R}^n$ as column vectors. It's also easy to check that $f(S^{n-1}) = S^{n-1}$ and in fact that f restricts to a diffeomorphism from $S^{n-1} \to S^{n-1}$. Note that f^{-1} is defined by $x \mapsto A^T x$.

Tangent spaces of manifolds

If a manifold is a submanifold of $\mathbb{R}^n$, we have an intuitive understanding of its tangent spaces as subspaces of $\mathbb{R}^n$. It's harder to define for general manifolds! We will describe two approaches:

I. Tangent vectors as equivalence classes of curves. Given a smooth n-manifold M, and a point $p \in M$, choose a chart (U, ϕ) such that $p \in U$. Define a curve in M through p to be a smooth map

$$\gamma : (-\epsilon, \epsilon) \to M \text{ such that } \gamma(0) = p.$$

In this definition, $\epsilon > 0$ is arbitrary. Let

$$\mathcal{C}_p = \{\text{all smooth curves in } M \text{ through } p\}.$$

An equivalence relation on curves through p

Define an equivalence relation on $\mathcal{C}_p$ by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \frac{d}{dt}(\phi \circ \gamma_1(t))|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2(t))|_{t=0}.$$

Define the tangent space of M at p to be

$$T_pM = \mathcal{C}_p / \sim \ \text{ as a set.}$$

There is a bijective map $T_pM \to \mathbb{R}^n$ defined by

$$[\gamma] \mapsto \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0}. \tag{1}$$

Using this map, transfer the structure of $\mathbb{R}^n$ as a vector space to T_pM, making T_pM a vector space.

- It is easily shown that this structure is independent of (U, ϕ).
- The zero element of T_pM is represented by the constant curve $\gamma(t) = p$.

Examples of Tangent Spaces

1. Let $M = \mathbb{R}^n$ and fix $p \in \mathbb{R}^n$. We obtain the isomorphism

$$T_p\mathbb{R}^n \cong \mathbb{R}^n$$

by $[\gamma] \mapsto \gamma'(0)$.

2. $M = S^n \subseteq \mathbb{R}^{n+1}$. There is a natural identification

$$T_pS^n \cong p^\perp := \{v \in \mathbb{R}^{n+1} : v \cdot p = 0\}$$

defined by $[\gamma] \mapsto \gamma'(0)$.

Proof that $\gamma'(0) \in p^\perp$:

$$|\gamma(t)| = 1 \text{ for all } t \Rightarrow \gamma(t)\cdot\gamma(t) = 1 \Rightarrow \gamma'(t)\cdot\gamma(t) = 0$$
$$\Rightarrow \gamma'(0)\cdot\gamma(0) = 0 \Rightarrow \gamma'(0)\cdot p = 0,$$

which implies that $\gamma'(0) \in p^\perp$.

Second way to define tangent vectors of M

II. Let $U \subseteq M$ be a neighborhood of p. Let

$$\mathcal{F}_p = \{\text{all smooth functions } U \to \mathbb{R}\}$$

Given a linear transformation $X : \mathcal{F}_p \to \mathbb{R}$, we call X a derivation at p if it satisfies

$$X(fg) = X(f)g(p) + f(p)X(g).$$

Think: Product rule!

Let $\mathcal{D}_p = \{\text{all derivations at } p\}$. Then it can be shown that the function $T_pM \to \mathcal{D}_p$

$$[\gamma] \mapsto \left[X(f) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}\right]$$

defines a linear isomorphism $T_pM \to \mathcal{D}_p$. In many treatments of this subject, T_pM is *defined* to be $\mathcal{D}_p$.

Submanifolds

Before giving a formal definition, we give two simple examples:

1. If $n > m$, then $\mathbb{R}^m \times \{0_{n-m}\}$ is a submanifold of $\mathbb{R}^n$.
2. S^n is a submanifold of $\mathbb{R}^{n+1}$.

Definition

Let M be an m-manifold. A subset $N \subseteq M$ is an n-dimensional submanifold of M if for all $p \in N$, there is a chart (U, ϕ) for M, with $p \in U$, such that $N \cap U = \phi^{-1}(\mathbb{R}^n \times \{0_{m-n}\})$.

Thus, every submanifold is locally just like example 1 above.

