Formal Reduction of a Class of Pfaffian Systems in Several Variables

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CIPSWNC (it's an acronym)

$$F = (F_1(x_1, x_2), F_2(x_1, x_2))$$

$$\frac{\partial}{\partial x_1} F_1 = x_1^{-3}((x_1^3 + x_2) F_1 + x_2^2 F_2)$$
$$\frac{\partial}{\partial x_1} F_2 = x_1^{-3}(-F_1 + (x_1^3 - x_2^2) F_2)$$

$$\frac{\partial}{\partial x_2} F_1 = x_2^{-1}(x_2 F_1 + x_2^2 F_2)$$
$$\frac{\partial}{\partial x_2} F_2 = x_2^{-1}(-2F_1 - 3F_2)$$

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$$\frac{\partial}{\partial x_2} F = x_2^{-1} \begin{pmatrix} x_2 & x_2^2 \\ -2 & -3 \end{pmatrix} F$$

Structures

$$x_1^{-3}\begin{pmatrix} x_1^3 + x_2 & x_2 \\ -1 & x_1^3 - x_2^2 \end{pmatrix}$$

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$$\{p \in R \mid p^{-1} \in \mathrm{Frac}(R) \text{ has finitely many poles}\}$$

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$$\{p \in R \mid p^{-1} \in \mathrm{Frac}(R) \text{ has finitely many poles}\}$$

- $\frac{1}{x_1+x_2} \notin R_L$

Formal Definition

Definition

Let $A_i \in M^\ell$ for $1 \leq i \leq \ell$. A **completely integrable Pfaffian system with normal crossings** is a system

$$\frac{\partial}{\partial x_i} F = x_i^{-(p_i+1)} A_i F,$$

with:

$$A_i A_j - A_j A_i = x_i \frac{\partial}{\partial x_i} A_j - x_j \frac{\partial}{\partial x_j} A_i$$

Formal Solutions

Formal Solutions

$\Phi(x_1^{1/s_1}, \ldots, x_n^{1/s_n})$		

▷ $\Phi \in R_L^{\ell\times\ell}$ invertible, $s_i \in \mathbb{N}^*$.

Formal Solutions

$\Phi(x_1^{1/s_1}, \ldots, x_n^{1/s_n})$	$\prod_{i=1}^n x_i^{\Lambda_i}$	

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- ▷ $\Lambda_i \in \mathbb{C}^{\ell \times \ell}$.

Formal Solutions

$$\boxed{\Phi(x_1^{1/s_1}, \ldots, x_n^{1/s_n})}\boxed{\prod_{i=1}^n x_i^{\Lambda_i}}\boxed{\prod_{i=1}^n \exp(Q_i)}$$

- $\Phi \in R_L^{\ell\times\ell}$ invertible, $s_i \in \mathbb{N}^*$.
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- Q_i diagonal matrices with entries in $\mathbb{C}[x_i^{-1/s_i}]$, commuting with the Λ_j.

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The Univariate Case

$$x^{-(p+1)}\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$t_1^{-(p_1+1)}(*)$$

$$t_2^{-1}\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\vdots$$

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Diagonalization Eigenvalue-shifting Rank-reduction Ramification

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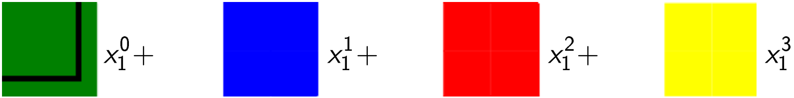
Rank Reduction

$$x_i^{-(p_i+1)} A_i \quad \xrightarrow{T} \quad T^{-1}\left(x_i^{-(p_i+1)} A_i T - x_i \frac{\partial}{\partial x_i} T\right)$$

Definition

A transformation $T \in GL_n(\mathrm{Frac}(R))$ is said to be compatible with a given system if the normal crossings are preserved and the Poincaré ranks of the individual subsystems are not increased.



