

Formal Reduction of a Class of Pfaffian Systems in Several Variables

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Joint work with M. A. Barkatou and Suzy S. Maddah



CIPSWNC (it's an acronym)

$$F = (F_1(x_1, x_2), F_2(x_1, x_2))$$

$$\frac{\partial}{\partial x_1} F_1 = x_1^{-3}((x_1^3 + x_2)F_1 + x_2^2 F_2)$$

$$\frac{\partial}{\partial x_1} F_2 = x_1^{-3}(-F_1 + (x_1^3 - x_2^2)F_2)$$

$$\frac{\partial}{\partial x_2} F_1 = x_2^{-1}(x_2 F_1 + x_2^2 F_2)$$

$$\frac{\partial}{\partial x_2} F_2 = x_2^{-1}(-2F_1 - 3F_2)$$

CIPSWNC (it's an acronym)

$$F = (F_1(x_1, x_2), F_2(x_1, x_2))$$

$$\frac{\partial}{\partial x_1} F = x_1^{-3} \begin{pmatrix} x_1^3 + x_2 & x_2 \\ -1 & x_1^3 - x_2^2 \end{pmatrix} F$$

$$\frac{\partial}{\partial x_2} F = x_2^{-1} \begin{pmatrix} x_2 & x_2^2 \\ -2 & -3 \end{pmatrix} F$$

Structures

$$x_1^{-3} \begin{pmatrix} x_1^3 + x_2 & x_2 \\ -1 & x_1^3 - x_2^2 \end{pmatrix}$$

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 $\{p \in R \mid p^{-1} \in \text{Frac}(R) \text{ has finitely many poles}\}$
- ▷ $\frac{1}{x_1+x_2} \notin R_L$

Formal Definition

Definition

Let $A_i \in M^\ell$ for $1 \leq i \leq \ell$. A **completely integrable Pfaffian system with normal crossings** is a system

$$\frac{\partial}{\partial x_i} F = x_i^{-(p_i+1)} A_i F,$$

with:

$$A_i A_j - A_j A_i = x_i \frac{\partial}{\partial x_i} A_j - x_j \frac{\partial}{\partial x_j} A_i$$

Formal Solutions



$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Formal Solutions

$$\boxed{\Phi(x_1^{1/s_1}, \dots, x_n^{1/s_n})}$$

▷ $\Phi \in R_L^{\ell \times \ell}$ invertible, $s_i \in \mathbb{N}^*$.

Formal Solutions

$$\boxed{\Phi(x_1^{1/s_1}, \dots, x_n^{1/s_n}) \left| \prod_{i=1}^n x_i^{\Lambda_i}\right.}$$

- ▷ $\Phi \in R_L^{\ell \times \ell}$ invertible, $s_i \in \mathbb{N}^*$.
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$$\boxed{\Phi(x_1^{1/s_1}, \dots, x_n^{1/s_n}) \mid \prod_{i=1}^n x_i^{\Lambda_i} \mid \prod_{i=1}^n \exp(Q_i)}$$

- ▷ $\Phi \in R_L^{\ell \times \ell}$ invertible, $s_i \in \mathbb{N}^*$.
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- ▷ Q_i diagonal matrices with entries in $\mathbb{C}[x_i^{-1/s_i}]$, commuting with the Λ_j .

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The Univariate Case

$$x^{-(p+1)} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{matrix} t_1^{-(p_1+1)}(*) \\ t_2^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \vdots \\ t_i^{-(p_i+1)}(*) \\ \vdots \\ t_k^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \end{matrix}$$

The Univariate Case

$$\begin{matrix}
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Diagonalization

Eigenvalue-shifting

Rank-reduction

Ramification

$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \quad | \quad A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

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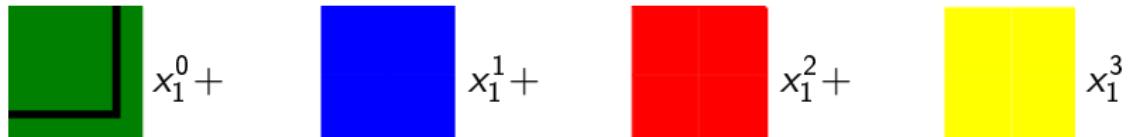
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Rank Reduction

$$x_i^{-(p_i+1)} A_i \quad \xrightarrow{T} \quad T^{-1} \left(x_i^{-(p_i+1)} A_i T - x_i \frac{\partial}{\partial x_i} T \right)$$

Definition

A transformation $T \in GL_n(\text{Frac}(R))$ is said to be compatible with a given system if the normal crossings are preserved and the Poincaré ranks of the individual subsystems are not increased.

