Exploring new symmetries in black hole mechanics

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Seminar IDP Tours
23$^{\text{th}}$ November

Based on
J. BA, E. R. Livine [JHEP '21]
J. BA, E. R. Livine, S. Mukohyama, J-P. Uzan [JHEP '22]
J. BA, D. Oriti, E. R. Livine, G. Piani, [JCAP '23]

Context and Motivations

- Black holes physics ↔ investigations of the symmetries in General Relativity (and beyond)
- Thermodynamical properties of black holes : quasi-local charges / flux balance laws

Different manifestations of symmetries in GR

- Space-time symmetries under diffeomorphisms :
  covariant phase space → well-defined formalism to associate flux -balance law for a diffeomorphism
  [Brown, Henneaux '86] ... [Wald, Zoupas '99]...[Barnich, Brandt '02]
  [Freidel, Geiller, Pranzetti '21] ... and many other
- Killing-Yano asymptotic charges [Kastor, Traschen '14]
- Non spacetime symmetries for 2d GR: integrable system
  → axi-symmetric phase space of GR, colliding waves : Ehlers/Matzner and Geroch groups
  [Geroch '72] ... [Nicolai, Samtleben '96] ... [Penna '22]
- Non spacetime symmetries for 1d GR : treated as a mechanical system
  Relevant for cosmology and black hole mechanics
  [**BA**, Livine '19 '20] [Geiller, Livine, Sartini '20] [**BA**, Livine, Oriti '23]

Context and Motivations

- Observations of astrophysical black holes have entered in a new era
  LIGO-Virgo '15 / Event Horizon Telescope '21/ NanoGRAV '22 / GRAVITY
- Trigger many efforts to further develop black hole perturbation theory [Regge, Wheeler '57]
  Non-linearities, spectral instability, environmental effects

- Can we identify manifestation of fundamental symmetries in compact objects perturbations ?
- Examples of universal behavior for compact objects:
  → universal behavior of transmission/reflexion coefficients for wave scattering
  → conformal symmetry of the wave operator for test fields near black holes
  [Maldacena, Strominger '97] [Castro, Maloney, Strominger '10] [Bertini '11]
  → equation-of-state independent relation for neutrons stars: I-Love-Q relations
  [Yagi, Nunes '17]
  → vanishing of tidal deformability of 4d GR black holes in vacuum
  [Damour, Nagar '09]

Main goal

- Review the notion of vanishing of Love numbers in 4d GR black holes
- Revisit and improve one proposal to explain it via symmetries using 1d mechanics
- Connect this to the non-spacetime symmetries appearing in 1d symmetry reduced GR
- Explain the techniques to identify these symmetries: embedding / Einsenhart-Duval lift

Outlines

- Vanishing of Love numbers and tidal deformation
- Symmetry protection for static perturbations
- Revealing non-standard symmetries of black hole mechanics

Mystery of the vanishing of black hole's Love numbers

Mystery of the vanishing of black holes Love numbers

Newtonian Love numbers

- Binary system (M, M') separated by a distance b
- M' generates a tidal environment : purely static quadrupole tide

$$U^{\text{tidal}} \simeq r^2 \mathcal{E} P_2(\cos\theta)$$

- In the region of size $R \ll b$ around M, the newtonian potential U reads

$$U = \frac{GM}{r} - \frac{1}{2}\left[r^2 + 2k_2 \frac{R^5}{r^3}\right] \mathcal{E} P_2(\cos\theta)$$

where k_2 is the Love number : coefficient of the decaying branch

Relativistic Love numbers

- Relativistic theory of tidal Love numbers [Damour, Nagar '09] [Binnington, Poisson '09]

$$g_{tt} = -f - f^2 \left[r^2 + 2\mathcal{C}(r) k_2^{\text{el}} \frac{R^5}{r^3}\right] \mathcal{E} P_2(\cos\theta)$$

- Main point: electric Love number k_2^{el} (gauge-invariant) counterpart to newtonian one k_2
  Also new magnetic Love number k_2^{mag}. Even more complicated when rotating
- Main difficulty: k_2 is defined at finite distance of the object
  No good definition of multipole moments at finite distance in GR : generate ambiguities
- Current progress [Poisson '20 '21]

Mystery of the vanishing of black holes Love numbers

Black holes's Love numbers

- Love numbers can be computed for a large class of self-gravitating objects
- For 4d vacuum solutions of GR, the Love numbers vanish at all order in the multipolar expansion

$$g_{tt} = -f - f^2 \left[r^2 + 2\mathcal{C}(r) k_2^{\text{el}} \frac{R^5}{r^3} \right] \mathcal{E} P_2(\cos\theta) \tag{1}$$

with

$$\mathcal{C}(r) = -\frac{15}{16} \frac{r^5}{M^5} \log f - \frac{5}{8} \frac{r(r-M)(3r^2 - 6Mr - 2M^2)}{M^4 f} \tag{2}$$

divergent as $f \to 0$, so $k_2^{\text{el}} = 0$ [Damour, Nagar '09] [Binnington, Poisson '09] [Poisson '21]

- This is no longer true in higher dimensions $d > 4$ or in modified gravity

Same results with other approaches

- EFT techniques: employ the worldline approach where Love numbers are coupling constant. Vanishing of LN appears as a fine tuning ! [Kol, Smolkin '12] [Porto '16] [Kälin, Porto '20]
- Test field approximation: compute the profil of spin-0, spin-1 and spin-2 test field on Schwarzschild or Kerr / No decaying profil is consistent [Hui, Joyce, Penco, Santoni, Salomon '21]
- This suggests that there might be a symmetry at play which protect BH deformability
  Which type of symmetry ?

