Matrix Models as Scalar Field Theories on Noncommutative Spaces

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Talk Plan

1 History and Overview

2 Origin from N.C. field theory

3 Set Up of N.C. Φ^3 model

4. Ward-Takahashi Ids & Schwinger-Dyson Eqs.

5. Large N, L limit

6. Q.F.T. (toy model)

7. Finite N Φ^3 model

8. Finite N Φ^3-Φ^4 mixed matrix model.

§1 History

▶ 90s' Matrix model

◎ 2D gravity $\Leftrightarrow$ random matrix

Brezin - Kazakov, Gross - Migdal, etc.

Kontsevich model (Witten conjecture)

$$Z[J] = \int \mathcal{D}\Phi \, \exp\left(-\mathrm{Tr}\left(\Lambda \Phi + \Phi^3 \right)\right)$$

Φ : Hermite matrix, $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \cdots)$

Makeenko - Semenoff solve this in $N \to \infty$

▶ 2000's

N.C. field Theory ⇒ Matrix model.

Grosse-Steinacker

Φ^3 model (Kontsevich model) basically Renormalizable

Grosse-Wulkenhaar

Φ^4 models in 2, 4 dim are Renormalizable

Φ^4 model is solvable

(SD-eq is recursivly determined.)

▷ Overview.

◎ We will see Φ^3 model closely.

Goal

- We will see every multi pts-function can be obtained explicitly.

  (Not only $N \to \infty$, but also finite N)

◎ Φ^4 is still difficult.

Φ^3-Φ^4 mixed model will be introduced.

Now we are trying to some problems.

§2. ~The Origin from N.C. field Theory~

$\mathbb{R}^2_\theta$: Moyal plane $\qquad [A,B] := AB - BA$

$$[x^1, x^2] = i\underset{\text{N.C. parameter}}{\underbrace{\theta}} \Leftrightarrow [z, \bar{z}] = 2\theta$$

- Annihilation $\left(a := \frac{z}{\sqrt{2\theta}} \right)$ Creation $\left(a^\dagger := \frac{\bar{z}}{\sqrt{2\theta}} \right)$ op.

$$\Rightarrow [a, a^\dagger] = 1, \quad [a, a] = [a^\dagger, a^\dagger] = 0$$

- $\frac{\partial}{\partial z} = -\frac{1}{\sqrt{2\theta}} [a^\dagger, \ ], \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2\theta}} [a, \ ]$

- Fock sp. $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$

$|0\rangle$: $a|0\rangle = 0$, $|n\rangle := \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$

Number op. $N := a^\dagger a$ $\quad N|n\rangle = n|n\rangle$

$\langle n|$ = dual of $|n\rangle$

$\langle n|m\rangle = \delta_{nm}$

- Scalar field $\Phi = \sum \Phi_{nm} |m\rangle\langle n|$

$\int d^2x \rightarrow \theta^2 \, \mathrm{Tr}$

Action

$$S_1 = \int d^4x \, -\phi\left(\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}\right)\phi$$

$$= \frac{\theta^2}{2\theta} \mathrm{Tr}\, \phi\, [a^\dagger, [a, \phi]] \qquad N = a^\dagger a$$

$$= \theta \;\; \mathrm{Tr}\,(\phi N \phi - \underline{\theta a^\dagger \phi a \phi})$$

Removing this term by a counter Lagrangian

Renormalizable model is obtained.

$$S_m = \theta \,\mathrm{Tr}\, \frac{\mu^2}{2} \phi^2$$

μ: const. (mass)

Interaction

$$S_{int} = \theta \frac{\lambda}{3} \operatorname{Tr} \phi^3$$

λ : const
coupling const

$$S = S_1 + S_m + S_{int} + \underbrace{\text{tadpole}}$$

$$= \theta \operatorname{Tr} \left(\phi E \phi - \overbrace{A\phi} + \frac{\lambda}{3} \phi^3 \right)$$

where $E_{nm} = \left(\frac{1}{2}\mu^2 + n\right)\delta_{nm}$

A : const

§3 Set Up of NC Φ^3 (2-dim case for simplicity)

Hermitian Matrix $\Phi = \Phi^\dagger \in M_N(\mathbb{C})$

Action $S = L\,\mathrm{Tr}(E\Phi^2 - A\Phi) + V(\Phi)$

$$V(\Phi) = L\frac{\lambda}{3}\mathrm{Tr}\,\Phi^3$$

$$E = (E_m \delta_{mn}) = \begin{pmatrix} E_1 & & 0 \\ 0 & E_2 & \\ & & \ddots \end{pmatrix}$$

$$E_m = \mu^2\left(\frac{1}{2} + e\left(\frac{m}{\mu^2 L}\right)\right),$$

θ

A, λ, L, μ : const. $e(0)=0$, C^∞-fun

$$S = L\left(\sum_{n,m}^{N} \frac{1}{2}\Phi_{nm}\Phi_{mn}H_{nm} - A\sum_{m=0}^{N}\Phi_{mm} + \frac{\lambda}{3}\sum_{k,l,m}^{N}\Phi_{kl}\Phi_{lm}\Phi_{mk}\right)$$

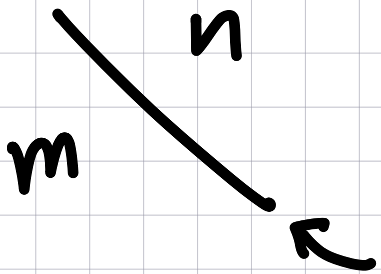
$$H_{mn} := E_m + E_n = \mu^2\left(1 + e\left(\frac{m}{\mu^2 L}\right) + e\left(\frac{n}{\mu^2 L}\right)\right)$$

$$Z[J] := \int \mathcal{D}\Phi\, e^{-S + L\,\mathrm{Tr}(J\Phi)}$$

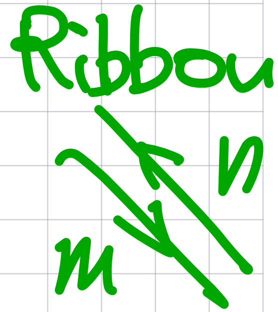
$$= K \exp\left(-V\left(\frac{1}{L}\frac{\partial}{\partial J}\right)\right) Z_{free}[J]$$

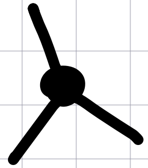
$$Z_{free} = e^{\sum \frac{L}{2}(\delta_{nm}A + J_{nm}) H_{nm}^{-1} (\delta_{nm}A + J_{mn})}$$

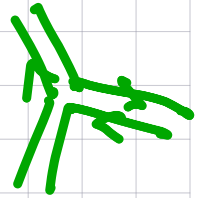
Remark) Correspondence with Graphs.