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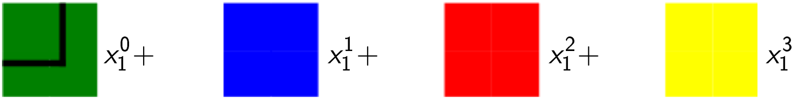

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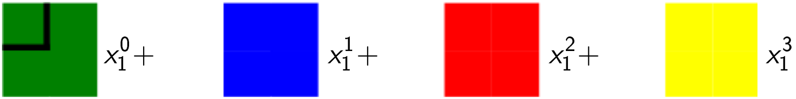

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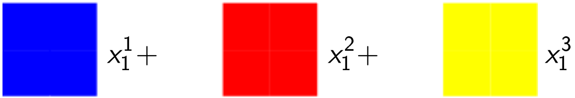

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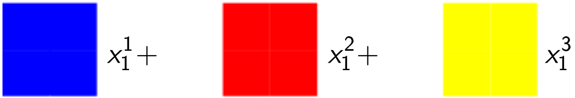
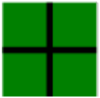
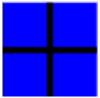
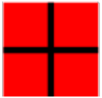
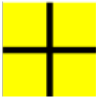


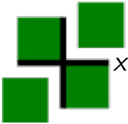
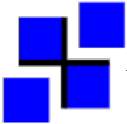
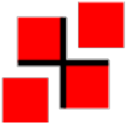
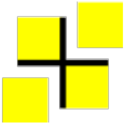
$$x_1^r \det(\lambda I + \frac{A_1^{(0)}}{x_1} + A_1^{(1)})|_{x_1=0}$$

Shearing

 $x_1^0 +$  $x_1^1 +$  $x_1^2 +$  x_1^3

$$\begin{pmatrix} x_1 &&&&&&& \\ & x_1 &&&&&& \\ && \ddots &&&&& \\ &&& x_1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& \ddots & \\ &&&&&&& 1 \end{pmatrix}$$

Shearing

 $x_1^0 +$  $x_1^1 +$  $x_1^2 +$  x_1^3

$$\begin{pmatrix} x_1 & & & & & & & \\ & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

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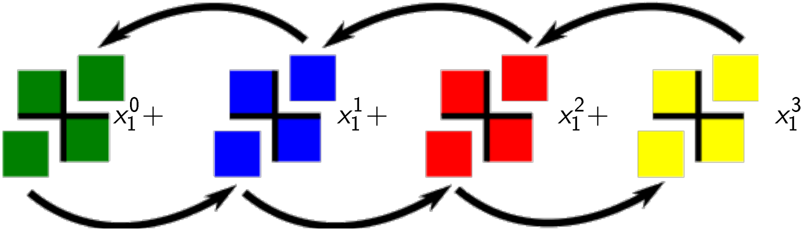

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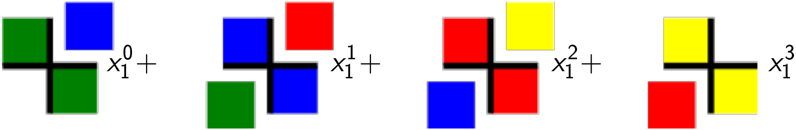

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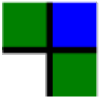
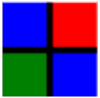
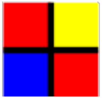
 $x_1^0 +$  $x_1^1 +$  $x_1^2 +$  x_1^3

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A Simple Illustration

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ a & b & 0 \\ \hline c & d & 0 \end{array}\right) + \left(\begin{array}{cc|c} * & * & 0 \\ * & * & e \\ \hline * & * & * \end{array}\right) x_1$$