Mystery of the vanishing of black holes Love numbers

Love symmetry from near-horizon conformal symmetry

- Near-horizon $SL(2,\mathbb{R})$ symmetry of test fields:
  → use the static limit of the near horizon symmetry of the wave operator
  → scalar modes organize into the discrete representation of $\mathfrak{sl}(2,\mathbb{R})$
  → valid only near the horizon
  → spacetime symmetry interpretation
  [Charalambous, Dubovsky, Ivanov (PRL) '21, '22]
- Near-horizon carrollian structure has been argued to also play a role [Penna '18]

Love symmetry from ladder structure

- New type of symmetry introduced by Hui, Joyce, Penco, Santoni and Salomon (HJPSS) [HJPSS '21]
- Inspired from previous work on ladders symmetries of de Sitter [Compton, Morrisson '03]
  → valid on in the full spacetime bulk
  → not spacetime symmetries: no well defined geometrical origin

- Goal: try to understand the geometrical origin of the HJPSS symmetry !

Symmetry of static black hole perturbations

Symmetry of static black hole perturbations

- Consider a static test scalar field

$$\varphi(r,\theta,\phi) = \sum_{\ell,m} \varphi_{\ell,m}(r) Y^{\ell}_{m}(\theta,\varphi) \tag{3}$$

- Equation of motion on the Schwarzschild background: $z = r(r - r_s)$

$$H_\ell \varphi_\ell = z\varphi''_\ell + z'\varphi'_\ell - \ell(\ell+1)\varphi_\ell = 0 \qquad \varphi_\ell(r) = A_\ell G_\ell(r) + B_\ell D_\ell(r) \tag{4}$$

- G_ℓ represents the tidal scalar field, D_ℓ the static response and B_ℓ the Love number
- Ladders operators allow to climb up and down the multipole tower

$$L^+_\ell = z\partial_r + \frac{\ell-1}{2} z' \qquad L^-_\ell = z\partial_r - \frac{\ell+2}{2} z' \,, \qquad \boxed{\varphi_\ell = L^+_{\ell-1} \ldots L^+_0 \varphi_0} \tag{5}$$

HJPSS Love symmetry

- HJPSS conserved charge for the ℓ-mode: Q_ℓ

$$Q_0 = \left(-z\partial_r + z'\right)\varphi_0 \,, \qquad Q_\ell \varphi_\ell = L^+_{\ell-1} \ldots L^+_1 Q_0 L^-_0 \ldots L^-_{\ell-1} \varphi_\ell \qquad [Q_\ell, H_\ell] = 0 \tag{6}$$

- HJPSS argument

$$\boxed{\delta_\ell G_\ell = Q_\ell G_\ell = 0 \,, \qquad \delta_\ell D_\ell = Q_\ell D_\ell \neq 0} \tag{7}$$

- Conservation of the charge implies that $B_\ell = 0$: trade regularity for symmetry criteria
- What is the geometrical origin of this symmetry ?

Symmetry of static black hole perturbations

- System is described by a Sturm-Liouville equation
- Look for the conformal symmetry of such 1d system to explain the HJPSS construction

Conformal symmetry for 1D system:

- Sturm-Liouville equation

$$\psi'' + V\psi = 0\,, \qquad \Rightarrow \qquad \psi = c_1\psi_1 + c_2\psi_2$$

- Wronskian is constant:

$$w = w[\psi_1, \psi_2] = \psi_1\psi_2' - \psi_2\psi_1' \qquad w' \simeq 0$$

- Two natural conserved charges:

$$w_1 = w[\psi_1, \psi] = \psi_1\psi' - \psi\psi_1'\,, \qquad w_2 = w[\psi_2, \psi] = \psi_2\psi' - \psi\psi_2' \tag{8}$$

- Any power of these conserved charges is a conserved charge.

$$\left|\begin{array}{l} Y_+ = w_1\,, \\ Y_- = w_2\,, \end{array}\right. \qquad \left|\begin{array}{l} Q_+ = w_1^2/2\,, \\ Q_- = w_2^2/2\,, \\ Q_0 = w_1 w_2/2\,. \end{array}\right. \tag{9}$$

- Action of the charge on the solutions space: dilate and squeeze the two branches

$$\delta_{Q_+}\psi \simeq -c_2 w\psi_1\,, \qquad \delta_{Q_-}\psi \simeq +c_1 w\psi_2\,, \qquad \delta_{Q_0}\psi \simeq \frac{1}{2}w(c_2\psi_2 - c_1\psi_1)\,. \tag{10}$$

Symmetry of static black hole perturbations

Hamiltonian formulation of the Wronskian charges

- Phase space:

$$p = \frac{\delta L}{\delta \psi'} = \psi' \,, \qquad H = p\psi' - L = \frac{1}{2}\left(p^2 + V(x)\psi^2\right) \,, \tag{11}$$

- Conformal Noether charges

$$Q[\xi, \psi] = \frac{1}{4}\xi''\psi^2 - \frac{1}{2}\xi'\psi p + \xi H \qquad \xi(x) = \alpha_+\psi_1^2 + \alpha_-\psi_2^2 + \alpha_0\psi_1\psi_2$$

  where (ψ_1, ψ_2) are the two linearly independent solutions of the dynamics
- Translation Noether charges

$$Y[\chi, \psi] = \chi p - \chi'\psi \qquad \chi = \eta_+\psi_1 + \eta_-\psi_2 \tag{12}$$

- Charge algebra for an arbitrary 1d particle in a x-potential

$$\{Q_+, Q_-\} = 2wQ_0 \,, \qquad \{Q_0, Q_+\} = -wQ_+ \,, \qquad \{Q_0, Q_-\} = wQ_- \,, \tag{13}$$

$$\{Q_0, Y_\pm\} = \mp\frac{w}{2}Y_\pm \,, \qquad \{Q_+, Y_-\} = wY_+ \,, \qquad \{Q_-, Y_+\} = -wY_- \,, \tag{14}$$

$$\{Y_+, Y_-\} = w \,. \tag{15}$$

- 1d Schrödinger algebra: $\mathfrak{sh}(1) = \mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R} \times \mathbb{R})$
- The conformal sector transforms solution onto solution with different energy

$$\{Q, H\} = \frac{\mathrm{d}Q}{\mathrm{d}x} - \frac{\partial Q}{\partial x} \neq 0 \tag{16}$$

- What is the action of the level of the action ?