propergator : 

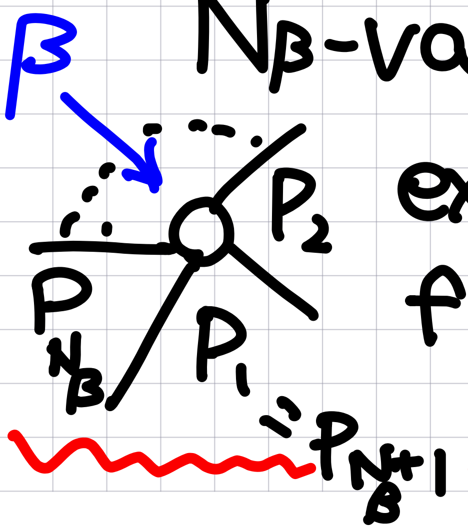
$\sim \dfrac{1}{H_{mn}}$



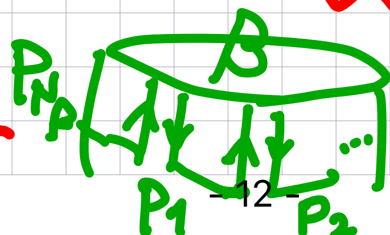
Black vertex :  $\sim \lambda \, \mathrm{tr} \, \Phi^3$



White vertex :



N_β-valence external field $\sim \mathbb{J}_{P_1 \cdots P_{N_\beta}} = \prod_{j=1}^{N_\beta} J_{P_j P_{j+1}}$



$(N_\beta + 1 \equiv 1)$

$$\log \frac{Z[J]}{Z[0]} = \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \cdots \le N_B}^{\infty} \sum_{\forall p_i^j = 0}^{N} L^{2-B} \frac{\overbrace{G_{|p_1^1 \cdots p_{N_1}^1| \cdots |p_1^B \cdots p_{N_B}^B|}}^{(N_1 + \cdots + N_B)\text{-point fun}}}{S(N_1, \cdots, N_B)} \prod_{\beta=1}^{B} \frac{J_{p_1^\beta \cdots p_{N_\beta}^\beta}}{N_\beta}$$

= Generating fun of connected graphs (Green fun.)

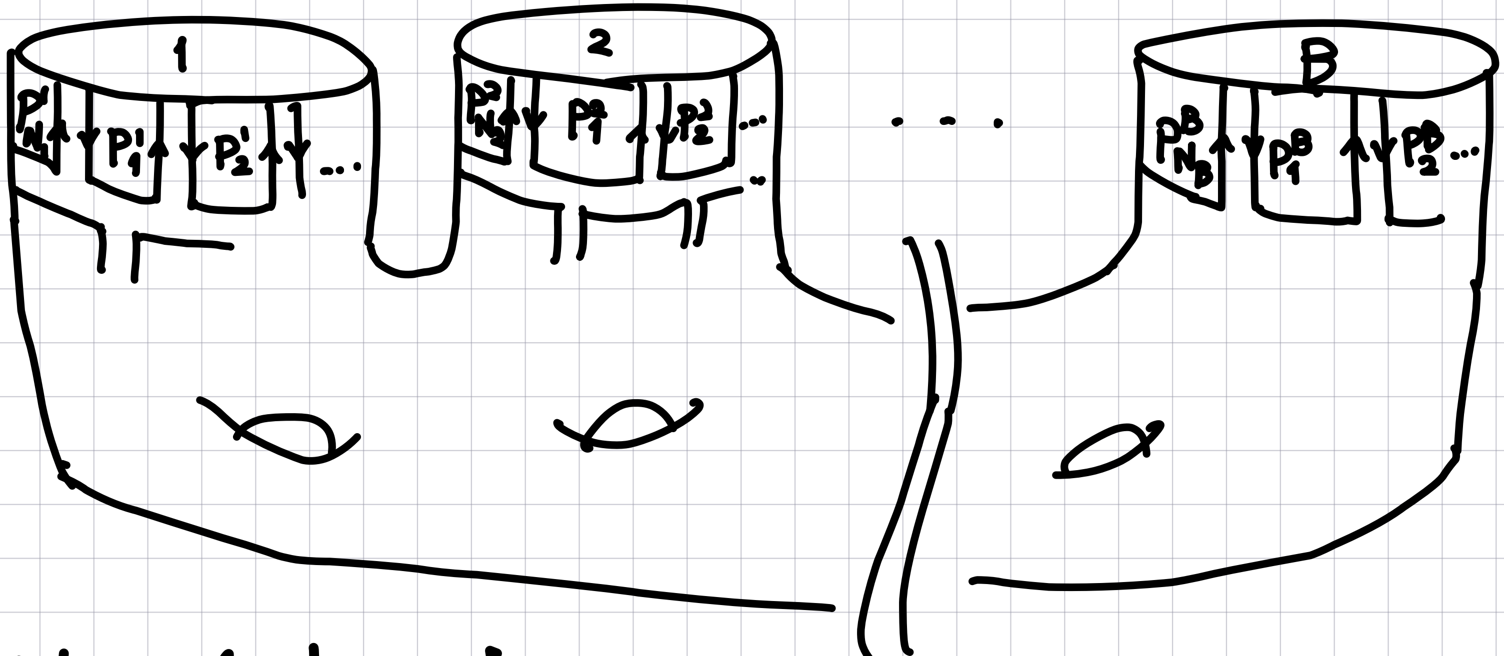
$$\left(\begin{array}{l} S(N_1, \cdots, N_B) := \prod_{i=1}^{s} \nu_i ! \quad \text{statistical factor} \\ \text{for } (N_1, \cdots, N_B) = (\underbrace{N_1', \cdots, N_1'}_{\nu_1}, \cdots, \underbrace{N_s', \cdots, N_s'}_{\nu_s}) \end{array} \right)$$

L^{2-B} : We choose this factor to obtain all Npt function as finite at Large (L, N) lim

