Heisenberg picture

$$UN|\psi\rangle = (UNU^\dagger)(U|\psi\rangle)$$

- applying U to the state transforms N to $UNU^\dagger$

$$N \rightarrow UNU^\dagger$$

: Heisenberg picture

- for large number of qubits, it is difficult to follow the unitary evolution of the state

Heisenberg picture

$$N|\psi\rangle \;\text{—}\boxed{U}\text{—}\; UN|\psi\rangle = (UNU^\dagger)(U|\psi\rangle)$$

- by following the evolution of a sufficiently large set of N's, we will be able to completely reconstruct the evolution of the state

- n-qubit operator can be expanded by tensor products of $\{I, X, Y, Z\}^{\otimes n}$

  ($\because$ n-qubit operator has 4^n independent elements)

→ it is convenient to follow the unitary evolution of the Pauli group

Pauli group

$\mathcal{P}_n = \{\pm, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$: n-qubit Pauli group

(overall phases $\{\pm, \pm i\}$ are necessary to complete the group)

e.g. $n = 2$: $\{\pm, \pm i\} \times \{II, IX, IY, IZ, XI, XX, XY, XZ,$
$YI, YX, YY, YZ, ZI, ZX, ZY, ZZ\}$

$MN \xrightarrow{U} (UMU^\dagger)(UNU^\dagger)$

it is sufficient to follow the evolution of a generating set of the Pauli group

$\{X_1, \ldots, X_n, Z_1, \ldots Z_n\}$: a generating set for the n-qubit Pauli group

e.g. $n = 2$ -> $\{XI, IX, ZI, IZ\}$

$X_1 = X \otimes I^{\otimes(n-1)}$ $Z_2 = I \otimes Z \otimes I^{\otimes(n-2)}$

Clifford group

- Clifford operators leave the group $\mathscr{P}_n$ fixed

U: Clifford operator

$$N \in \mathscr{P}_n \qquad N \xrightarrow{U} UNU^\dagger \in \mathscr{P}_n$$

- Clifford operators form a group: Clifford group

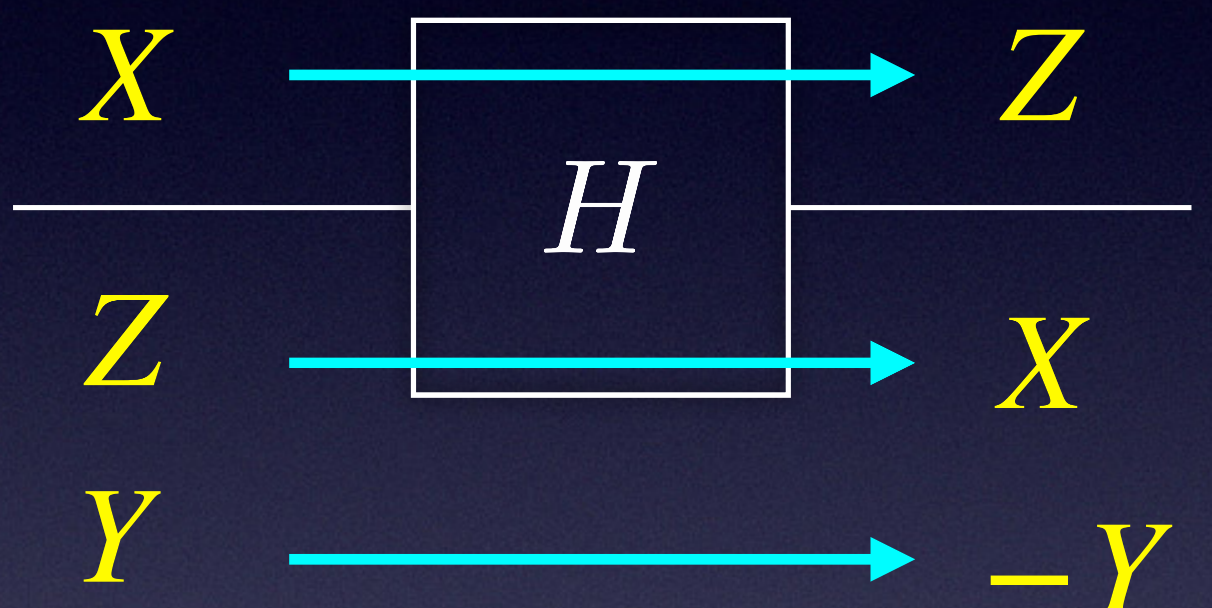
- $\{H, S, \mathrm{CNOT}\}$ are generators of the Clifford group