Conformal transformation at the level of the action

- Consider 1d field Ψ in a time-dependent potential

$$S[\psi] = \int \mathrm{d}x \, L[x,\psi] \,, \qquad \text{with} \quad L[x,\psi] = \frac{1}{2}\Big[(\psi')^2 - V(x)\psi^2\Big] \,,$$

- Consider the finite symmetry transformations

$$\begin{aligned} x \quad &\mapsto \quad \tilde{x} \quad = \quad f(x) \,, \\ \psi(x) \quad &\mapsto \quad \tilde{\psi}(\tilde{x}) \quad = \quad f'(x)^{1/2}\psi(x) \,, \end{aligned} \tag{17}$$

- Non-standard because $\Psi_\ell(r)$ does not transform as a scalar quantity
- Action transform as

$$\Delta S = \frac{1}{2}\int \mathrm{d}x \left\{ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f''}{f'}\psi^2\right) - \left[\frac{1}{2}\mathrm{Sch}[f] + (f')^2(V\circ f) - V\right]\psi^2 \right\}$$

$$\mathrm{Sch}[f] = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \,,$$

- Noether symmetry if

$$\boxed{\mathrm{Sch}[f] = 2V - 2(f')^2(V\circ f)} \tag{18}$$

- General feature for such 1d system: conformal symmetry for any form of x-potential generated by $(Q_\pm, Q_0)$
- Additonnal symmetry under a translation in solution space: generated by $Y_\pm$

$$x \to \tilde{x} = x \,, \tag{19}$$

$$\psi \to \tilde{\psi}(\tilde{x}) = \psi(x) + \chi(x) \,, \tag{20}$$

- Schrödinger symmetry of mechanical system generated by $(Q_\pm, Q_0, Y_\pm)$

Symmetry of static black hole perturbations

What does it mean for the HJPSS argument ?

- HJPSS argument is not complete:

$$\delta_+ G_\ell = Q_+ G_\ell = 0 \qquad \delta_+ d_\ell = Q_+ D_\ell \neq 0$$

$$\delta_- G_\ell = Q_- G_\ell \neq 0 \qquad \delta_- d_\ell = Q_- D_\ell = 0$$

  miss part of the symmetry
- Need an additional criteria to select the growing branch

Criteria

- The action has to remain finite under the transformation: compute the boundary term

$$\delta S = \frac{1}{2} \int_{r_s}^{r} \mathrm{d}x \frac{\mathrm{d}B}{\mathrm{d}x} < +\infty \tag{21}$$

$$B(r) \simeq \left[\alpha_+ \left(G'_\ell \right)^2 + \alpha_- \left(D'_\ell \right)^2 + \alpha_0 G'_\ell D'_\ell \right] \left(c_1 G_\ell + c_2 D_\ell \right)^2 \ . \tag{22}$$

  with D_ℓ diverging as a log at the r_s.
- Criteria selects the HJPSS symmetry: only (Q_+, Y_+) generate finite symmetry transformation

$$\alpha_- = \alpha_0 = 0 \ , \qquad \text{and} \qquad c_2 = 0 \, . \tag{23}$$

- Provide a symmetry protection for the vanishing of Love numbers for 4d GR black holes [**BA**, Livine, Mukohyama, Uzan '21]

Symmetry of static black hole perturbations

Resulting Love symmetry

- Love symmetry: abelian sub-algebra of Schrödinger

$$\{Q_+, Y_+\} = 0 \tag{24}$$

- Transformation of the ℓ-mode Ψ_ℓ complicated
- Transformation of the zero mode Ψ_0 for Schwarzschild black hole (change of mass)

$$r \to \tilde{r} = \frac{\lambda r_s r}{(\lambda - 1)r + r_s} \tag{25}$$

$$\psi_0 \to \tilde{\Psi}_0(\tilde{r}) = \frac{\sqrt{\lambda} r_s}{(\lambda - 1)r + r_s}\psi(r) + \eta_+ \sqrt{r(r - r_s)} \tag{26}$$

- Can reproduce this construction for static test field with spin-s
- Can reproduce this for tests fields on Kerr
- Generalize to physical static perturbations

Symmetry of static black hole perturbations

Results

- Provide a geometrical origin for the HJPSS symmetry
- Provide a suitable criteria to select the HJPSS symmetry as being the Love symmetry
- It can be identified with non-standard conformal symmetry common to any 1D mechanical systems (free particle, harmonic oscillator ...)

Open questions

- Can we restore the symmetry on the boundary ? By adding new boundary d.o.f ?
- Symmetry for each ℓ-multipole: can we resum it ?
- Full symmetry is infinite dimensional

$$w_1 := w[\psi_1, \psi] = \psi_1\psi' - \psi\psi_1' , \qquad w_2 := w[\psi_2, \psi] = \psi_2\psi' - \psi\psi_2' \qquad \{w_1, w_2\} = w . \tag{27}$$

with

$$\{w_1^{n_1} w_2^{n_2}, w_1^{m_1} w_2^{m_2}\} = w(n_1 m_2 - m_1 n_2)\, w_1^{n_1+m_1-1} w_2^{n_2+m_2-1} , \tag{28}$$

What is the interpretation of this symmetry ? Charges ?

- How can we generalize to dynamical perturbations ? To quasi-normal modes ? [work in progress]

- Which lessons for black hole mechanics ?