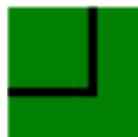


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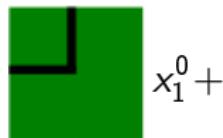
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$$x_1^1 + \quad x_1^2 + \quad x_1^3$$

$$x_1^r \det(\lambda I + \frac{A_1^{(0)}}{x_1} + A_1^{(1)})|_{x_1=0}$$

$$\mathcal{R} = \mathbb{C}[[x_1, \dots, x_n]], \quad \mathcal{R}_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = \mathcal{R}^\ell \quad | \quad A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Shearing

$$\begin{array}{c|c} \text{green} & \\ \hline & \text{x}_1^0 + \end{array}$$

$$\begin{array}{c|c} \text{blue} & \\ \hline & \text{x}_1^1 + \end{array}$$

$$\begin{array}{c|c} \text{red} & \\ \hline & \text{x}_1^2 + \end{array}$$

$$\begin{array}{c|c} \text{yellow} & \\ \hline & \text{x}_1^3 \end{array}$$

$$\left(\begin{array}{ccccccccc} x_1 & & & & & & & & \\ & x_1 & & & & & & & \\ & & \ddots & & & & & & \\ & & & x_1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & \ddots & & \\ & & & & & & & 1 & \end{array} \right)$$

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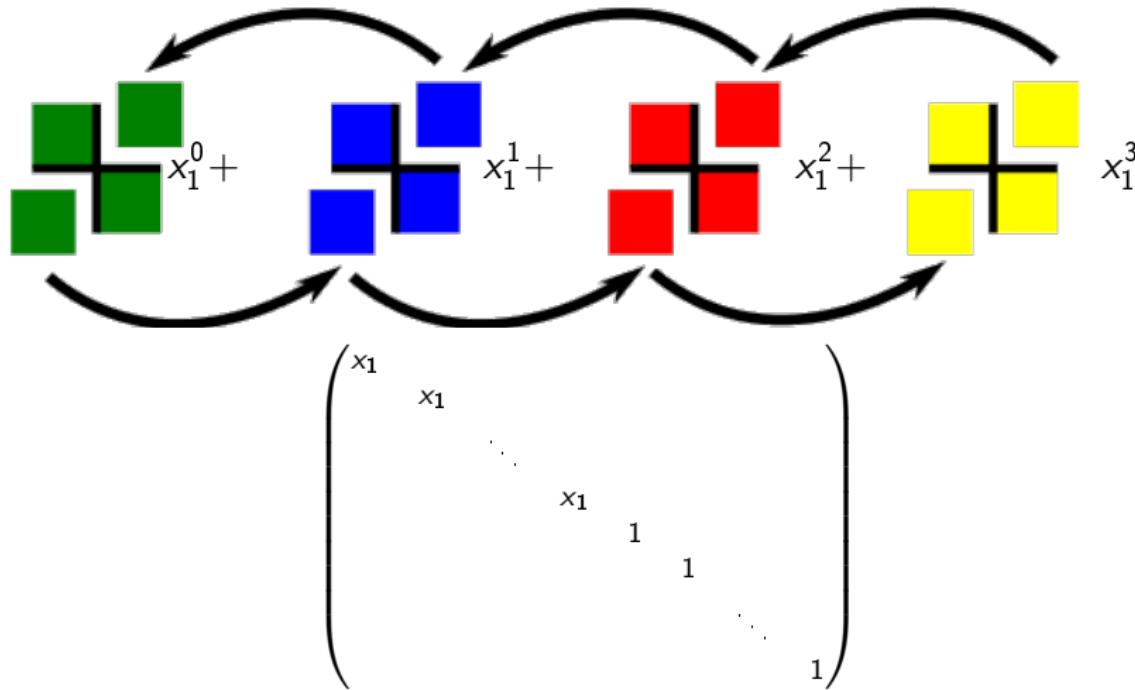
Shearing