Differential of a smooth mapping $M \to N$

Let $f : M \to N$ be a smooth mapping, $p \in M$. The differential of f at p is the linear transformation

$$df_p : T_pM \to T_{f(p)}N \text{ defined by } df_p([\gamma]) = [f \circ \gamma].$$

Example: Let $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, with $p \in M$, and assume $f : M \to N$ is smooth. Write $f(x) = (f_1(x), \ldots, f_n(x))$. As previously mentioned, identify $T_pM \cong \mathbb{R}^m$ and $T_{f(p)}N \cong \mathbb{R}^n$. Hence we can think of df_p as a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$. It's easily seen that the matrix representing df_p is $\left(\frac{df_i}{dx_j}|_{x_j=p_j}\right)_{i,j}$, i.e., the standard Jacobian matrix of f.

Example of the differential

Let $M = M_{n\times n}(\mathbb{R}) = \{n \times n \text{ matrices over } \mathbb{R}\} \cong \mathbb{R}^{n^2}$.

Define $f : M \to M$ by $f(A) = AA^T$. Given $B \in M$ and $V \in T_B M \cong M$, we will compute $df_B(V)$.
To represent V by a path, define $\gamma(t) = B + tV$. Note that $\gamma(0) = B$ and $\gamma'(0) = V$. Then compute:

$$df_B(V) = \frac{d}{dt} f(B + tV)|_{t=0} = \frac{d}{dt}((B + tV)(B + tV)^T)|_{t=0}$$

$$= BV^T + VB^T \in T_{BB^T} M \cong M.$$

So $df_B(V) = BV^T + VB^T$. We will use this computation later.

Chain Rule for differentials

Theorem

If

$$M \xrightarrow{f} N \xrightarrow{g} P$$

are smooth maps, $p \in M$*, then*

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Proof.

Trivial from the definitions! ☐

Critical and Regular Points and Values

Definition

Given a smooth $f : M \to N$, a point $p \in M$ is called a critical point of f if df_p is not surjective; it is called a regular point if df_p is surjective.

If $p \in M$ is a critical point of f, then $f(p) \in N$ is called a critical value of f. Any point $q \in N$ that is not a critical value of f is called a regular value of f.

Notes:

1. It follows from the above that if $q \in N - f(M)$, then q is a regular value.
2. If $q \in f(M)$, then q is a regular value if and only if df_p is surjective for all $p \in f^{-1}(q)$.
3. If $\dim(M) < \dim(N)$, then all points $p \in M$ are critical points.

Critical and Regular Points and Values (continued)

Example: Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(p) = p_1^2 + \cdots + p_n^2$. Think of $\mathbb{R}^n$ as column vectors. Then the matrix representing df_p is $(2p_1, \ldots, 2p_n)$. Hence 0 is a critical point of f, but all other points $p \in \mathbb{R}^n$ are regular points. Furthermore, $0 \in \mathbb{R}$ is a critical value, but all other $q \in \mathbb{R}$ are regular values.

Theorem (Regular Value Theorem)

Suppose M is an m-manifold and N is an n-manifold, $m \geq n$, $f : M \to N$ is smooth, and $y \in N$ is a regular value of f. Then $f^{-1}(y)$ is a submanifold of M of dimension $m - n$. Also, for all $p \in f^{-1}(y)$,

$$T_p(f^{-1}(y)) = \ker(df_p).$$

Proof.

This is an application of the inverse function theorem, and we omit the details. □

Examples of this Theorem

Example 1: Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(p) = p_1^2 + \cdots + p_n^2$.

As we observed earlier, $1 \in \mathbb{R}$ is a regular value. This provides a second proof that $S^{n-1} = f^{-1}(1)$ is a smooth manifold, without explicitly constructing any charts!

Example 2: Recall that $O(n) = \{A \in M_{n\times n} : AA^T = I\}$. We will now use the inverse image theorem to prove that $O(n)$ is a submanifold of $M_{n\times n}$.

Let $S(n)$ denote the set of $n \times n$ symmetric matrices. Clearly,

$$S(n) \cong \mathbb{R}^{\frac{1}{2}(n^2+n)};$$

therefore for all $A \in S(n)$,

$$T_A S(n) \cong \mathbb{R}^{\frac{1}{2}(n^2+n)}$$

as well.