A Simple Illustration

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Column Reduction

$$A=\begin{array}{c} \\ \\ \end{array}\begin{array}{c}\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}\\ \begin{pmatrix} 1 & -8y & -2y-2x^2-x^2y-2x^4 & -2x+2y+2x^2+xy+2x^3+x^2y+2x^4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1-x-x^2-x^3 & 1+2x+2x^2+x^3 \\ 0 & 0 & -x-x^3 & x+x^2+x^3 \end{pmatrix}\end{array}$$

$$+\mathcal{O}(x,y)^5$$

$$\text{rank}(A)=3$$

Question

Is there a unimodular transformation to reduce one column to zero? How to construct it?

$$p_1 A_1+p_2 A_2+p_3 A_3+1 A_4=0$$

"Free At Last, Free At Last, Thank God Almighty We Are Free At Last" ©Estate of Martin Luther King, Jr., Inc.

Lemma

Let $S = \langle A_1, \dots, A_\ell \rangle$ be a free submodule of M. Then the matrix rank of A in $\mathrm{Frac}(R) \otimes_R M$ is equal to the module rank of S.

How do we get the module basis into our matrix?
Can this be done via a unimodular transformation?

Nakayama's Lemma

Theorem

Let S be a finite R-module and $\mathcal{I}$ an ideal of R. If $S = \mathcal{I}S$ then there exists an $a \in R$ such that $aS = 0$ and $a \equiv 1 \mod \mathcal{I}$. If in addition $\mathcal{I} \subset \mathrm{rad}(R)$, then $S = 0$.

Corollary

Let $(R, \mathcal{M})$ be a local ring and S a free R-module. Let $\tilde{S} := S/\mathcal{M}S$ and let B be a vector space basis of $\tilde{S}$ over $R/\mathcal{M}$. For each element in B, take one element of its preimage under the canonical homomorphism. Then these elements form a basis of S and every basis is obtained that way.

Column Reduction

$$
\begin{array}{cc}
 & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\
A= & \begin{pmatrix}
1 & -8y & -2y-2x^2-x^2y-2x^4 & -2x+2y+2x^2+xy+2x^3+x^2y+2x^4 \\
0 & 3 & 0 & 0 \\
0 & 0 & -1-x-x^2-x^3 & 1+2x+2x^2+x^3 \\
0 & 0 & -x-x^3 & x+x^2+x^3
\end{pmatrix}
\end{array}
$$

$$+\mathcal{O}(x,y)^5$$

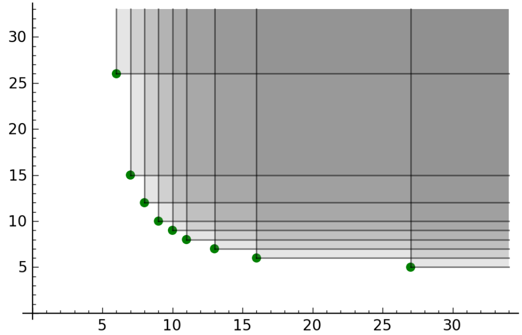
Gröbner Bases

Given $p, p_1, \ldots, p_\ell \in \mathbb{C}[x_1, \ldots, x_n]$, decide if $p \in \langle p_1, \ldots, p_\ell \rangle$.

$$x_1 < x_2 < \cdots < x_n$$
$$x_1^4 < x_1^2 x_2 < x_2^3 < x_1 x_2^3$$

$$t_1 \mid t_2 \Rightarrow t_1 \leq t_2$$

$$p = q_1 p_1 + q_2 p_2 + \cdots + r \text{ with } r = 0 \text{ or } \mathrm{lt}(p_i) \nmid r_j$$



Gröbner Bases Variants

Standard Bases

Standard bases = Gröbner bases for power series.
Replace *leading* by *trailing* coefficient and truncate.

Gröbner Bases for Modules

Given $p, p_1, \ldots, p_\ell \in \mathbb{C}[x_1, \ldots, x_n]^k$, decide if $p \in \langle p_1, \ldots, p_\ell \rangle$.

$$\begin{pmatrix} 1 + 3x_2 \\ -2x_1x_2 + x_3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} x_2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ x_1x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ x_3 \end{pmatrix}$$

Gröbner Bases Variants

Standard Bases

Standard bases = Gröbner bases for power series.
Replace *leading* by *trailing* coefficient and truncate.