$$\begin{array}{c}
 \text{green square} \quad \text{green square} \\
 \text{green square} \quad \text{green square} \\
 \text{black cross} \\
 \text{green square} \quad \text{green square}
 \end{array} x_1^0 +
 \begin{array}{c}
 \text{blue square} \quad \text{blue square} \\
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 \text{blue square} \quad \text{blue square}
 \end{array} x_1^1 +
 \begin{array}{c}
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 \text{red square} \quad \text{red square} \\
 \text{black cross} \\
 \text{red square}
 \end{array} x_1^2 +
 \begin{array}{c}
 \text{yellow square} \quad \text{yellow square} \\
 \text{yellow square} \\
 \text{black cross}
 \end{array} x_1^3$$

$$\left(\begin{array}{ccccccccc}
 x_1 & & & & & & & & \\
 & x_1 & & & & & & & \\
 & & \ddots & & & & & & \\
 & & & x_1 & & & & & \\
 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & & & & \ddots & & \\
 & & & & & & & 1 & \\
 & & & & & & & & 1
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$$\begin{array}{c}
 \text{Diagram 1: } \\
 \begin{array}{c} \text{Green} \quad \text{Blue} \\ \text{Blue} \quad \text{Green} \end{array} x_1^0 +
 \end{array}
 \quad
 \begin{array}{c}
 \text{Diagram 2: } \\
 \begin{array}{c} \text{Blue} \quad \text{Red} \\ \text{Green} \quad \text{Blue} \end{array} x_1^1 +
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 \quad
 \begin{array}{c}
 \text{Diagram 3: } \\
 \begin{array}{c} \text{Red} \quad \text{Yellow} \\ \text{Blue} \quad \text{Red} \end{array} x_1^2 +
 \end{array}
 \quad
 \begin{array}{c}
 \text{Diagram 4: } \\
 \begin{array}{c} \text{Yellow} \quad \text{Black} \\ \text{Red} \quad \text{Yellow} \end{array} x_1^3
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 \text{[Image: A 2x2 matrix with green top-left, blue top-right, green bottom-left, and black bottom-right blocks.]} \\
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 \text{[Image: A 2x2 matrix with yellow top-left, black top-right, red bottom-left, and yellow bottom-right blocks.]} \\
 x_1^3
 \end{array}$$

$$\left(\begin{array}{ccccccccc}
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A Simple Illustration

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ a & b & 0 \\ \hline c & d & 0 \end{array} \right) + \left(\begin{array}{cc|c} * & * & 0 \\ * & * & e \\ \hline * & * & * \end{array} \right) x_1$$

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Column Reduction

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 1 & -8y & -2y - 2x^2 - x^2y - 2x^4 & -2x + 2y + 2x^2 + xy + 2x^3 + x^2y + 2x^4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 - x - x^2 - x^3 & 1 + 2x + 2x^2 + x^3 \\ 0 & 0 & -x - x^3 & x + x^2 + x^3 \end{pmatrix} + \mathcal{O}(x,y)^5$$

$$\text{rank}(A)=3$$

Question

Is there a **unimodular** transformation to reduce one column to zero?
How to construct it?

$$p_1 A_1 + p_2 A_2 + p_3 A_3 + 1 A_4 = 0$$

$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

“Free At Last, Free At Last, Thank God Almighty We Are Free At Last” ©Estate of Martin Luther King, Jr., Inc.

Lemma

Let $S = \langle A_1, \dots, A_\ell \rangle$ be a free submodule of M . Then the matrix rank of A in $\text{Frac}(R) \otimes_R M$ is equal to the module rank of S .

How do we get the module basis into our matrix?
Can this be done via a unimodular transformation?

$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Nakayama's Lemma

Theorem

Let S be a finite R -module and \mathcal{I} an ideal of R . If $S = \mathcal{I}S$ then there exists an $a \in R$ such that $aS = 0$ and $a \equiv 1 \pmod{\mathcal{I}}$. If in addition $\mathcal{I} \subset \text{rad}(R)$, then $S = 0$.