Proof that $O(n)$ is a manifold

Define $f : M_{n \times n} \to S(n)$ by $f(A) = AA^T$. Clearly, $O(n) = f^{-1}(I)$. To apply the theorem, we need to show that I is a regular value of f. First, we will show that I is a regular *point* of f. Based on an earlier computation, we know that

$$df_I(V) = V + V^T$$

for all $V \in T_I M_{n \times n}$. Clearly, df_I is onto, since for all $W \in S(n)$, $W = df_I(\frac{1}{2}W)$.
Now, let $A \in O(n)$ be arbitrary. Define $R : M_{n \times n} \to M_{n \times n}$ by $R(X) = XA$. We know that R is a diffeomorphism, since its inverse is $\tilde{R}(X) = XA^T$. It follows that $dR_I : T_I M_{n \times n} \to T_A M_{n \times n}$ is a linear isomorphism since, by the chain rule, $(dR_I)^{-1} = d\tilde{R}_A$.

Proof that $O(n)$ is a manifold (continued)

Claim: $f = f \circ R$.

Proof.
(Proof of claim)
$f \circ R(X) = f(XA) = XAA^T X^T = XX^T = f(X)$. □

By the chain rule, $df_I = df_A \circ dR_I$. Since df_I is onto, df_A is also onto!

Therefore, for all $A \in f^{-1}(I) = O(n)$, A is a regular point of f. Hence by the Regular Value Theorem $O(n)$ is a submanifold of $M_{n \times n}$ of dimension $n^2 - \frac{1}{2}(n^2 + n) = \frac{1}{2}(n^2 - n)$.

What is $T_I O(n)$?

By the Regular Value Theorem, $T_I O(n) = \ker df_I$. Since $df_I(V) = V + V^T$, it follows that

$$T_I O(n) = \{V \in M_{n \times n} : V + V^T = 0\} = \{\text{anti-symmetric matrices}\}.$$

This set is commonly denoted by $o(n)$, "the Lie algebra of the Lie group $O(n)$". We will learn more about this nomenclature later.

Tools from geometry: Riemannian metrics

Smooth manifolds as such have no concept of distance, length or angles. If we try to measure these quantities using charts, the answer will always depend on which charts we choose! A Riemannian metric on a manifold M makes possible all of these measurements in an elegant and unified way.

In $\mathbb{R}^n$, we can determine lengths and angles by the “dot product”. Riemann taught us to generalize this concept to smooth manifolds.

Incidentally here are some concepts associated with Riemann's name: Riemann integral, Riemannian metric, Riemann curvature, Reimann surface, Riemann Roch theorem, Riemann mapping theorem, Cauchy-Riemann equations, Riemann zeta function, Riemann hypothesis.

And Riemann only lived to be 39. How did he do that?!

Basic Definitions

Definition

A bilinear form on a vector space V is a function $\Phi : V \times V \to \mathbb{R}$, satisfying the following properties (for all $\alpha, \beta \in \mathbb{R}$ and for all $v, v_1, v_2, w, w_1, w_2 \in V$.)

1. $\Phi(\alpha v_1 + \beta v_2, w) = \alpha\Phi(v_1, w) + \beta\Phi(v_2, w)$
2. $\Phi(v, \alpha w_1 + \beta w_2) = \alpha\Phi(v, w_1) + \beta\Phi(v, w_2)$

Thus, Φ is linear in each variable if we hold the other variable constant.

Φ is symmetric if $\Phi(v, w) = \Phi(w, v)$ (for all v, w).

Φ is positive definite if $\Phi(v, v) \geq 0$ (for all v) and $\Phi(v, v) = 0 \Leftrightarrow v = 0$.

Standard case: $V = \mathbb{R}^n$

Let $V = \mathbb{R}^n$ (which we think of as column vectors); let $A \in M_{n \times n}$. Define

$$\Phi : V \times V \to \mathbb{R} \text{ by } \Phi(v, w) = v^T A w.$$

It's easy to prove that every bilinear form on $\mathbb{R}^n$ is of this form. Note:

- Φ is symmetric $\Leftrightarrow$ A is symmetric.
- Φ is symmetric and positive definite $\Leftrightarrow$ A is symmetric and all of its eigenvalues are > 0.

Denote by $PDSM(n)$ the set of positive definite symmetric $n \times n$ matrices with entries in $\mathbb{R}$.

Riemannian metrics

Definition

A Riemannian metric on a smooth manifold M is a function Φ that assigns to each $p \in M$ a positive definite symmetric bilinear form Φ_p on T_pM. This form must vary smoothly in the following sense:

Let (U, ϕ) be a chart on M and use $d\phi$ to identify each tangent space with $\mathbb{R}^n$. For each $p \in U$, Φ induces a positive definite symmetric bilinear form on $\mathbb{R}^n$. Thus ϕ gives a function $U \to PDSM(n) \subseteq M_{n\times n}$. If this induced function is smooth for all charts (U, ϕ), then we say Φ is a Riemannian metric.