Symmetry of static black hole perturbations

- The $\ell = 0$ mode of the static perturbations corresponds to a perturbative change of the mass
- The $\ell = 1$ mode of the static perturbations corresponds to a perturbative change of the angular momentum

- We have changed the mass of the Schwarzschild black hole by a non-standard symmetry: $\rightarrow$ not a spacetime-symmetry

- Look for a conformal symmetry of black hole mechanics which changes the Schwarzschild mass at the non-perturbative level

Möbius covariance of black hole mechanics

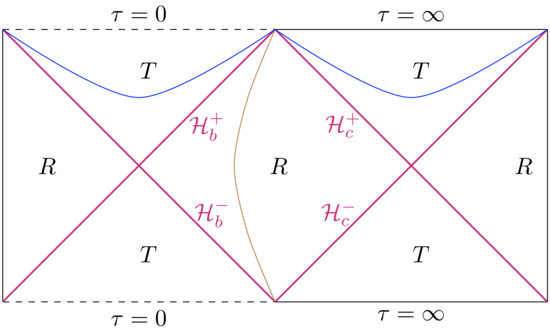
A new symmetry for black hole mechanics

Black hole mechanics

- Consider the Schwarzschild-de Sitter geometry

$$\mathrm{d}s^2 = -\left(1 - \frac{\ell_M}{r} + \frac{r^2}{\ell_\Lambda}\right)\mathrm{d}t^2 + \left(1 - \frac{\ell_M}{r} + \frac{r^2}{\ell_\Lambda}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \qquad (29)$$

- Penrose diagram



- Mechanical system which encodes the geometries of both the T and R regions at once
- Symmetry reduction: each slide is homogeneous

Black hole mechanics

- Action

$$S[g] = \frac{1}{\ell_P^2} \int_{\mathcal{M}} \mathrm{d}^4 x \left[\mathcal{R} - 2\Lambda \right] , \tag{30}$$

- Metric decomposition:

$$\mathrm{d}s^2 = \epsilon \left(-N^2(r)\mathrm{d}r^2 + \gamma_{tt}(r)\mathrm{d}t^2 \right) + \gamma_{\theta\theta}(r)\mathrm{d}\Omega^2 , \tag{31}$$

- Homogeneous slice Σ_ϵ: timelike if $\epsilon = -1$ (T-region) / spacelike if $\epsilon = +1$ (R-region)

$$\mathrm{d}s^2 = \epsilon\gamma_{tt}(r)\mathrm{d}t^2 + \gamma_{\theta\theta}(r)\mathrm{d}\Omega^2 \tag{32}$$

- Introduce the fields and proceed to gauge fixing:

$$\gamma_{tt} := \frac{2\beta(r)}{\alpha(r)} , \qquad \gamma_{\theta\theta} := \ell_s^2 \alpha(r) , \qquad \mathrm{d}\tau = \sqrt{\frac{2\beta}{\alpha}} N(r)\mathrm{d}r \tag{33}$$

- Reduced action for black hole mechanics:

$$\frac{1}{\ell_P^2} \int_{\mathcal{M}} \mathrm{d}^4 x \left[\mathcal{R} - 2\Lambda \right] = S_\epsilon[\alpha, \beta] = \epsilon c \ell_P \int \mathrm{d}\tau \left[\frac{\beta\dot{\alpha}^2 - 2\alpha\dot{\alpha}\dot{\beta}}{2\alpha^2} + \frac{\epsilon}{\ell_s^2} - \frac{\epsilon\alpha}{\ell_\Lambda^2} \right] , \tag{34}$$

- Role of the overall constant: IR/UV cut-off (information on the boundary)

$$c = \frac{1}{\ell_p^3} \int_{t_i}^{t_f} \mathrm{d}t \oint \ell_s^2 \mathrm{d}\Omega = \frac{\ell_0 \ell_s^2}{\ell_p^3} , \tag{35}$$

A new symmetry for black hole mechanics

Reformulating black hole mechanics.....

- Phase space and hamiltonian

$$p_\alpha = \frac{\epsilon c \ell_P}{\alpha^2}(\beta\dot{\alpha} - \alpha\dot{\beta}) \,, \qquad p_\beta = -\epsilon c \ell_P \frac{\dot{\alpha}}{\alpha} \tag{36}$$

  and

$$H^{(\Lambda)} = H^{(0)} + \frac{c\ell_P}{\ell_\Lambda^2}\alpha - \frac{c\ell_P}{\ell_s^2} \qquad H^{(0)} = -\frac{1}{\epsilon c \ell_P}\left[\alpha p_\alpha p_\beta + \frac{1}{2}\beta p_\beta^2\right] \tag{37}$$

.... as a particle in a potential

- New canonical pair

$$X_\pm = \frac{1}{\sqrt{2}}\left(\frac{\beta}{\sqrt{\alpha}} \pm 2\sqrt{\alpha}\right) \,, \quad \{X_\pm, P_\pm\} = 1\,. \tag{38}$$

- Black hole dynamics: 2d particle with non-standard kinetic term

$$H^{(\Lambda)} + \frac{c\ell_P}{\ell_s^2} = H^{(0)} + \frac{c\ell_P}{\ell_\Lambda^2}\alpha = \frac{\epsilon}{2c\ell_P}(P_+^2 - P_-^2) + \frac{c\ell_P}{8\ell_\Lambda^2}(X_+ + X_-)^2 \tag{39}$$

- Schwarzschild mechanics, i.e $\ell_\Lambda \to +\infty$, is 2d free particle (up to the minus sign)
- Schwarzschild-(A)dS mechanics is a 2d harmonic oscillator (up to the minus sign)

Can known symmetries for the 2d particle be realized in black hole mechanics ?