Feynman Graph

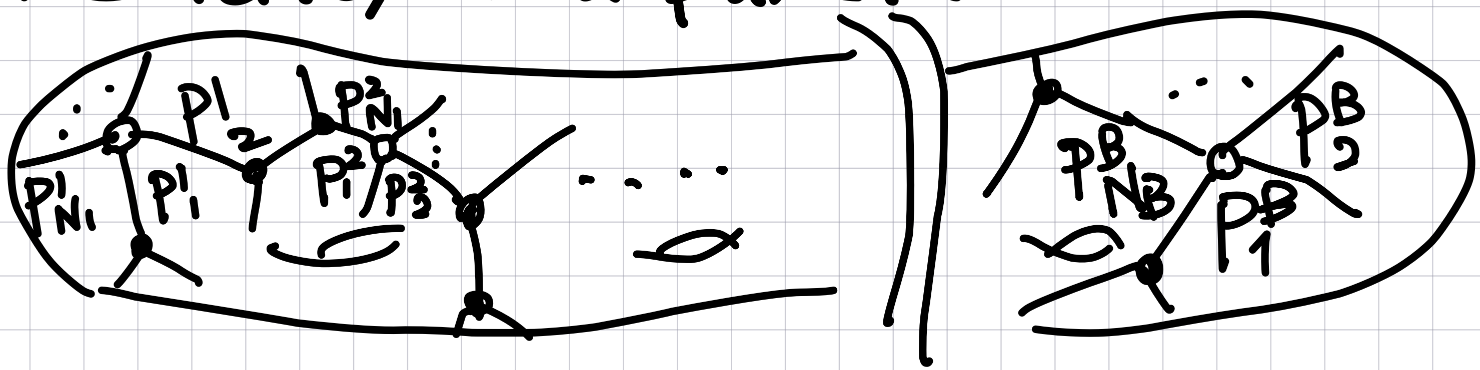
$$G|P_1^1 P_2^1 \cdots P_{N_1}^1 | P_1^2 P_2^2 \cdots P_{N_2}^2 | \cdots | P_1^B P_2^B \cdots P_{N_B}^B|$$

$= \sum$



○ : white vertex is a puncture.

$= \sum$



~ $\mathcal{N} := \sum_{i=1}^{B} N_i$ - external ribbon case ~

Let us count the power of L in connected Feynman diagram

$$\langle \Phi_{a^1_1 a^1_2} \cdots \Phi_{a^1_{N_1} a^1_1} \Phi_{a^2_1 a^2_2} \cdots \Phi_{a^2_{N_2} 1} \cdots \Phi_{a^B_1 a^B_2} \cdots \Phi_{a^B_{N_B} a^B_1} \rangle_c$$

$$= \frac{1}{L^{\mathcal{N}}} \frac{\partial}{\partial J_{a^1_2 a^1_1}} \cdots \frac{\partial}{\partial J_{a^B_1 a^B_{N_B}}} \log Z[J] \Big|_{J=0}$$

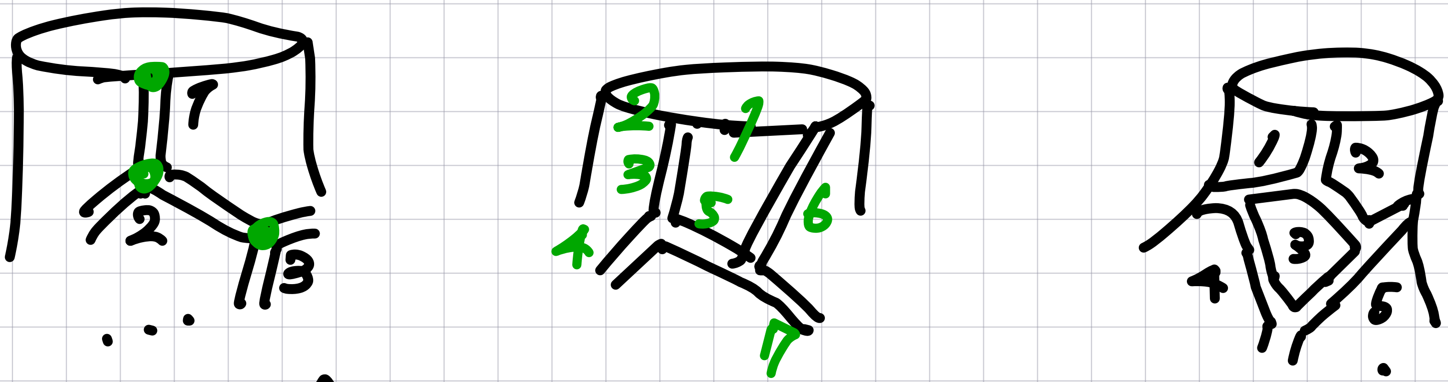
(For simplicity, a^i_j $(i=1,\cdots,B,\ j=1,\cdots,N_i)$ pairwise different)

k_3: # of $\frac{1}{3}\lambda \, \mathrm{tr}\Phi^3$ interaction

graph $\sim L^{k_3 - \frac{3k_3+\mathcal{N}}{2}}$ $\left\{ \begin{array}{l} \text{propagator } \left(\frac{1}{L}\right)^{\frac{3k_3+\mathcal{N}}{2}} \\ \text{Vertex } L^{k_3} \end{array} \right.$

Euler $\chi = 2-2g-B$ $\quad$ (g : genus , B : boundaries)

$$= \underbrace{(k_3+N)}_{\text{Vertex}} - \underbrace{\left(\frac{3k_3+N}{2}+N\right)}_{\text{Edge}} + \underbrace{(N+\underbrace{\Sigma}_{\text{loop}})}_{\text{Face}}$$



$$\langle \cdots \rangle_c \sim L^{k_3 - \frac{3k_3+N}{2}} = L^{\chi - N - \Sigma} = L^{2-2g-B-N-\Sigma}$$

$$\langle \cdots \rangle_c = L^{2-N-B} \underbrace{G|a_1^1 \cdots a_{N_1}^1| \cdots |a_1^B \cdots a_{N_B}^B|}$$

This $G|\cdots|$ is given by the sum over all Feynman graph with fixed $B, \{N_1, \cdots, N_B\}$

When we expand it by g.

$$G_{|a_1^1 \cdots a_{N_1}^1| \cdots |a_1^B \cdots a_{N_B}^B|} = \sum_{g=0}^{\infty} L^{-2g} \, G^{(g)}_{|a_1^1 \cdots a_{N_1}^1| \cdots |a_1^B \cdots a_{N_B}^B|}$$

(See for ex "Laplacian to compute Intersection Numbers on $\overline{\mathcal{M}}_{g,n}$ and Correlation Functions in NCQFT" Hock - Grosse - Wulkenhaar)

§4. ~W.T. Id & SD Eqs.~

~ Ward Takahashi like Id. ~

$\Phi \to \Phi' = \Phi + [u, \Phi]$ ⇐ $Z[J]$ is inv.

WT-Id.

$$\boxed{\sum_m \frac{\partial^2 Z[J]}{\partial J_{am}\partial J_{mb}} = \sum_m \frac{L}{E_a - E_b}\left(J_{ma}\frac{\partial}{\partial J_{mb}} - J_{bm}\frac{\partial}{\partial J_{am}}\right) Z[J]}$$

2pt functions are reduced to 1pt functions by the WT-Id.