A manifold M endowed with a Riemannian metric is called a Riemannian manifold.

Examples of Riemannian manifolds

1. $M = \mathbb{R}^n$ (column vectors). For all $p \in \mathbb{R}^n$, $T_p\mathbb{R}^n \cong \mathbb{R}^n$. Then the standard "dot product"

$$\Phi_p(v, w) = v^T w$$

defines a Riemannian metric on $\mathbb{R}^n$. This is called the "Euclidean metric" because it gives rise to Euclidean geometry!

2. $M = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Again, for all $(x, y) \in M$, $T_{(x,y)}M \cong \mathbb{R}^2$. Define

$$\Phi_{(x,y)}(v, w) = \frac{1}{y^2}(v^T w)$$

for all $(x, y) \in M$ and for all $v, w \in T_{(x,y)}M$.

With this metric, M is called the "Poincaré upper half plane", and provides a model for hyperbolic geometry, a non-Euclidean geometry.

Spheres as Riemannian manifolds

3. Let $M = S^n \subseteq \mathbb{R}^{n+1}$, $p \in S^n$, $v, w \in T_pS^n$
(In other words, $p \in S^n$ and $v \perp p$ and $w \perp p$.)
Define

$$\Phi_p(v, w) = v^T w.$$

In other words, the Riemannian metric on S^n is just the restriction of the Riemannian metric on $\mathbb{R}^{n+1}$. This metric induces standard spherical geometry on S^n, another famous non-Euclidean geometry.

One more example!

4. Let $M = O(n)$, $A \in O(n)$, and $Y, Z \in T_A O(n)$.
Note: $T_A O(n) \subseteq T_A M_{n\times n} \cong M_{n\times n}$, so $Y, Z \in M_{n\times n}$.
Define

$$\Phi_A(Y, Z) = \mathrm{tr}(YZ^T).$$

One important property of this metric on $O(n)$ is that it is bi-invariant, i.e., it is invariant under left and right translation in the group $O(n)$. More specifically, fix a $B \in O(n)$. Then:

- $\Phi_{AB}(XB, YB) = \Phi_A(X, Y)$
- $\Phi_{BA}(BX, BY) = \Phi_A(X, Y)$.

Isometries of Riemannian manifolds

Notation: In the following, I will often use the notation $\langle v, w \rangle_p$ instead of $\Phi_p(v, w)$. This is to avoid complications resulting from having Riemannian metrics on several different manifolds!

Definition

Let M, N be Riemannian manifolds. A diffeomorphism $f : M \to N$ is called an isometry if for all $p \in M$ and $v, w \in T_pM$

$$\langle v, w \rangle_p = \langle df_p(v), df_p(w) \rangle_{f(p)}.$$

(Note: in this formula, the LHS is a Riemannian metric on M, while the RHS is a Riemannian metric on N.)

Examples of Isometries

Let $A \in GL(n, \mathbb{R})$. Define

$$f : \mathbb{R}^n \to \mathbb{R}^n \text{ by } f(x) = Ax.$$

Then f is an isometry $\Leftrightarrow A \in O(n)$.

Also, if $A \in O(n)$, then $f(S^{n-1}) = S^{n-1}$, and f induces an isometry $S^{n-1} \to S^{n-1}$. In low dimensions, this isometry will be a composition of reflections and rotations.

Basic Geometry in a Riemannian manifold

If M is Riemannian manifold, $p \in M$, and $u, v \in T_pM$, define

- Norm of u: $\|u\| = \sqrt{\langle u, u \rangle_p}$.
- Angle between u and $v = \cos^{-1}\left(\frac{\langle u,v\rangle_p}{\|u\|\|v\|}\right)$

Recall: In $\mathbb{R}^n$, if $\gamma : [a, b] \to \mathbb{R}^n$ is a parametrized smooth path, its length is

$$L(\gamma) = \int_a^b \|\gamma'(t)\| dt.$$

Analogously: Let M be a Riemannian manifold, and $\gamma : [a, b] \to M$ a smooth path. For each $t \in [a, b]$, define

$$\gamma'(t) = d\gamma_t(1) \text{ thinking of } 1 \in T_t[a, b] = \mathbb{R}.$$

Pathlength and Riemannian distance

Define the length of $\gamma : [a, b] \to M$ by

$$L(\gamma) = \int_{t=a}^{b} \|\gamma'(t)\|_{\gamma(t)} dt = \int_{t=a}^{b} \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} dt.$$

If M is a path connected Riemannian manifold, define a distance function on M by

$$d(p, q) = \inf_{\alpha} L(\alpha)$$

where the infimum is taken over all smooth paths $\alpha : [0, 1] \to M$ such that $\alpha(0) = p$ and $\alpha(1) = q$. This function $d(p, q)$ is called the Riemannian distance function on M.