Well known symmetries of the free particle

- Action for a 2d free particle

$$S[t, X^a] = \frac{m}{2} \int \mathrm{d}t \, \dot{X}^a \dot{X}_a \qquad a \in \{1, 2\} \tag{40}$$

with

$$P_a = m\dot{X}_a, \qquad H = \frac{1}{2m}\delta_{ab}P^a P^b \,, \qquad \{X^a, P_b\} = \delta^a_b \tag{41}$$

- Conserved charges: charges for galilean relativity + conformal extension

$$\begin{aligned} &J = X_1 P_2 - X_2 P_1 \\ &B_1 = \tfrac{1}{m}\left[mX_1 - tP_1\right] \\ &B_2 = \tfrac{1}{m}\left[mX_2 - tP_2\right] \\ &P_1 \\ &P_2 \end{aligned}$$

$$\begin{aligned} &Q_+ = mH \,, \\ &2Q_0 = X_1 P^1 + X_2 P^2 - 2Ht \,, \\ &2mQ_- = m(X_1^2 + X_2^2) - 2t(X_1 P^1 + X_2 P^2) + 2t^2 H \,. \end{aligned}$$

- Form the 2d Schrödinger algebra:

$$\mathfrak{sh}(2) = (\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \times \mathbb{R}^2) \tag{42}$$

- Conformal extension of the galileean symmetry of mechanics for massive system

A new symmetry for black hole mechanics

What role does this symmetry play ?

- Classically, conformal extension of the galileean symmetry : charge changes the energy
- Quantum mechanically, it is a symmetry of the free Schrödinger equation

$$i\partial_t \Psi = -\frac{\hbar}{2m}\nabla^2\Psi \tag{43}$$

- Schrödinger symmetry is preserved in specific non-linear extension and potential

$$i\partial_t \Psi = -\frac{\hbar}{2m}\nabla^2\Psi + g|\Psi|^n\Psi^* + V(\vec{X}, t)\Psi \tag{44}$$

  where g encodes the atom-atom interaction strength
- Schrödinger symmetry selects quantum corrections in the many-body context

Characterizing quantumness

- QM is a field theory: constrain the dynamics of the higher moments of the wave function
- 2d Schrödinger casimirs encode the deviations from classical mechanics.
  Classically

$$\mathcal{C}_2 = P_+B_- - P_-B_+ - nJ = 0 \tag{45}$$

$$\mathcal{C}_3 = n\left(Q_0^2 - Q_+Q_- - \frac{1}{4}J^2\right) - B_+B_-Q_+ - P_+P_-Q_-$$

$$-(B_-P_+ + B_+P_-)Q_0 + \frac{1}{2}(B_-P_+ - B_+P_-)J = 0\,.$$

- Quantum mechanically, $\mathcal{C}_2 \neq 0$ encodes the squared incertitude

A new symmetry for black hole mechanics

Schrödinger-like symmetry for Schwarzschild black hole mechanics

- Action

$$S_\epsilon[\alpha,\beta] = \epsilon c\ell_P \int \mathrm{d}\tau \left[\frac{\beta\dot{\alpha}^2 - 2\alpha\dot{\alpha}\dot{\beta}}{2\alpha^2} + \frac{\epsilon}{\ell_s^2}\right], \tag{46}$$

  and phase space: $\{\alpha, p_\alpha\} = \{\beta, p_\beta\} = 1$

$$\tilde{H} = H + \frac{c\ell_P}{\ell_s^2} = -\frac{1}{\epsilon c\ell_P}\left[\alpha p_\alpha p_\beta + \frac{1}{2}\beta p_\beta^2\right] \tag{47}$$

- Schrödinger charges are given by $J = 2\alpha p_\alpha$ and

$$\begin{aligned} P_+ &= \sqrt{\alpha}p_\alpha + \frac{\beta p_\beta}{2\sqrt{\alpha}}, \quad & c\ell_P B_+ &= \epsilon c\ell_P \frac{\beta}{\sqrt{\alpha}} + \tau P_+, \\ P_- &= \sqrt{\alpha}p_\beta, & c\ell_P B_- &= \epsilon c\ell_P 2\sqrt{\alpha} + \tau P_-. \end{aligned} \tag{48}$$

  and

$$Q_+ = c\ell_P \tilde{H} \qquad Q_0 = D - \tau\tilde{H}, \qquad c\ell_P Q_- = -2\epsilon c\ell_P\beta - 2\tau D + \tau^2\tilde{H}$$

- Key difference with the free particle:

$$\mathrm{sch}(2) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2)) \oplus_s (\mathbb{R}^2 \oplus \mathbb{R}^2) \quad \to \quad \boxed{\mathrm{sch}(2) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1)) \oplus_s (\mathbb{R}^2 \oplus \mathbb{R}^2)}$$

  J is not a rotation but a boost [**BA**, Livine, Oriti '23] [**BA**, Livine, Oriti, Piani '22]
- Mean that the black hole particle is propagating on a Lorenztian 2d manifold

Algebraic characterization of the black hole mass

- Where is the black hole mass in the Schrödinger algebra ?

$$M \propto J P_- \tag{49}$$

- Which charge can change the mass ?

$$\{Q_0, P_\pm\} = \frac{1}{2} P_\pm \,, \qquad \{Q_-, P_\pm\} = -B_\pm \qquad \{J, P_\pm\} = \pm P_\pm$$

- Therefore, the conformal sector (Q_0, Q_-) and the boost J can shift the Schwarzschild mass

Finite transformation

- Finite conformal transformations of the metric components (α, β):

$$\tilde{\tau} = f(\tau) = \frac{a\tau + b}{c\tau + d} \,, \qquad \tilde{\alpha}(\tilde{\tau}) = f(\tau)\alpha(\tau) \qquad \tilde{\beta}(\tilde{\tau}) = f(\tau)\beta(\tau) \tag{50}$$