~ Schwinger-Dyson Eqs. ~

▷ 1pt function

$$G_{|a|} := \frac{1}{L} \frac{\partial \log Z[J]}{\partial J_{aa}}\bigg|_{J=0}$$

$$= H^{-1}_{aa}\left(A - \lambda G^2_{|a|} - \frac{\lambda}{L}\sum_{m=0}^{N} G_{|am|} - \frac{\lambda}{L^2} G_{|a|a|}\right) \quad \text{— ①}$$

▷ 2pt fun.

$$G_{|ab|} := \frac{1}{L} \frac{\partial^2 \log Z[J]}{\partial J_{ab}\partial J_{ba}}\bigg|_{J=0}$$

Using W-T id.
2pt → 1pt

$$= H^{-1}_{ab}\left(1 + \lambda \frac{G_{|a|} - G_{|b|}}{E_a - E_b}\right) \quad \text{— ②}$$

▷ Renormalization Condition

$$\boxed{G_{|0|} = 0 \iff A = \frac{\lambda}{L}\sum_{m=0}^{N} G_{|0m|} + \frac{\lambda}{L^2} G_{|0|0|}}$$

⇓ Remove A in ① by using this condition

$$G_{|a|} = H^{-1}_{aa}\Big\{-\lambda G^2_{|a|} - \frac{\lambda}{L}\sum_m (H^{-1}_{am} - H^{-1}_{om}) - \frac{\lambda}{L^2}(G_{|a|0|} - G_{|0|0|})$$
$$- \frac{\lambda^2}{L}\sum_m \Big(H^{-1}_{am}\frac{(G_{|a|} - G_{|m|})}{E_a - E_m} - H^{-1}_{om}\frac{G_{|m|}}{E_m - E_0}\Big)\Big\} \quad \cdots ①'$$

⇓ using $\frac{W_{|a|}}{2\lambda} := G_{|a|} + \frac{H_{aa}}{2\lambda} = G_{|a|} + \frac{E_a}{\lambda}$

①′ ② are simplified.

Schwinger - Dyson Eqs. for 1pt, 2pt fun.

- $$W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{L^2}\left(G_{|a|a|} - G_{|0|0|}\right)$$
$$- \frac{2\lambda^2}{L}\sum_{m=0}^{N}\left(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|m|} - W_{|0|}}{E_m^2 - E_0^2}\right)$$

- $$G_{|ab|} = \frac{1}{2}\,\frac{W_{|a|} - W_{|b|}}{E_a^2 - E_m^2}$$

§5 ~Large (N, L)-lim & Solutions~

matrix size & N.C. parameter

$N, L \longrightarrow \infty$ with fixing $\dfrac{N}{L} = \mu^2\Lambda^2$

$$\boxed{\begin{aligned}\lim \frac{1}{L}\sum_{m=0}^{N} f\left(\frac{m}{L}\right) &= \mu^2\Lambda^2\int_0^1 f(\mu^2\Lambda^2 x)\,dx \\ &= \mu^2\int_0^{\Lambda^2} f(\mu^2 x)\,dx\end{aligned}}$$

$$\mu^2 W(x) = \lim W_{|L\mu^2 x|}$$

$$G(x) = \lim G_{|L\mu^2 x|} \quad \text{etc.}$$

$$X := (2e(x)+1)^2$$

Using these expression

Schwinger-Dyson eq for 1pt fun.

$$W^2(X) + \int_1^{\square} dY\, \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \underbrace{\int_1^{\square} dY\, \rho(Y) \frac{W(1)-W(Y)}{1-Y}}_{\text{const.}}$$

where $\rho(Y) = \dfrac{2\tilde{\lambda}^2}{\sqrt{Y}\, e'\left(e^{-1}\left(\frac{\sqrt{Y}-1}{2}\right)\right)}$, $\square = \left(1+2e(\lambda^2)\right)^2$

$$\tilde{\lambda} = \frac{\lambda\mu}{2}$$

Makeenko-Semenoff solved similar type

Solution

$$W(X) := \sqrt{X+c} + \frac{1}{2}\int_1^{\square} dZ \frac{\rho(Z)}{\left(\sqrt{X+c}+\sqrt{Z+c}\right)\sqrt{Z+c}}$$

ex). N.C. scalar ϕ^3 field theory

$$e(x) = x, \quad X = (2x+1)^2, \quad \rho(Y) = \frac{2\tilde{\lambda}^2}{\sqrt{Y}}$$

$$\Downarrow$$

$$W(X) = \sqrt{X+c} + \frac{2\tilde{\lambda}^2}{\sqrt{X}} \log\left(\frac{(\sqrt{X}+1)(\sqrt{X+c}+\sqrt{X})}{\sqrt{X+cX} + \sqrt{X+c}} \right)$$

$$\Updownarrow$$

$$G(x) = \frac{\sqrt{(2x+1)^2+c} - (2x+1)}{2\tilde{\lambda}} + \frac{\tilde{\lambda}}{2x+1} \log\left(\frac{(2x+2)\left(\sqrt{(2x+1)^2+c} + 2x+1\right)}{(2x+1)\sqrt{1+c} + \sqrt{(2x+1)^2+c}} \right)$$

~ $(N_1+\cdots+N_B)$-pt function ~

$$G|a^1_1\cdots a^1_{N_1}|\cdots|a^B_1\cdots a^B_{N_B}| = L^{B-2}\frac{\partial^{N_1}}{\partial J_{a^1_1\cdots a^1_{N_1}}}\cdots\frac{\partial^{N_B}}{\partial J_{a^B_1\cdots a^B_{N_B}}}\log\frac{Z[J]}{Z[0]}$$

where $\dfrac{\partial^N}{\partial J_{a_1\cdots a_N}} = \dfrac{\partial}{\partial J_{a_1a_2}}\dfrac{\partial}{\partial J_{a_2a_3}}\cdots\dfrac{\partial}{\partial J_{a_{N-1}a_N}}\dfrac{\partial}{\partial J_{a_Na_1}}$

Feymman graph

$N, L \to \infty$

(Using W-T Id.)