any Clifford operator can be created by a product of H, S, and CNOT

we focus on a quantum circuit that consists of only Clifford operators

Single-qubit Clifford gate

- Hadamard gate $HXH = Z, \quad HZH = X$



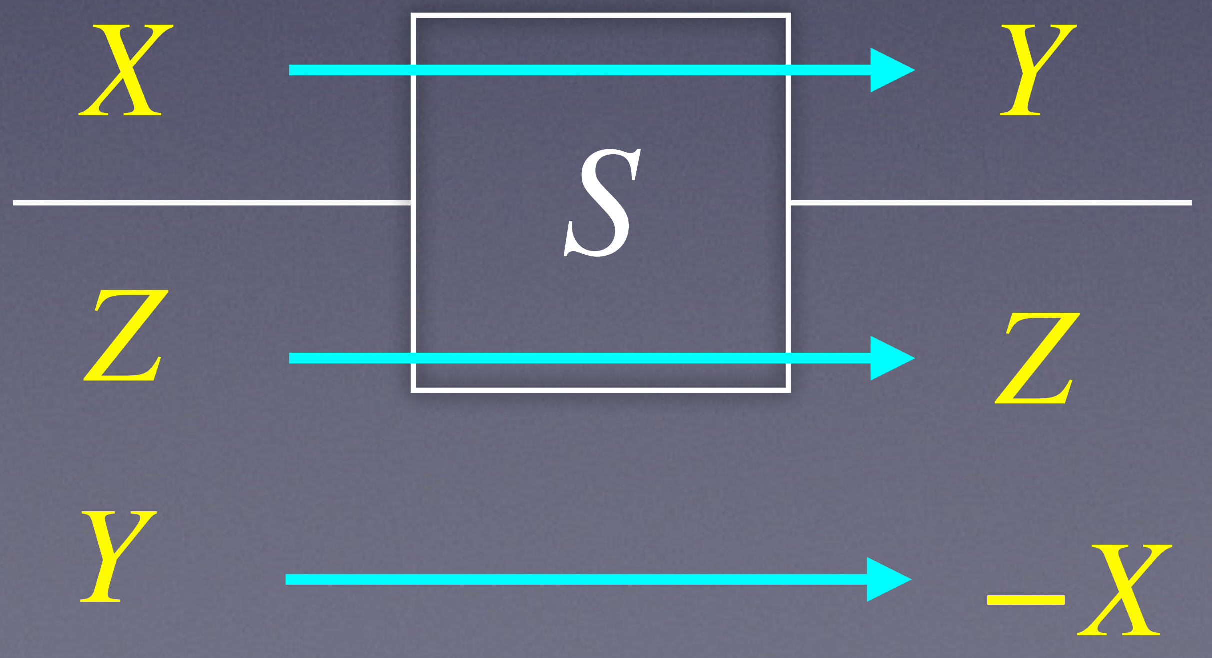
$\Rightarrow HYH = -Y$

enough to study the generators of the Pauli group

- Phase gate

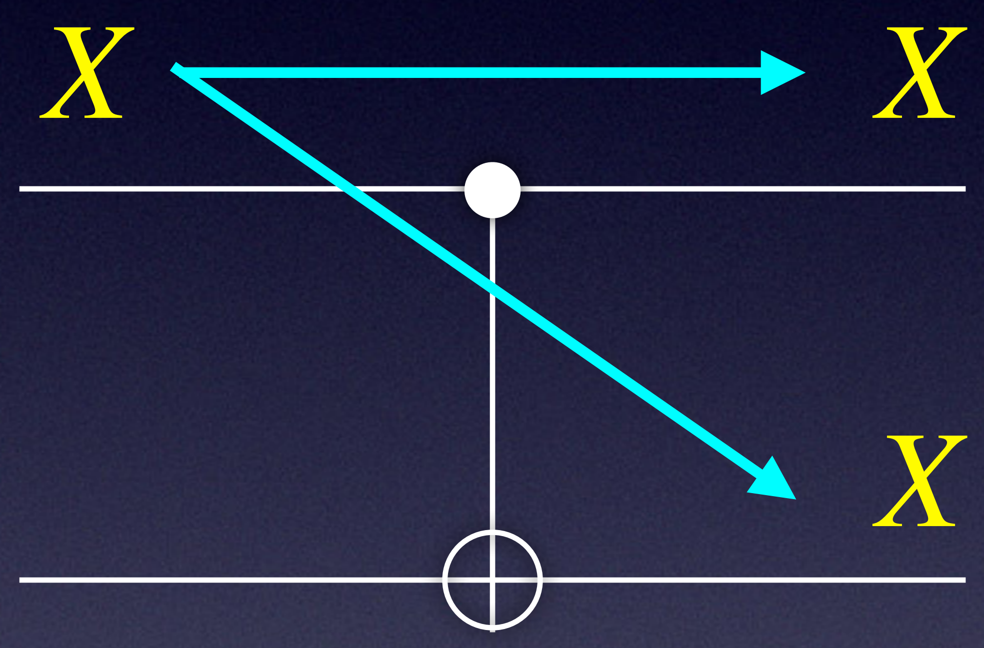
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$SXS^\dagger = Y, \quad SZS^\dagger = Z$$



two-qubit Clifford gate

- CNOT gate $\mathrm{CNOT} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$



$$X \otimes I \to X \otimes X$$
$$I \otimes X \to I \otimes X$$
$$Z \otimes I \to Z \otimes I$$
$$I \otimes Z \to Z \otimes Z$$