- Solution for (α, β):

$$-2\epsilon\beta = \frac{1}{\ell_s^2}(\tau - \tau_0)(\tau - \tau_1) \qquad \alpha = k^2(\tau - \tau_0)^2 \tag{51}$$

- Symmetry of the action leads to Möbius covariance of the Schwarzschild solution

$$\tilde{\tau}_1 = f(\tau_1) \,, \qquad \tilde{\tau}_0 = f(\tau_0) \qquad \tilde{\ell}_s = \frac{\ell_s}{\sqrt{f(\tau_1) f(\tau_0)}} \qquad \tilde{k} = \frac{k}{\sqrt{f(\tau_0)}} \tag{52}$$

- Correspond to the change of mass by a Mobius transformation

$$M = k\ell_s(\tau_1 - \tau_0) \qquad \rightarrow \qquad \tilde{\ell}_m = \ell_m \frac{f(\tau_1) - f(\tau_0)}{f(\tau_0)\sqrt{f(\tau_1)}(\tau_1 - \tau_0)} \tag{53}$$

Summary

- Schwarzschild black hole mechanics → Schrödinger-like symmmetry as any mechanical system
- Conformal sector: the Schwarzschild solution enjoys a Möbius covariance

$$\tau \to f(\tau) = \frac{a\tau + b}{c\tau + d} \qquad \alpha \to \dot{f}\alpha \qquad \beta \to \dot{f}\beta \tag{54}$$

with

$$\gamma_{rr}(\tau) = \frac{2\beta}{\alpha} \qquad \gamma_{\theta\theta}(\tau) = \ell_s^2 \alpha \tag{55}$$

- Not a standard diffeomorphism: metric transforms through an anisotropic Weyl rescaling
- Conformal sector : Schwarzschild black hole → Schwarzschild black hole at different masses
- Realize the initial expectation from the symmetry of $\ell = 0$ mode of static perturbations

What about Schwarzschild-(A)dS mechanics ?

What if we turn on the cosmological constant ?

- Action

$$S_\epsilon[\alpha,\beta] = \epsilon c \ell_P \int \mathrm{d}\tau \left[\frac{\beta\dot{\alpha}^2 - 2\alpha\dot{\alpha}\dot{\beta}}{2\alpha^2} + \frac{\epsilon}{\ell_s^2} - \frac{\epsilon\alpha}{\ell_\Lambda^2} \right], \tag{56}$$

and phase space: $\{\alpha, p_\alpha\} = \{\beta, p_\beta\} = 1$

$$\tilde{H}^{(\Lambda)} = H^{(\Lambda)} + \frac{c\ell_P}{\ell_s^2} = -\frac{1}{\epsilon c \ell_P}\left[\alpha p_\alpha p_\beta + \frac{1}{2}\beta p_\beta^2\right] + \frac{c\ell_P}{\ell_\Lambda^2}\alpha \tag{57}$$

- Schrödinger charges are given by $J = 2\alpha p_\alpha$ and

$$P_+^{(\Lambda)} = \sqrt{\alpha}p_\alpha + \frac{\beta p_\beta}{2\sqrt{\alpha}} - \epsilon\frac{c^2\ell_P^2}{\ell_\Lambda^2}\frac{\sqrt{\alpha}}{p_\beta}, \qquad P_- = \sqrt{\alpha}p_\beta,$$

$$c\ell_P B_+ = \epsilon c\ell_P \frac{1}{\sqrt{\alpha}}\left[\beta - \epsilon\frac{2c^2\ell_P^2}{3\ell_\Lambda^2}\frac{\alpha}{p_\beta^2}\right] + \tau P_+^{(\Lambda)},$$
$$c\ell_P B_- = \epsilon c\ell_P 2\sqrt{\alpha} + \tau P_-$$
$$J^\Lambda = 2\alpha p_\alpha - \epsilon\frac{4c^2\ell_P^2}{3\ell_\Lambda^2}\frac{\alpha}{p_\beta} \tag{58}$$

and

$$Q_+ = c\ell_P\tilde{H}^{(\Lambda)} \qquad Q_0 = D - \epsilon\frac{4c^2\ell_P^2}{3\ell_\Lambda^2}\frac{\alpha}{p_\beta} - \tau\tilde{H}^{(\Lambda)}, \tag{59}$$
$$c\ell_P Q_- = -2\epsilon c\ell_P\left(\beta - \epsilon\frac{2c^2\ell_P^2}{3\ell_\Lambda^2}\frac{\alpha}{p_\beta^2}\right) - 2\tau\left(D - \epsilon\frac{4c^2\ell_P^2}{3\ell_\Lambda^2}\frac{\alpha}{p_\beta}\right) + \tau^2\tilde{H}^{(\Lambda)}$$

- Symmetry is preserved for dS and AdS: $\mathrm{sch}(2) = (\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1)) \oplus_s (\mathbb{R}^2 \oplus \mathbb{R}^2)$

A new symmetry for black hole mechanics

Additional structures: the conformal bridge

- Consider the generator

$$\Lambda = \frac{\alpha p_\alpha}{2} - \beta p_\beta \tag{60}$$

- Consider the transformation of the hamiltonian under its flow

$$\tilde{H}^{(\Lambda)} = H^{(0)} + \frac{c\ell_P}{\ell_\Lambda^2}\alpha \qquad \rightarrow \qquad \delta_\epsilon \tilde{H}^{(\Lambda)} = \{\Lambda, \tilde{H}^{(\Lambda)}\} = -\epsilon \frac{c\ell_P}{\ell_\Lambda^2}\alpha \tag{61}$$

- Generate rescaling of the cosmological constant !
  Can freely deform the phase space from Schwarzschild to Schwarzschild-(A)dS
- Algebraically, when ℓ_Λ is finite, the generator Λ^2 is the Casimir of the $\mathfrak{sl}(2,\mathbb{R})$ algebra
- When $\ell_\Lambda \rightarrow +\infty$, it reduces to a Dirac observable: the mass of the black hole

- Well known structure of "conformal bridge" relating un-trapped and trapped mechanical systems by conformal transformation
  Arnold map : free particle $\leftrightarrow$ harmonic oscillator
  [Inzunza, Plyushchay, Wipf '19 '21]

A new symmetry for black hole mechanics

- These symmetry transformations does not act as standard diffeomorphism, i.e. not as Lie derivative on the spacetime metric
- Can we find a geometrical understanding for these symmetries ?
- Can we view them as diffeomrophism on an auxiliary manifold ?