Geodesics

If there is a path $\hat{\alpha}$ that achieves the above infimum, then $\hat{\alpha}$ is called a geodesic in M from p to q.
Note: There can be other geodesics besides the type just defined. Here is a more general definition:

Definition

Let D be any interval in $\mathbb{R}$ (open or closed, finite or infinite). A smooth path $\alpha : D \to M$ is called a geodesic if for all $t \in D$, there exists $\epsilon > 0$ such that for all $s \in (t - \epsilon, t + \epsilon) \cap D$, $L(\alpha|[s, t])$ (or $[t, s]$ if $t < s$) achieves the shortest path length between $\alpha(t)$ and $\alpha(s)$.

Thus, a geodesic only needs to be locally length-minimizing.

Geodesics (continued)

Example: Define

$$\alpha : [0, 3\pi/2] \to S^2 \text{ by } \alpha(t) = (\cos t, \sin t, 0).$$

α is a geodesic, even though it does not minimize length between $\alpha(0)$ and $\alpha(3\pi/2)$.

Another approach to geodesics: "Energy"

Let M be a Riemannian manifold, with $p, q \in M$. Define $\mathcal{P}_{p,q}$ to be the set of all smooth paths $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Define $E : \mathcal{P}_{p,q} \to \mathbb{R}$ by

$$E(\gamma) = \int_0^1 \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)} dt.$$

Note: We are not integrating the norm of the velocity vector, but rather the square of its norm. "Kinetic energy..."

Theorem

γ is a critical point for E on $\mathcal{P}_{p,q}$ if and only if γ is a constant speed parametrized geodesic in M starting at p and ending at q.

Proof.

We omit the proof of this Theorem. □

Geodesics (continued)

Note:

1. There are other important characterizations of geodesics besides these.
2. For certain M, p, q, there do not exist any geodesics from p to q in M, even if M is path connected.

Example: Let $M = \mathbb{R}^2 - \{(0,0)\}$ with the standard Riemannian metric. Then there is no geodesic in M from $(1,0)$ to $(-1,0)$.

Examples of Geodesics

Finding geodesics in an arbitrary Riemannian manifold is, in general, a difficult problem! But in certain cases, this problem has easy solutions.

Example 1. If $M = \mathbb{R}^n$ with the standard Riemannian metric, then the geodesics are precisely the straight lines.
Proof: Given $p, q \in \mathbb{R}^n$, we can parametrize the straight line from p to q by

$$\gamma(t) = p + t(q - p), \text{ where } 0 \leq t \leq 1.$$

Then $\gamma'(t) = q - p$ and therefore

$$L(\gamma) = \int_0^1 \|q - p\| dt = \|q - p\|.$$

We now need to prove that an arbitrary path from p to q cannot have shorter length than $\|q - p\|$. So, let $\alpha : [0, 1] \to \mathbb{R}^n$ be an arbitrary smooth path from p to q.

Define $w = \frac{q-p}{\|q-p\|}$. Clearly $\|w\| = 1$. Then

$$L(\alpha) = \int_0^1 \|\alpha'(t)\| dt \geq \int_0^1 |\alpha'(t) \cdot w| dt \geq \left| \int_0^1 \alpha'(t) \cdot w \, dt \right|$$

$$= \left| \left(\int_0^1 \alpha'(t) \, dt \right) \cdot w \right| = |(\alpha(1) - \alpha(0))| \cdot w$$

$$= \left| \frac{(q-p) \cdot (q-p)}{\|q-p\|} \right| = \|q-p\|,$$

proving the $L(\alpha) \geq \|q-p\|$. This proves that the straight line is the shortest path from p to q! Looking at the above inequalities carefully, it's easy to see that if $\alpha(t)$ is *not* a straight line, equality does not hold.