Corollary

Let (R, \mathcal{M}) be a local ring and S a free R -module. Let $\tilde{S} := S/\mathcal{M}S$ and let B be a vector space basis of \tilde{S} over R/\mathcal{M} . For each element in B , take one element of its preimage under the canonical homomorphism. Then these elements form a basis of S and every basis is obtained that way.

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Column Reduction

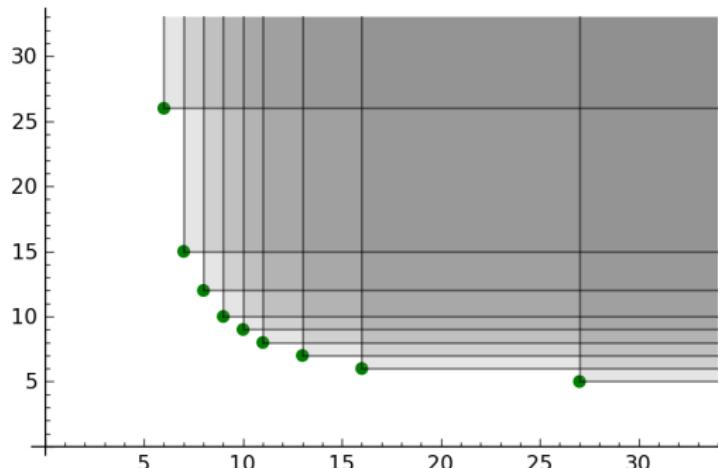
$$\begin{array}{cccc}
 A_1 & A_2 & A_3 & A_4 \\
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 A = & & & + \mathcal{O}(x,y)^5
 \end{array}$$

Gröbner Bases

Given $p, p_1, \dots, p_\ell \in \mathbb{C}[x_1, \dots, x_n]$, decide if $p \in \langle p_1, \dots, p_\ell \rangle$.

$$\begin{array}{l} x_1 < x_2 < \dots < x_n \\ x_1^4 < x_1^2 x_2 < x_2^3 < x_1 x_2^3 \end{array} \quad t_1 \mid t_2 \Rightarrow t_1 \leq t_2$$

$$p = q_1 p_1 + q_2 p_2 + \dots + r \text{ with } r = 0 \text{ or } \text{lt}(p_i) \nmid r_j$$



$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Gröbner Bases Variants

Standard Bases

Standard bases = Gröbner bases for power series.

Replace *leading* by *trailing* coefficient and truncate.

Gröbner Bases for Modules

Given $p, p_1, \dots, p_\ell \in \mathbb{C}[x_1, \dots, x_n]^k$, decide if $p \in \langle p_1, \dots, p_\ell \rangle$.

$$\begin{pmatrix} 1 + 3x_2 \\ -2x_1x_2 + x_3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} x_2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_3 \end{pmatrix}$$

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Replace *leading* by *trailing* coefficient and truncate.

Gröbner Bases for Modules

Given $p, p_1, \dots, p_\ell \in \mathbb{C}[x_1, \dots, x_n]^k$, decide if $p \in \langle p_1, \dots, p_\ell \rangle$.

New term orders:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \end{pmatrix} <_{TOP} \begin{pmatrix} 0 \\ x_3x_4^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \end{pmatrix} >_{POT} \begin{pmatrix} 0 \\ x_3x_4^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Gröbner Bases Variants

Standard Bases for Modules

Use TOP/POT orders with trailing coefficients.

WARNING

A GB (StdB) is in general NOT a module basis. It is, however, a set of generators of the module.

$$A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ \begin{matrix} 1 & -8y & -2y - 2x^2 - x^2y - 2x^4 & -2x + 2y + 2x^2 + xy + 2x^3 + x^2y + 2x^4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 - x - x^2 - x^3 & 1 + 2x + 2x^2 + x^3 \\ 0 & 0 & -x - x^3 & x + x^2 + x^3 \end{matrix} \end{pmatrix} + \mathcal{O}(x,y)^5$$

$$G = \{A_1, A_2, A_3, B_1, B_2\}$$

$$B_1 = (0, -6, -8 - 8x - 8x^2 - 8x^3, -8x - 8x^3) + \mathcal{O}(x,y)^5, \quad B_2 = (0, -3/4, 0, 0) + \mathcal{O}(x,y)^5$$