Point

$$G(x^1_1,\cdots,x^1_{N_1}|\cdots|x^B_1,\cdots,x^B_{N_B})$$

$$=\tilde{\lambda}^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}\underbrace{G(x^1_{k_1}|\cdots|x^B_{k_B})}\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4}{x^\beta_{k_\beta}-x^\beta_{l_\beta}}$$

If we obtain this, then every $(N_1+\cdots+N_B)$-pt function is solved!!

$$G|a^1|\cdots|a^B| = L^{B-2}\frac{\partial}{\partial J_{a_1a_1}}\cdots\frac{\partial}{\partial J_{a_Ba_B}}\log\frac{Z[J]}{Z[0]}\Bigg|_{J=0}$$

SD-eq. for this
$L, N \to \infty$ lim

Similar process as previous discussions

S-D eq

$$W(x^1)G(x^1|X^{\{2,\cdots,B\}}) + \frac{1}{2}\int_1^\infty dT\,\rho(T)\frac{G(x^1|X^{\{2,\cdots,B\}}) - G(T|X^{\{2,\cdots,B\}})}{(x-T)}$$
$$= -\tilde{\lambda}\sum_{\beta=2}^{B}G(x^1,x^\beta,x^\beta|X^{\{2,\cdots,B\}}) - \tilde{\lambda}\sum_{\substack{J\subset\{2,\cdots,B\}\\ 1\le|J|\le B-2}}G(x^1|X^J)G(x^1|X^{\{2,\cdots,B\}\setminus J})$$

where $G(X|Y^J) = G(X|Y^{j_1}|Y^{j_2}|\cdots|Y^{j_p})$ for $J = \{j_1,\cdots,j_p\}$

▷ Solution for (1+1)-pt fun.

S-D eq.

$$W(x)G(x|Y) = -\tilde{\lambda}\, G(x,Y,Y) - \frac{1}{2}\int_1^\infty dz\,\rho(z)\,\frac{G(x|Y)-G(z|Y)}{x-z}$$

Solution

$$\boxed{G(x|Y) = \frac{4\tilde{\lambda}^2}{\sqrt{x+c}\sqrt{Y+c}\left(\sqrt{x+c}+\sqrt{Y+c}\right)^2}}$$

▷ Solution for $B \geq 3$

$$G(x^1|\cdots|x^B) = \left(-2\tilde{\lambda}\right)^{3B-4}\left(\frac{d}{dt}\right)^{B-3}\left.\left(\frac{\left(\frac{1}{\sqrt{x^1+c-2t}}\right)^3\cdots\left(\frac{1}{\sqrt{x^B+c-2t}}\right)^3}{\left(1-\int_1^\infty dT\,\rho(T)\frac{1}{\sqrt{T+c}\sqrt{T+c-2t}\left(\sqrt{T+c}+\sqrt{T+c-2t}\right)}\right)^{B-2}}\right)\right|_{t=0}$$

Every N-pt function is solved exactly!

Comments

- For 4-dim, 6-dim cases

$$\boxed{\begin{array}{l}\text{2-dim action} \\ S = L\,\mathrm{Tr}(E\Phi^2 - A\Phi) + L\frac{\lambda}{3}\mathrm{Tr}\Phi^3\end{array}}$$

$\Rightarrow$ 4, 6-dim action

$$S = V\,\mathrm{Tr}\left(zE\Phi^2 + (\kappa + \nu E + \zeta E^2)\Phi + \frac{\lambda}{3} z^{\frac{3}{2}} \Phi^3\right)$$

for renormalization

~ 2, 4, 6 - dim ~

$\boxed{\text{Every } (N_1 + \cdots + N_B)\text{-pt fun } G(x^1_1 \cdots x^1_{N_1} | \cdots | x^B_1 \cdots x^B_{N_B}) \text{ is given explicitly by solving S-D eq.}}$

$V, N \to \infty$

$$\log \frac{Z[J]}{Z[0]} =: \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \cdots \le N_B}^{\infty} \sum_{\forall p^j_i = 0}^{N} V^{2-B} \frac{G(|p^1_1 \cdots p^1_{N_1}| \cdots |p^B_1 \cdots p^B_{N_B}|)}{S(N_1, \cdots, N_B)} \prod_{\beta=1}^{B} \frac{J_{p^\beta_1} \cdots J_{p^\beta_{N_\beta}}}{N_\beta}$$

This Φ^3_d Q.F.T. is completely solved!

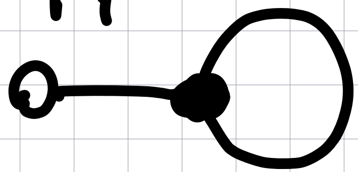
$d = 2, 4, 6$

§6. ~ Which kind of Quantum Field Theory ~

@ 2-dim case

Γ_i : planar graph on S^2

ex. Γ_1

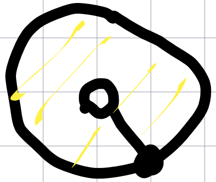


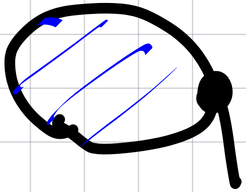
○ : white vertex $\sim \mathbb{J}_{P_1 \cdots P_N} = J_{P_1P_2} J_{P_2P_3} \cdots J_{P_NP_1}$

↑ puncture, external vertex,

● : black vertex ← internal vertex

Face : A number of "○" touching a face is 1 or 0 .

 : Face with "o" : "external" $\leftarrow x$

 : Face without "o" : "internal" $\leftarrow y$

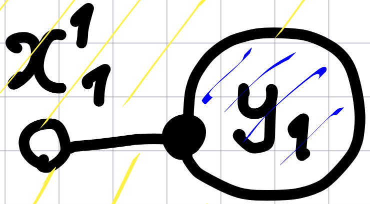
x^{i}_{j}

$i \leftarrow$ label for "o" $(1 \sim B)$

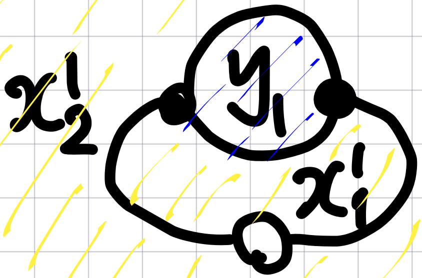
$j \leftarrow$ j-th face in N_i faces touching i-th "o" $(1 \sim N_i)$

ex)

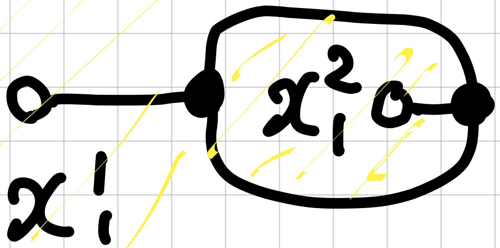
Γ_1 :



Γ_2 :



Γ_3 :

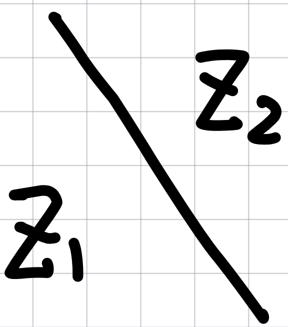


$\Leftarrow$ (Dual graph of a triangulation of S^2 with B-puncture)