Geodesics in the Hyperbolic Upper Half-Plane

In the hyperbolic upper half plane described earlier ($M = \mathbb{R}^2_+$ with the hyperbolic metric) it can be shown that geodesics come in two "flavors":

1. Vertical straight lines ($x = \text{constant}$)
2. Half circles in $\mathbb{R}^2_+$ centered on the x-axis.

We won't give the proof here, but it follows the same pattern as the above proof that the geodesics in $\mathbb{R}^n$ are straight lines.

Geodesics in Spheres

We assume that $S^n \subseteq \mathbb{R}^{n+1}$ is endowed with the usual Riemannian metric, as described earlier.

Definition

A great circle in S^n is a subset of the form $S^n \cap P$, where P is any 2-dimensional linear subspace of $\mathbb{R}^{n+1}$.

Theorem

The geodesics in S^n are precisely the great circles. The geodesic distance between two points p and q in S^n is

$$d(p, q) = \cos^{-1}(p \cdot q),$$

where $p \cdot q$ refers to the standard dot product in $\mathbb{R}^{n+1}$.

Note: If $p = -q$, then there is not a unique shortest geodesic between p and q. Otherwise, there is one.

Geodesics in $O(n)$

Recall the bi-invariant Riemannian metric on $O(n)$ defined earlier as

$$\langle V, W \rangle_A = \mathrm{tr}(VW^T).$$

The standard matrix exponential map, $\exp : M_{n\times n} \to M_{n\times n}$ is given by the formula

$$\exp(X) = I + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{k!}X^k + \cdots.$$

It is well known that this series converges for all $X \in M_{n\times n}$.
Note: The formula $\exp(A + B) = \exp(A)\exp(B)$ no longer holds for all A, B, but it does hold if $AB = BA$.

The following two facts will prove useful:

1. If $A \in GL(n, \mathbb{R})$ and $B \in M_{n\times n}$, then $\exp(ABA^{-1}) = A\exp(B)A^{-1}$.
2. For all $X \in M_{n\times n}$, $\exp(X^T) = (\exp(X))^T$.

Theorem

Let $X \in T_I O(n)$ (i.e. $X + X^T = 0$). Then $\exp(X) \in O(n)$.

Proof.

$$\exp(X)\exp(X)^T = \exp(X)\exp(X^T) = \exp(X + X^T)$$

(using the fact that, since $X^T = -X$, $XX^T = X^T X$)

$$= \exp(0) = I.$$

This proves that $\exp(X) \in O(n)$. □

Hence, $\exp : T_I O(n) \to O(n)$ and of course $\exp(0) = I$.

Geodesics in $O(n)$ (continued)

Theorem

If $X \in T_I O(n)$, then $\gamma : \mathbb{R} \to O(n)$ defined by $\gamma(t) = \exp(tX)$ is the unique geodesic in $O(n)$ with $\gamma(0) = I$ and $\gamma'(0) = X$.

Similarly, if $A \in O(n)$ and $X \in T_I O(n)$, then $AX \in T_A O(n)$ and $\gamma(t) = A \exp(tX)$ is the unique geodesic with $\gamma(0) = A$ and $\gamma'(0) = AX$.

We omit the proof of this theorem too!

General Theorems about Geodesics

We now gather together some general theorems about geodesics. For their proofs see any book on Differential Geometry!

Theorem (Local Existence of Geodesics)

Let M be a Riemannian manifold, $p \in M$, and $v \in T_pM$. Then for a sufficiently small $\epsilon > 0$, there exists a unique constant speed parametrized geodesic $\alpha_{p,v} : (-\epsilon, \epsilon) \to M$ such that $\alpha_{p,v}(0) = p$ and $\alpha'_{p,v}(0) = v$.

The proof of this theorem reduces to the existence and uniqueness theorem for ODE's, but we won't give the details here.

Note: As the above theorem says, ϵ must be sufficiently small! (Think about $M = \mathbb{R}^2 - \{0\}$.)

Theorem

If M is complete with respect to the Riemannian distance function, then the above theorem holds with $\epsilon = \infty$ for all p, v.

Again, no proof given here!

The Riemannian exponential map

Theorem

If M is a Riemannian manifold and $p \in M$, there exists an open set $U \subseteq T_pM$ with $0 \in U$, and a mapping

$$\exp_p : U \to M$$

such that $\exp_p(v) = \alpha_{p,v}(1)$ for all $v \in U$ ($\alpha_{p,v}$ is as defined in the Theorem on the Local Existence of Geodesics.) Furthermore, $\exp_p$ maps a neighborhood of 0 in T_pM diffeomorphically onto a neighborhood of p in M.