Geometrizing the dynamics ... and the symmetries

- Consider a mechanical system with potential and n degrees of freedom χ^a

$$S[\chi^a, \dot{\chi}^a, \tau] = cL_P \int \mathrm{d}\tau \left(\frac{1}{2} g_{ab}(\chi)\dot{\chi}^a\dot{\chi}^b - V(\chi) \right)$$

- From this kinetic part of the Lagrangian, construct the super-space : $X^a = \chi^a$

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}X^a\mathrm{d}X^b = cL_P g_{ab}(\chi)\mathrm{d}\chi^a\mathrm{d}\chi^b \tag{62}$$

- Key Idea: Need to treat time on the same foot as the other dynamical field !
- Need new coordinates (u, w) play the role of the time and its conjugated momentum
- Consider the $n+2$ extended field space : $X^A = (u, w, \chi^a)$

$$\mathrm{d}s^2 = \mathcal{G}_{AB}\mathrm{d}X^A\mathrm{d}X^B = cL_P g_{ab}(\chi)\mathrm{d}\chi^a\mathrm{d}\chi^b + 2\mathrm{d}u\mathrm{d}w - 2cL_P V(\chi)\mathrm{d}u^2$$

  [Eisenhart '28] [Duval, Gibbons, Horvathy '85 '91 '00][Bekaert, Morand '14]

- Null geodesics reproduce the Euler-Lagrange equations of the initial mechanical system
- Conformal isometries of the ED lift ↔ field space symmetries of the mechanical systems

$$\xi = \xi^A\partial_A\ , \qquad \mathcal{L}_\xi\mathcal{G}_{AB} = \Omega^2\mathcal{G}_{AB} \tag{63}$$

- Schrödinger observables identified as the conformal isometries of the lift which commute with null vector ∂_w [**BA**, Livine, Oriti, Piani '22].
- Additional symmetries: find the full $\mathfrak{so}(4,2)$ algebra of charges [**BA**, Livine, Stankiewicz]
- Symmetries of superspace g_{ab} and $\mathcal{G}_{AB}$ are different ! Need both

Conclusion and perspectives

Conclusion

- Static black hole perturbations describe by 1D mechanical system
- Inherit the conformal symmetry inherent to such system ... also related to the symmetry of the Sturm-Liouville system
- Generator acts in an auxiliary space (the Einsehart-Duval lift): not spacetime symmetries !
- Provide the origin for the Hui-Joyce-Penco-Salomon-Santoni (HJPSS) symmetry
- Improve their construction and provide a clean criteria to select the Love symmetry out of the Schrodinger one

- Can we restore the full Schrodinger symmetry at the horizon ?
- Can we generalize to dynamical perturbations ? to QNM ?

- The symmetry found for the $\ell = 0$ mode, the shift of mass, extends to black hole mechanics
- Realization of this symmetry : reformulate black hole mechanics as a 1d system
- Enjoy a Schrödinger symmetry
- Mobius covariance of black hole mechanics : change the mass with conformal transformation
- Extend to Schwarzschild-(A)dS

- Suggest a deeper structure to explore
- Current progress towards generalizations to i) QNM, ii) axi-symmetric GR phase space
- Path to construct non-linear Wheeler-de Witt dynamics for quantum black hole using the symmetry : quantum black hole as many-body systems $\rightarrow$ identify analogue models

Towards non-linear WdW

- How can we use the Schrödinger symmetry to go beyond the standard WdW quantization ?
- Quantum black hole (just as quantum cosmology) should be understood as many-body quantum systems emerging from a suitable mean field approximation of quantum gravity
- Concretely, the wave function of the black hole should be regarded as a collective wave function
- Need to include information on the existence of these quanta and their interaction in the quantum dynamics

More on non-linear Wheeler-de Witt dynamics for quantum black hole

Schrödinger symmetry in many-body condensed matter systems

- In general, non-linear Schrödinger equations describing Bose-Einstein condensates does not enjoy any symmetry

$$i\hbar\partial_t\Psi(\vec{x},t) = \frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Psi(\vec{x},t) + gV[\Psi,\Psi^*] \qquad V[\Psi,\Psi^*] = |\Psi|^{2n} \tag{64}$$

- Preserving the Schrödinger symmetry selects uniquely the non-linear extension [Gosh '06]

$$V[\Psi,\Psi^*] = |\Psi|^{2n}\,, \qquad d(n-1) = 2 \tag{65}$$

- In $d = 1$, conformal invariance selects the Tonks-Gerardeau equation: $V[\Psi,\Psi^*] = |\Psi|^6$
- In $d = 2$, conformal invariance selects the Gross-Pitaevskii equation: $V[\Psi,\Psi^*] = |\Psi|^4$

$$i\hbar\partial_t\Psi(\vec{x},t) = \frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Psi(\vec{x},t) + g|\Psi|^2\Psi^* \tag{66}$$

- Moreover, exact solutions to these non-linear equations can be found via the underling symmetry

Suggest that there is family of symmetry-protected UV non-linear corrections to the WdW equation to be explored.