Feynman rules

● : 3-point interaction $\longleftrightarrow (-\tilde{\lambda})$

○ : external vertex $\longleftrightarrow 1$

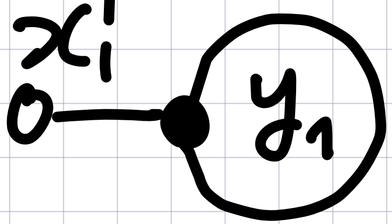


: border line between z_1-face and z_2-face

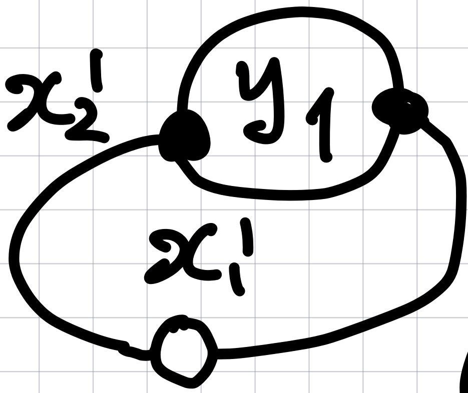
$$\longleftrightarrow \frac{1}{z_1+z_2+1}$$

y_i : internal face variables

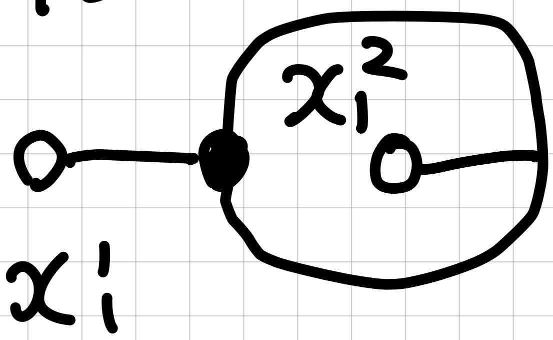
$$\longleftrightarrow \int_0^{\Lambda^2} dy_i$$

Γ_1 : 

$$\tilde{G}^{\Lambda}_{\Gamma_1}(x_1^1) = \frac{(-\tilde{\lambda})}{2x_1^1+1}\int_0^{\Lambda^2} \frac{dy_1}{x_1^1+y_1+1}$$

Γ_2 : 

$$\tilde{G}^{\Lambda}_{\Gamma_2}(x_1^1, x_2^1) = \frac{(-\tilde{\lambda})^2}{(x_1^1+x_2^1+1)^2}\int_0^{\Lambda^2} \frac{dy_1}{(x_1^1+y_1+1)(x_2^1+y_1+1)}$$

Γ_3 : 

$$\tilde{G}^{\Lambda}_{\Gamma_3}(x_1^1|x_1^2) = \frac{(-\tilde{\lambda})}{(2x_1^1+1)(2x_1^2+1)(x_1^1+x_1^2+1)^2}$$

§7 Finite Φ^3-model.

$$S = V\,\mathrm{tr}\left(E\Phi^2 + K\Phi + \frac{\lambda}{3}\Phi^3\right) - V\,\mathrm{tr}\,J\Phi$$

E: diag$(E_1, E_2, \cdots, E_N)$, $E_i > 0$ $\quad$ K: const $\quad$ J: external field $\quad J^\dagger = J$

<u>Remark</u>.

When we put $K = 0$ & $J = 0$,

this model becomes Kontsevich model.

$\Rightarrow$

$Z(E)$ corresponds to τ function of KdV hierarchy.

▷ Partition function for finite N

$$Z = \int d\Phi \, \exp\left(-i\, V \operatorname{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right)\right) \exp\left(iV \operatorname{tr} J\Phi\right)$$

$\Phi = X - E$

$$= \exp\left(-i V \operatorname{tr}\left(\frac{2}{3\lambda^2}E^3 - \frac{\kappa}{\lambda}E + \frac{1}{\lambda}JE\right)\right)$$

$$\int \mathcal{D}X \, \exp\left(-i\frac{\lambda V}{3} \operatorname{tr} X^3\right) \exp\left(i\lambda V \kappa \operatorname{tr}\{(M - I + K)UXU^{-1}\}\right)$$

, where $M = \frac{E^2}{\kappa}$, $K = \frac{J}{\lambda\kappa}$

IZ (HC) integral

eigen value of A & B

Hermitian

$$\int_{U(n)} \exp t \operatorname{tr} AUBU^{-1} dU = C \frac{\det_{1 \le i,j \le n} \exp\left(t\lambda_i(A)\lambda_j(B)\right)}{t^{\frac{n^2-n}{2}} \Delta(\lambda(A))\, \Delta(\lambda(B))}$$

Vandermonde det on $\lambda_i(A)$ & $\lambda_i(B)$

$$Z[J] = C \exp\left(-i \frac{V}{\lambda^2} \mathrm{tr}\left(\frac{2}{3}E^3 - \lambda \kappa E + \lambda JE\right)\right) \frac{\det\left(\phi_i(s_j)\right)}{\prod_{1 \le t < u \le N} (s_u - s_t)},$$

where s_i is an eigenvalue of $\frac{E^2}{\lambda \kappa} - I + \frac{J}{\kappa}$

$$\phi_k(z) = 2\pi i \left(\frac{i}{(\lambda V)^{\frac{1}{3}}}\right)^{k-1} \left(\frac{d}{dy}\right)^{k-1} A_i[y] \Bigg|_{y = -\frac{V\kappa z}{(\lambda V)^{\frac{1}{3}}}}$$

Airy fun.

or using

$$A_N(y_1, \cdots, y_N) = \prod (\partial_{y_i} - \partial_{y_j}) A_i(y_1) \cdots A_i(y_N) = \det\left(A_i^{(j-1)}(y_i)\right)$$

$$y_i = -\frac{V\kappa}{(\lambda V)^{\frac{1}{3}}} s_i$$

$$Z[J] = C' e^{\frac{-iV}{\lambda} \mathrm{tr}(JE)} \frac{A_N(y_1, \cdots y_N)}{\prod_{1 \le t < u \le N} (s_u - s_t)}$$

Thm.