Note: exp maps each line through the origin of T_pM to a geodesic in M through p with unit speed.

Examples of the Riemannian exponential map

1. In $\mathbb{R}^n$ with the standard metric, it is elementary to see that $\exp_p(v) = p + v$. Since $\mathbb{R}^n$ is complete, the domain of exp_p is all of $T_p\mathbb{R}^n \cong \mathbb{R}^n$.
2. In S^n, if $p \in S^n$ and $v \in T_pS^n$ (i.e., $v \perp p$), then

$$\alpha_{p,v}(t) = \cos(t|v|)p + \sin(t|v|)\frac{v}{|v|}.$$

Hence,

$$\exp_p(v) = \cos(|v|)p + \sin(|v|)\frac{v}{|v|}.$$

Since S^n is complete, the domain of $\exp_p$ is all of T_pS^n, but it is only a diffeomorphism on a smaller neighborhood of 0!

The Riemannian exponential for $O(n)$

3. Let $A \in O(n)$, and $V \in T_I O(n)$ (i.e., $V + V^T = 0$), hence $AV \in T_A O(n)$. Then

$$\exp_A(AV) = A \exp(V).$$

Note: The exponential on the left is the Riemannian exponential map; the exponential on the right is the matrix exponential. Perhaps this formula hints at where the Riemannian exponential map gets its name!

The Tangent Bundle

Given a smooth n-manifold M, the tangent bundle of M is

$$TM = \{(p, v) : p \in M \text{ and } v \in T_pM\}.$$

TM is a smooth $2n$-dimensional manifold. To see this, construct charts for TM as follows: Given a chart (U, ϕ) on M, let

$$\mathcal{U} = \{(p, v) : p \in U \text{ and } v \in T_pM\}.$$

Define $\psi : \mathcal{U} \to \mathbb{R}^{2n}$ by

$$\psi(p, v) = (\phi(p), d\phi_p(v)).$$

The charts on TM constructed in this manner cover TM and are smoothly compatible, giving TM the structure of a smooth $2n$-manifold.

Lie Groups

Definition

A group G is a Lie Group if
(i) It is a smooth manifold and
(ii) the functions

$$G \times G \to G \text{ defined by } (g, h) \mapsto gh$$

and

$$G \to G \text{ defined by } g \mapsto g^{-1}$$

are smooth functions.

Here are some examples of Lie groups:

- $\mathbb{R}^n$ under addition
- $GL(n, \mathbb{R})$ and $O(n)$ under matrix multiplication
- $\mathbb{R} - \{0\}$ under multiplication

More Lie groups

- $S^1 \subseteq \mathbb{C}$ under complex multiplication

S^2 admits no Lie group structure!
S^3 has a Lie group structure using quaternion multiplication:

$$S^3 \subseteq \mathbb{R}^4 \cong \mathbb{H}$$

where $\mathbb{H}$ denotes $\mathbb{R}^4$ with quaternion multiplication.

In fact, S^0, S^1 and S^3 are the only spheres that admit Lie group structures!

Group Actions

Given a manifold M and Lie group G (with identity element e), a left group action of G on M is a map

$$G \times M \to M \text{ written as } g * p$$

such that

1. $g_1 * (g_2 * p)) = (g_1 g_2) * p$ for all $g_1, g_2 \in G$ and $p \in M$
2. $e * p = p$ for all $p \in M$.

We say G acts smoothly on M if $(g, p) \mapsto g * p$ is smooth.
Note: Right group actions $M \times G \to M$ are defined similarly.

Actions by Isometries

Definition

If M is a Riemannian manifold, G acts smoothly on M and for all $g \in G$, the map $M \to M$ given by $p \mapsto g * p$ is an isometry, we say that G acts on M by isometries.

If G acts by isometries, it follows that

$$d(x, y) = d(g * x, g * y) \text{ for all } x, y \in M, g \in G,$$

and if $\gamma : (a, b) \to M$ is a geodesic, then $g * \gamma$ is also a geodesic.

Basic Definitions about Group Actions

Definition

Assume a group G with identity element e acts on a manifold M. Here are some important terms:

1. If $p \in M$, the orbit of p under G is defined by

$$[p] = Gp = \{g * p : g \in G\}.$$

2. If $p \in M$ and $[p] = M$, we say the action is transitive.
3. If for all $g \in G$ and $p \in M$,

$$g * p = p \Rightarrow g = e,$$

   we say the action is free.