$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Linear Combinations

$$A_4 = p_1 A_1 + p_2 A_2 + p_3 A_3 + p_4 B_1 + p_5 B_2$$

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$$A_4 = (p_1 + p_4 c_1 + p_5 d_1) A_1 + (p_2 + p_4 c_2 + p_5 d_2) A_2 + (p_3 + p_4 c_3 + p_5 d_3) A_3$$

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$$A = \begin{pmatrix} 1 & -8y & -2y - 2x^2 - x^2y - 2x^4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 - x - x^2 - x^3 & 0 \\ 0 & 0 & -x - x^3 & 0 \end{pmatrix} + \mathcal{O}(x,y)^5$$

The Multivariate Case

$$\begin{matrix}
 & & t_{i,1}^{-(p_{i,1}+1)}(*) \\
 & & t_{i,2}^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\
 x_i^{-(p_i+1)} \begin{pmatrix} * & * \\ * & * \end{pmatrix} & \swarrow & \vdots \\
 & & t_{i,j}^{-(p_{i,j}+1)}(*) \\
 & & \vdots \\
 & & t_{i,k}^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix}
 \end{matrix}$$

Diagonalization

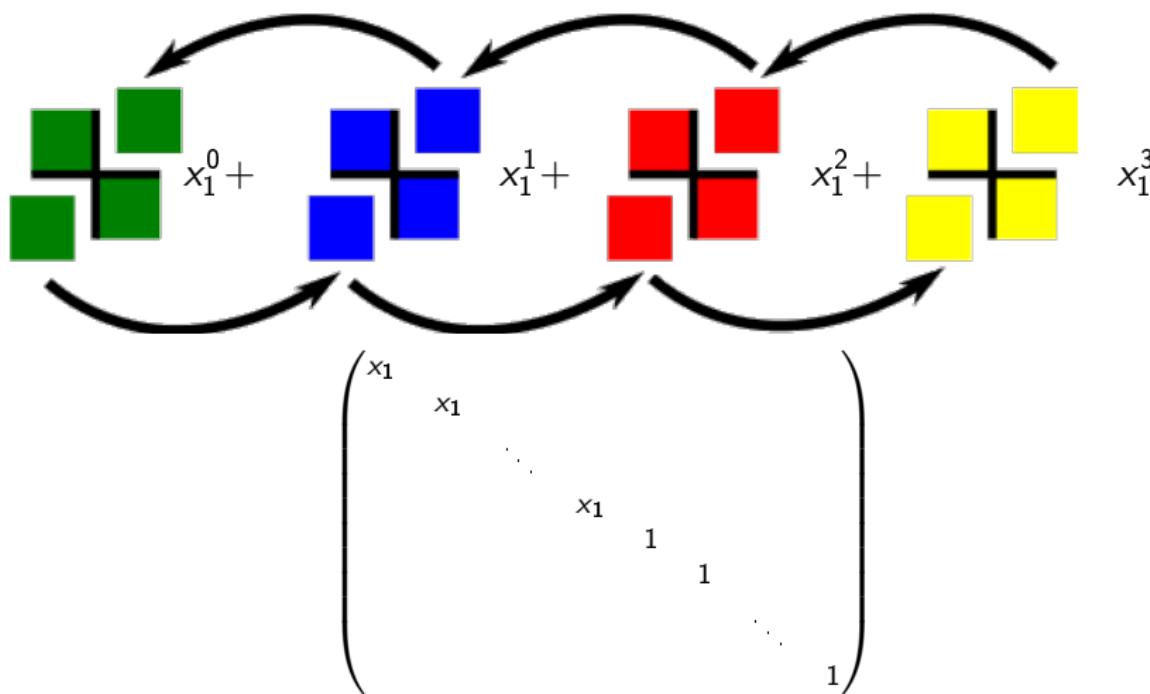
Eigenvalue-shifting

Rank-reduction

Ramification

$$R = \mathbb{C}[[x_1, \dots, x_n]], \quad R_L = \mathbb{C}((x_1, \dots, x_n)), \quad M = R^\ell \mid A \in M^\ell, \quad S = \langle A_1, \dots, A_\ell \rangle$$

Shearing



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