$G_{|a_1^1 \cdots a_{N_1}^1| \cdots |a_1^B \cdots a_N^B|}$ ← Every N pt-function is described by $G_{|a|b|\cdots|c|}$ type.

$$= \lambda^{N-B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G_{|a^1_{k_1}|\cdots|a^B_{k_B}|} \left(\prod_{\substack{l_1=1 \\ l_1 \neq k_1}}^{N_1} \frac{1}{E^2_{a^1_{k_1}} - E^2_{a^1_{l_1}}} \right) \cdots \left(\prod_{\substack{l_B=1 \\ l_B \neq k_B}}^{N_B} \frac{1}{E^2_{a^B_{k_B}} - E^2_{a^B_{l_B}}} \right)$$

$N = N_1 + N_2 + \cdots + N_B$

⇓

We should do cal. of $G_{|a|b|\cdots|c|}$ type n pt-function.

$$G_{|a_1|a_2|\cdots|a_n|} = (iV)^{n-2} \frac{\partial^n}{\partial J_{a_1 a_1} \cdots \partial J_{a_n a_n}} \log Z[J] \Big|_{J=0}$$

⇓

diagnal J is enough.

"eigenvalue of $\frac{E^2}{\lambda K} - I + \frac{J}{K}$" $= \frac{E_i^2}{\lambda K} - 1 + \frac{J_{ii}}{K}$ $\quad i = 1 \sim N$

$$\Rightarrow G|a| = -\frac{E_a}{\lambda} - \frac{\lambda}{iV}\sum_{i\neq a}\frac{1}{E_a^2 - E_i^2} + \frac{i}{(\lambda V)^{\frac{1}{3}}}\partial_a \log A_N(z)$$

$$z_i = -\frac{V E_i^2}{(\lambda V)^{\frac{2}{3}}\lambda} + \frac{V\kappa}{(\lambda V)^{\frac{1}{3}}}$$

Similarly

$$G|a_1|a_2|\cdots|a_n| = (iV)^{n-1} c \left(\frac{V^{\frac{2}{3}n}}{\lambda^{\frac{1}{3}n}}\right)\partial_{a_1}\cdots\partial_{a_k}\log A_N(z_1,\cdots,z_N)$$

$$(n \geq 3)$$

We can obtain every N-point function from above.

§8 Φ^3-Φ^4 mixed matrix model (finite N)

Φ^4 (Grosse-Wulkenhaar) model is still not enogh to be calculated, and its properties are not unveiled.

⇘ Integrable model Φ^3-Φ^4 mixed model might help to reveal properties of Φ^4 model.

coupling const.

$$S[\Phi] = V \,\mathrm{tr}\Big(E\Phi^2 + \frac{1}{2} M\Phi M\Phi + \sqrt{\lambda}\, M\Phi^3 + \frac{\lambda}{4}\Phi^4 \Big)$$

const.

$$\underbrace{\begin{pmatrix} E_1 & & 0 \\ & E_2 & \\ 0 & & \ddots \end{pmatrix} \qquad M^2 = E, \quad M = \begin{pmatrix} \sqrt{E_1} & & 0 \\ & \sqrt{E_2} & \\ 0 & & \ddots \end{pmatrix}}_{\text{constant matrix}}$$

$$Z[J] = \int D\Phi \; e^{-S} \; e^{V \mathrm{tr} J\Phi}$$

▷ Feynman Rules

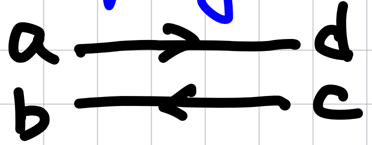
$$S_{\text{free}} = V\,\mathrm{tr}\left(E\Phi^2 + \frac{1}{2}M\Phi M\Phi\right)$$

$$Z_{\text{free}}[J] := \int D\Phi\, e^{-S + V\,\mathrm{tr}\,J\Phi}$$

$$= C \exp\left(\frac{V}{2}\sum_{n,m=1}^{N} J_{mn}\frac{1}{E_n+E_m+\sqrt{E_nE_m}}J_{nm}\right)$$

const. $Z_{\text{free}}[0]$

- **Propagator**

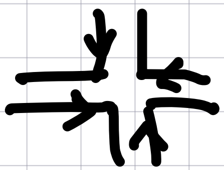


$$:= \langle \Phi_{ba}\Phi_{dc}\rangle = \frac{1}{V}\frac{\delta_{ad}\delta_{bc}}{E_a+E_b+\sqrt{E_aE_b}}$$

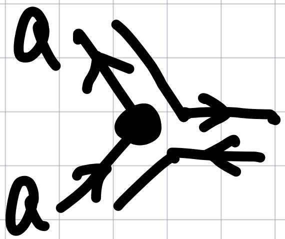
$$Z[J] = \int D\Phi \, e^{-S_{free}} \, e^{-S_{int}} \, e^{V \mathrm{tr} J\Phi}$$

$$S_{int} = V \mathrm{tr}\left(\sqrt{\lambda} M \Phi^3 + \frac{\lambda}{4} \Phi^4 \right)$$

- 4pt Interaction

 $= -\dfrac{V\lambda}{4}$ (without statistical factors)

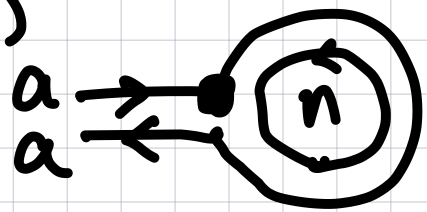
- 3pt Interaction



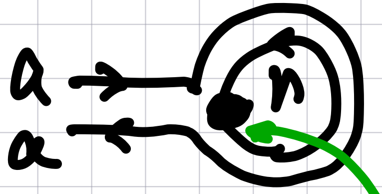
$= -V\sqrt{\lambda} E_a$ (without statistical factors)

- loop $\quad \displaystyle\sum_{n=1}^{N}$

Ex)

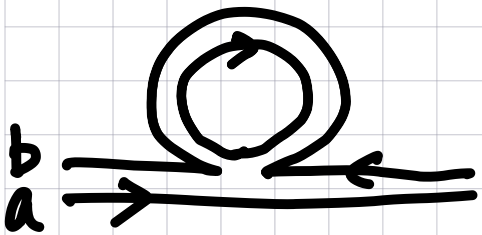


$$= \frac{1}{\sqrt{3E_a}} \times \left(-V\sqrt{\lambda E_a}\right) \sum_{n=1}^{N} \frac{1}{\sqrt{(E_a+E_n+\sqrt{E_aE_n})}}$$



$$= \frac{1}{\sqrt{3E_a}} \times \sum_{n=1}^{N} \frac{-V\sqrt{\lambda E_n}}{\sqrt{(E_a+E_n+\sqrt{E_aE_n})}}$$

on the loop



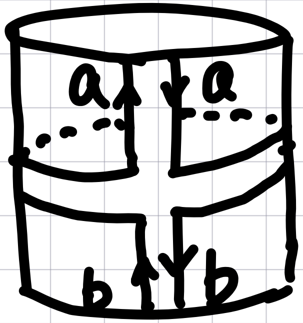
$$= -\frac{\lambda}{4V^2} \frac{1}{(E_a+E_b+\sqrt{E_aE_b})^2}$$