Examples

1. $\mathbb{R}^n$ acts on $\mathbb{R}^n$ by translation: $x * y = x + y$. This is an action by isometries and it is free and transitive.
2. Define the scale group $\mathbb{R}^x = \{x \in \mathbb{R} : x > 0\}$, under multiplication. $\mathbb{R}^x$ acts on $\mathbb{R}^n$ by multiplication: $a * x = ax$.
   $[0] = \{0\}$
   If $x \neq 0, [x] = \{ax : a > 0\}$ = the open ray from the origin through x. This action is not by isometries, it is not free and it is not transitive.
3. $O(n)$ acts on $\mathbb{R}^n$ by $A * x = Ax$. This action is by isometries.
   The orbit $[0] = \{0\}$
   If $p \neq 0$, $[p] = \{x \in \mathbb{R}^n : |x| = |p|\}$, the sphere of radius $|p|$.
   This action is neither free nor transitive.

Quotients by Group Actions

Definition

If G acts on M, and $p, q \in M$, define a relation on M:

$$p \sim q \Leftrightarrow p \in [q].$$

It is easy to verify that this is an equivalence relation.

Define the quotient of M by the action of G by

$$M/G = M/\sim = \{[p] : p \in M\}.$$

M/G inherits a quotient space topology from M, but it is not always a manifold, even if M is a manifold and G acts smoothly.

Actions of Product Groups

If G and H are Lie groups, then $G \times H$ is a also a Lie group, with multiplication defined by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Suppose G and H are groups and both act on M; also, assume that these actions commute, i.e., that

$$g * (h * p)) = h * (g * p) \text{ for } g \in G, h \in G, \text{ and } p \in M.$$

Then we can define an action of $G \times H$ on M by $(g, h) * p = g * (h * p)$. (Note that if the actions don't commute, this does not define an action of the product group.)

Example: $O(n)$ and $\mathbb{R}^x$ act on $\mathbb{R}^n$ and these actions commute; so we obtain an action of $O(n) \times \mathbb{R}^x$ on $\mathbb{R}^n$ with the formula:

$$(A, a) * x = aAx.$$

Semidirect Products

If the actions of G and H on M don't commute with each other, we don't get an action of $G \times H$ on M. But we sometimes obtain an action of a semidirect product of G and H. Here is the set-up:

We say that H acts on G by isomorphisms if H acts on G and if, for all $h \in H$, the function $g \mapsto h * g$ is a group isomorphism $G \to G$. For example if G is a normal subgroup of H, then H acts on G by isomorphisms via conjugation: $h * g = hgh^{-1}$.

If H acts on G by isomorphisms, we can define a new group operation on the set $G \times H$ as follows:

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \cdot (h_1 * g_2), h_1 \cdot h_2).$$

With this operation, $G \times H$ is called the semidirect product of G with H and is denoted $G \rtimes H$.

Note: $G \rtimes H$ contains $G = G \times \{e\}$ as a normal subgroup; hence the notation.

Examples

As we have seen, R^x, $O(n)$, and $\mathbb{R}^n$ all act on $\mathbb{R}^n$. The actions of R^x and $O(n)$ commute, so they imply an action of $\mathbb{R}^x \times O(n)$ on $\mathbb{R}^n$.

But the actions of $\mathbb{R}^n$ and $O(n)$ do not commute, since $Ax + a \neq A(x + a)$, so we do not have an action of $\mathbb{R}^n \times \mathbb{R}^x \times O(n)$. However, $\mathbb{R}^x \times O(n)$ acts on the translation group $\mathbb{R}^n$ in the same way that it acts on the vector space $\mathbb{R}^n$. Hence we can form the semidirect product $\mathbb{R}^n \rtimes (\mathbb{R}^x \times O(n))$ with group operation

$$(v_1, a_1, A_1)(v_2, a_2, A_2) = (v_1 + a_1 A_1 v_2, a_1 a_2, A_1 A_2).$$

We then have an action of $\mathbb{R}^n \rtimes (\mathbb{R}^x \times O(n))$ on $\mathbb{R}^n$ given by

$$(v, a, A) * x = aAx + v.$$

This is the group that preserves shapes of objects in $\mathbb{R}^n$. It is generated by isometries and rescalings.