Gröbner Bases for Modules

Given $p, p_1, \ldots, p_\ell \in \mathbb{C}[x_1, \ldots, x_n]^k$, decide if $p \in \langle p_1, \ldots, p_\ell \rangle$.
New term orders:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \end{pmatrix} <_{TOP} \begin{pmatrix} 0 \\ x_3 x_4^3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_1^2 \\ 0 \end{pmatrix} >_{POT} \begin{pmatrix} 0 \\ x_3 x_4^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Gröbner Bases Variants

Standard Bases for Modules

Use TOP/POT orders with trailing coefficients.

WARNING

A GB (StdB) is in general NOT a module basis. It is, however, a set of generators of the module.

$$
\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}
$$

$$
A=\begin{pmatrix} 1 & -8y & -2y-2x^2-x^2y-2x^4 & -2x+2y+2x^2+xy+2x^3+x^2y+2x^4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1-x-x^2-x^3 & 1+2x+2x^2+x^3 \\ 0 & 0 & -x-x^3 & x+x^2+x^3 \end{pmatrix}
$$

$$
+\mathcal{O}(x,y)^5
$$

$$
G=\{A_1,A_2,A_3,B_1,B_2\}
$$

$$
B_1=(0,-6,-8-8x-8x^2-8x^3,-8x-8x^3)+\mathcal{O}(x,y)^5,\ B_2=(0,-3/4,0,0)+\mathcal{O}(x,y)^5
$$

Linear Combinations

$$A_4 = p_1 A_1 + p_2 A_2 + p_3 A_3 + p_4 B_1 + p_5 B_2$$

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$$B_1 = c_1 A_1 + c_2 A_2 + c_3 A_3$$

$$B_2 = d_1 A_1 + d_2 A_2 + d_3 A_3$$

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$$B_2 = d_1 A_1 + d_2 A_2 + d_3 A_3$$

$$A_4 = (p_1 + p_4 c_1 + p_5 d_1) A_1 + (p_2 + p_4 c_2 + p_5 d_2) A_2 + (p_3 + p_4 c_3 + p_5 d_3) A_3$$

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$$A_4 = (p_1 + p_4 c_1 + p_5 d_1) A_1 + (p_2 + p_4 c_2 + p_5 d_2) A_2 + (p_3 + p_4 c_3 + p_5 d_3) A_3$$

$$A = \begin{pmatrix} 1 & -8y & -2y - 2x^2 - x^2 y - 2x^4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 - x - x^2 - x^3 & 0 \\ 0 & 0 & -x - x^3 & 0 \end{pmatrix} + \mathcal{O}(x,y)^5$$

The Multivariate Case

$$x_i^{-(p_i+1)} \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$t_{i,1}^{-(p_{i,1}+1)}(*)$$

$$t_{i,2}^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\vdots$$

$$t_{i,j}^{-(p_{i,j}+1)}(*)$$

$$\vdots$$

$$t_{i,k}^{-1} \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

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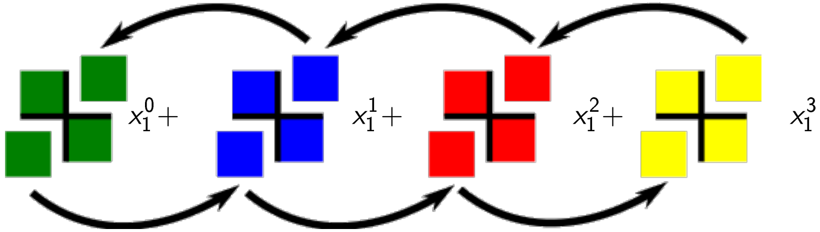
Shearing



$$\begin{pmatrix} x_1 & & & & & & & \\ & x_1 & & & & & & \\ & & \ddots & & & & & \\ & & & x_1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$