$$\times \sum_{n=1}^{N} \frac{1}{E_b+E_n+\sqrt{E_bE_n}}$$

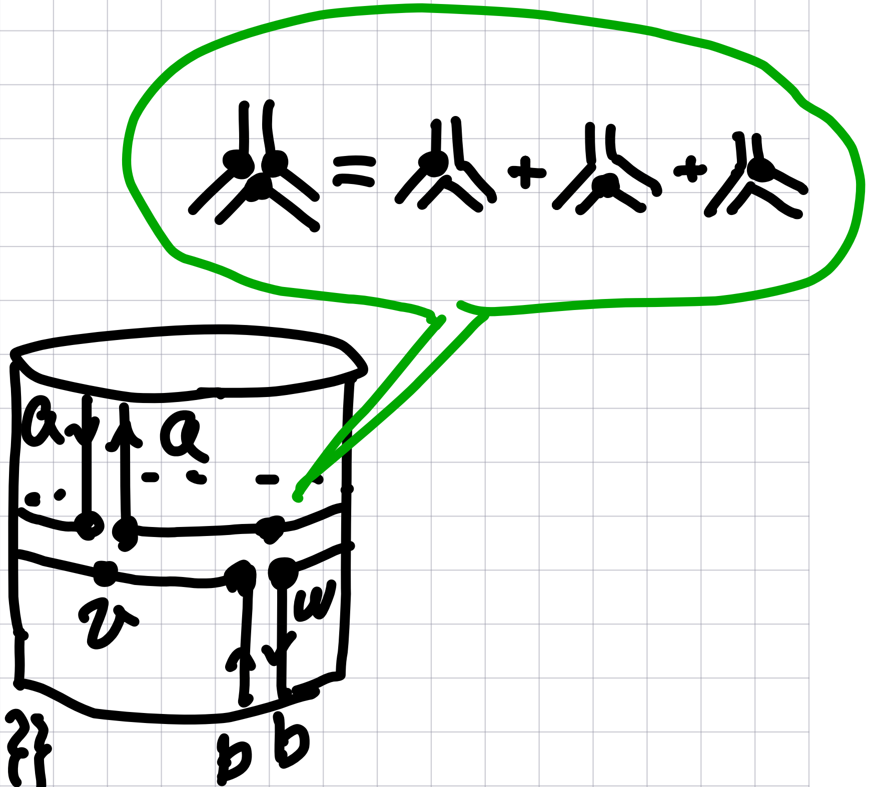
▷ Ex.

$$G_{|a|b|} = \frac{\partial^2 \log Z[J]}{\partial J_{aa}\, \partial J_{bb}} \Bigg|_{J=0}$$

$$= 4V^2$$



$$+ V^2 \sum_{\substack{w \\ \notin \\ \{\{b,b,a\}\}}} \sum_{\substack{v \\ \in \\ \{\{a,a,b\}\}}}$$



$$= -\frac{\lambda}{9E_aE_b\left(E_a+E_b+\sqrt{E_aE_b}\right)}$$

$$+ \sum_{w}\sum_{v} \frac{\lambda \sqrt{E_v}\sqrt{E_w}}{9E_aE_b\left(E_a+E_b+\sqrt{E_aE_b}\right)^2} + O(\lambda^2)$$

▷ Exact calculation.

Using IZ integral

$$Z[J] = C\,\frac{e^{-\frac{V}{\sqrt{\lambda}}\mathrm{tr}(JM)}\,P_N(s_1, s_2, \cdots, s_N)}{\prod_{1\le t<u\le N}(s_u - s_t)}$$

where $P_N(s_1, \cdots, s_N) := \det\left(P^{(j-1)}(s_i)\right) = \prod_{1\le i<j\le N}(\partial_{s_i} - \partial_{s_j})P(s_1)\cdots P(s_N)$,

$$P(z) := \int_{-\infty}^{\infty} dx\, e^{-\frac{\lambda V}{4}x^4 + V\lambda z} \quad (\text{like Pearcey integral})$$

s_t : eigenvalues of $\frac{1}{\sqrt{\lambda}}M^3 + J$

ex). $G|a|b| = \frac{\partial^2}{\partial J_{aa} \partial J_{bb}} \log Z[J] \Big|_{J=0}$

We can chose $J = \mathrm{diag}(J_{11}, \cdots, J_{NN})$

$\Rightarrow \quad S_t = \frac{1}{\sqrt{\lambda}} E_t^{\frac{3}{2}} + J_{tt}$

$$\boxed{G|a|b| = \frac{\partial_a \partial_b P_N(z_1, \cdots; z_N)}{P_N(z_1, \cdots; z_N)} - \frac{\partial_a P_N(z_1, \cdots; z_N)}{P_N(z_1, \cdots; z_N)} \frac{\partial_b P_N(z_1, \cdots; z_N)}{P_N(z_1, \cdots; z_N)} - \frac{\lambda}{(E_a\sqrt{E_a} - E_b\sqrt{E_b})^2}}$$

$\downarrow$ If we use saddle point approximation then we get the previous result.

Summary

- Scalar field theories on Moyal plane
  $\Rightarrow$ 1-Matrix model

  KdV τ

  Kontsevich type

  renormalized ϕ^3 $\longrightarrow$ Grosse-Steinacker model

  " ϕ^4 $\longrightarrow$ Grosse-Wulkenhaar model

  Every $G|a^1_1 \cdots a^1_{N_1}| \cdots |a^b_1 \cdots a^b_{N_b}|$ for finite N

  or $G(x^1_1 \cdots x^1_{N_1}| \cdots |x^b_1 \cdots x^b_{N_b})$ for $N \to \infty$ ($\frac{L}{N}$ fixed)

  is calculated exactly.

Note: The relation between $G|a^1_1 \cdots a^1_{N_1}| \cdots |a^b_1 \cdots a^b_{N_b}|$

& $G(x^1_1 \cdots x^1_{N_1}| \cdots |x^b_1 \cdots x^b_{N_b})$ is unknown.

∘ Φ^3-Φ^4 hyblid model?

Φ^4 Grosse-Wulkenhaar model ← More difficult than Φ^3 model

Φ^3-Φ^4 hyblid model is more easy than Φ^4.

↳ Indeed it is an integrable model.

(Itykson-Zuber 92. generalized Kontsevich, W3-alg. higher KP.)

For finite N,

- pertubation theory
- exact solutions

Note: Renormalization, $N \to \infty$, etc have not been studied yet.