More on non-linear Wheeler-de Witt dynamics for quantum black hole

2d Gross-Pitaevskii condensate

- In $d = 2$, conformal invariance selects the Gross-Pitaevskii equation:

$$i\hbar\partial_t\Psi(\vec{x},t) = \frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Psi(\vec{x},t) + g|\Psi|^2\Psi^* \tag{67}$$

- Consider stationary solution: $\Psi(\vec{x},t) = \Phi(\vec{x})e^{i\mu t}$

$$\frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Phi(\vec{x}) + g|\Phi|^2\Phi^* = \mu\Phi(\vec{x}) \tag{68}$$

- Many well-known properties and solutions related to generation of vortex (rotation): Berezinskii-Kosterlitz-Thouless phase transition

Non-linear WdW dynamics for the quantum black hole

- BH wave function as a function on the superspace: $\Psi := \Psi(\alpha,\beta)$
- Standard procedure, quantize the (shifted) hamiltonian

$$\hat{H}\Psi = \mathcal{G}^{ij}\nabla_i\nabla_j\Psi(\alpha,\beta) = \mu\Psi(\alpha,\beta) \tag{69}$$

where $\mathcal{G}^{ij}$ is the superspace metric, $\mu = \frac{c\ell_p}{\ell_s^2}$

More on non-linear Wheeler-de Witt dynamics for quantum black hole

2d Gross-Pitaevskii condensate

- In $d = 2$, conformal invariance selects the Gross-Pitaevskii equation:

$$i\hbar\partial_t \Psi(\vec{x}, t) = \frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Psi(\vec{x}, t) + g|\Psi|^2\Psi^* \tag{70}$$

- Consider stationary solution: $\Psi(\vec{x}, t) = \Phi(\vec{x})e^{i\mu t}$

$$\frac{\hbar^2}{2m}\delta^{ij}\nabla_i\nabla_j\Phi(\vec{x}) + g|\Phi|^2\Phi^* = \mu\Phi(\vec{x}) \tag{71}$$

- Many well-known properties and solutions related to generation of vortex (rotation): Berezinskii-Kosterlitz-Thouless phase transition

Non-linear WdW dynamics for the quantum black hole

- BH wave function as a function on the superspace: $\Psi := \Psi(\alpha, \beta)$
- Standard procedure, quantize the hamiltonian

$$\hat{H}\Psi = \mathcal{G}^{ij}\nabla_i\nabla_j\Psi(\alpha, \beta) + g|\Psi|^2\Psi^*(\alpha, \beta) = \mu\Psi(\alpha, \beta) \tag{72}$$

where $\mathcal{G}^{ij}$ is the superspace metric, $\mu = \frac{c\ell_p}{\ell_s^2}$

- Can we import adapt the known properties of 2d GP BEC to quantum black hole ?
Work in progress

Black hole mechanics

- Reduced action

$$S_\epsilon[\alpha,\beta] = \epsilon c \ell_P \int \mathrm{d}\tau \left[\frac{\beta\dot{\alpha}^2 - 2\alpha\dot{\alpha}\dot{\beta}}{2\alpha^2} + \frac{\epsilon}{\ell_s^2} - \frac{\epsilon\alpha}{\ell_\Lambda^2} \right] , \tag{73}$$

- Metric

$$\mathrm{d}s^2 = -\epsilon \frac{\alpha}{2\beta}\mathrm{d}\tau^2 + \epsilon \frac{2\beta}{\alpha}\mathrm{d}t^2 + \ell_s^2 \alpha \mathrm{d}\Omega^2 , \tag{74}$$

- Solutions to the dynamics

$$-2\epsilon\beta = \frac{1}{\ell_s^2}(\tau-\tau_0)(\tau-\tau_1) - \frac{k^2}{3\ell_\Lambda^2}(\tau-\tau_0)^4 \tag{75}$$

$$\alpha = k^2(\tau-\tau_0)^2 \tag{76}$$

with (τ_0, τ_1, k) constant of integration

- Standard form of Schwarzschild-(A)dS: rescaling and translation

$$r = k\ell_s(\tau-\tau_0) , \qquad \tilde{t} = t/k\ell_s \tag{77}$$

gives

$$-\epsilon\frac{\alpha}{2\beta} = f(r) = 1 - \frac{\ell_M}{r} - \frac{r^2}{3\ell_\Lambda^2} \quad \text{with} \qquad \boxed{\ell_M = k\ell_s(\tau_1-\tau_0)} \tag{78}$$

with ℓ_m the mass of the black hole

Black hole mechanics: Schwarzschild solution and black hole mass

Mass of the black hole as a charge

- Metric

$$\mathrm{d}s^2 = -\epsilon\frac{\alpha}{2\beta}\mathrm{d}\tau^2 + \epsilon\frac{2\beta}{\alpha}\mathrm{d}t^2 + \ell_s^2\alpha\mathrm{d}\Omega^2\,, \tag{79}$$

- Solutions to the dynamics

$$-2\epsilon\beta = \frac{1}{\ell_s^2}(\tau-\tau_0)(\tau-\tau_1) - \frac{k^2}{3\ell_\Lambda^2}(\tau-\tau_0)^4 \tag{80}$$

$$\alpha = k^2(\tau-\tau_0)^2 \tag{81}$$

with (τ_0, τ_1, k) constant of integration

- Mass ℓ_m of the black hole

$$\boxed{\ell_M = k\ell_s(\tau_1-\tau_0)} \tag{82}$$

- Where is the black hole mass in the Schrödinger algebra ?

$$J = \frac{c\ell_P}{\ell_s^2}(\tau_1-\tau_0)\,,\qquad P_- = -\epsilon 2c\ell_P k \qquad \rightarrow \qquad \boxed{\ell_M = -\epsilon\frac{2\ell_s^3}{c^2\ell_P^2}\,JP_-}$$

$\rightarrow$ black hole mass as a conserved charge