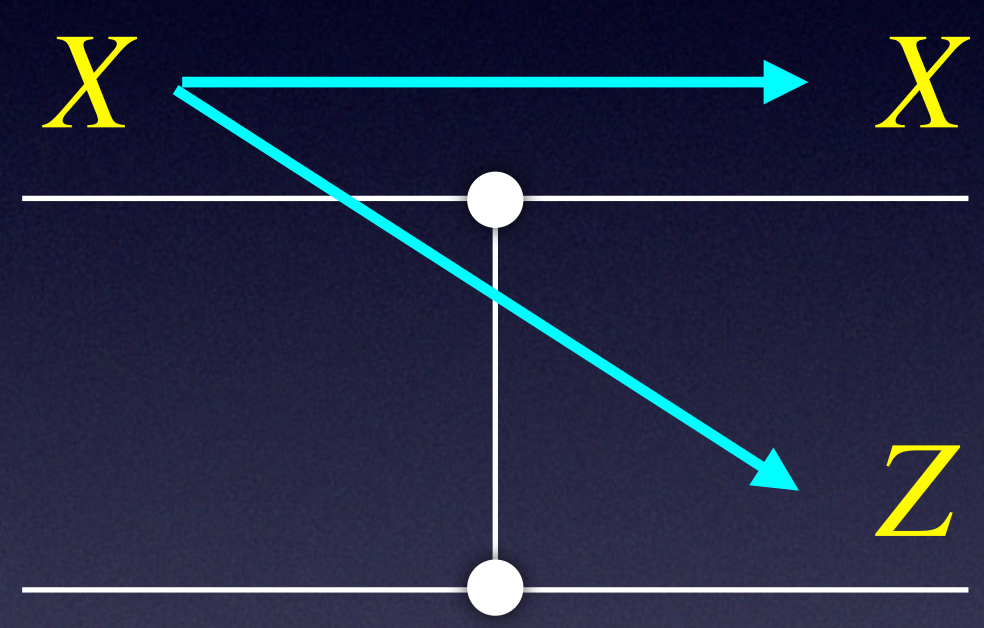
Exercise

Confirm the above evolution by CNOT gate

two-qubit Clifford gate

- CZ gate

$$CZ = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z$$

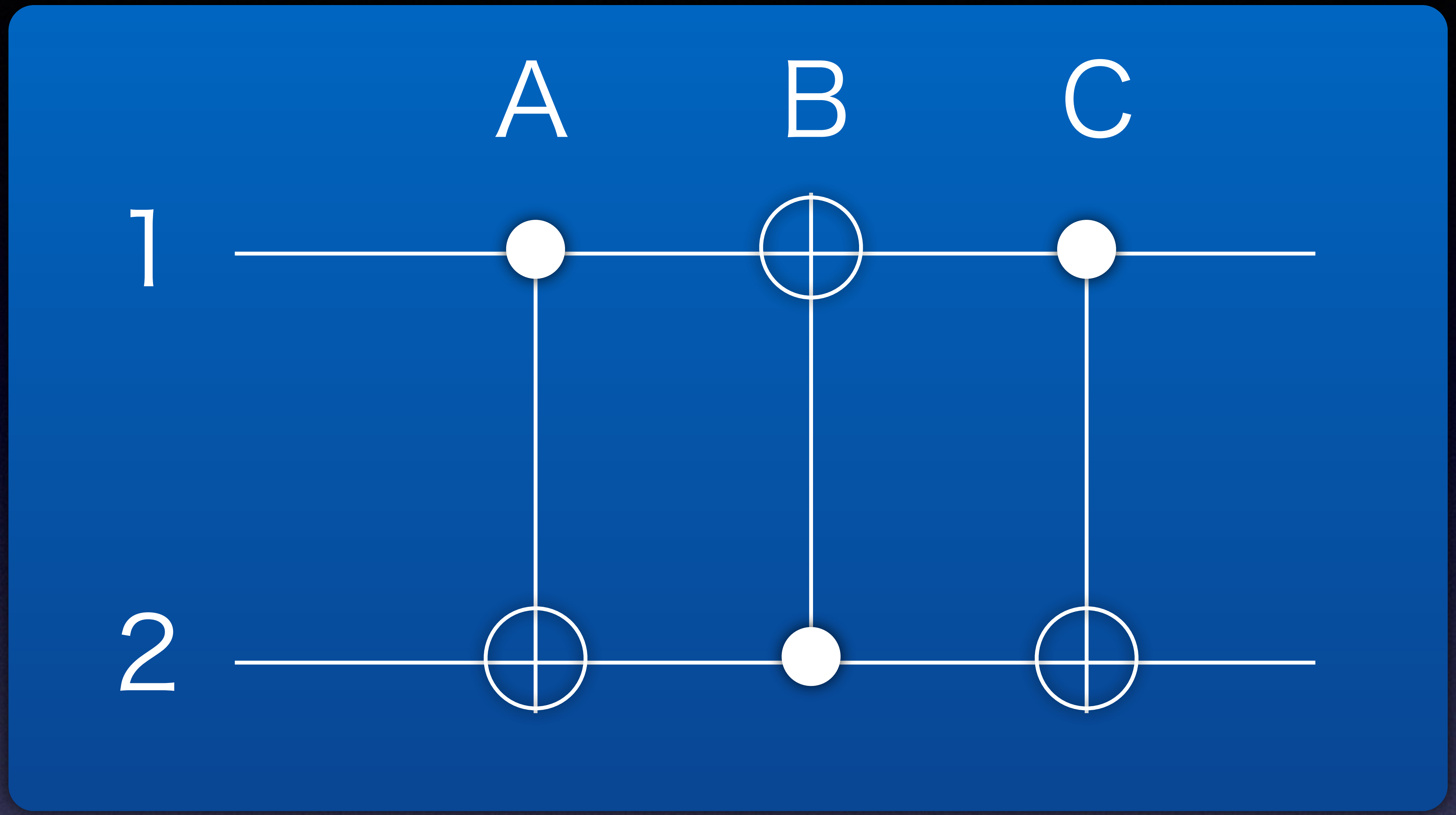


$$X \otimes I \to X \otimes Z$$
$$I \otimes X \to Z \otimes X$$
$$Z \otimes I \to Z \otimes I$$
$$I \otimes Z \to I \otimes Z$$

Exercise

Confirm the above evolution by CZ gate

Application of the Heisenber picture I



$\bar{X}_i$, $\bar{Z}_i$: logical operators

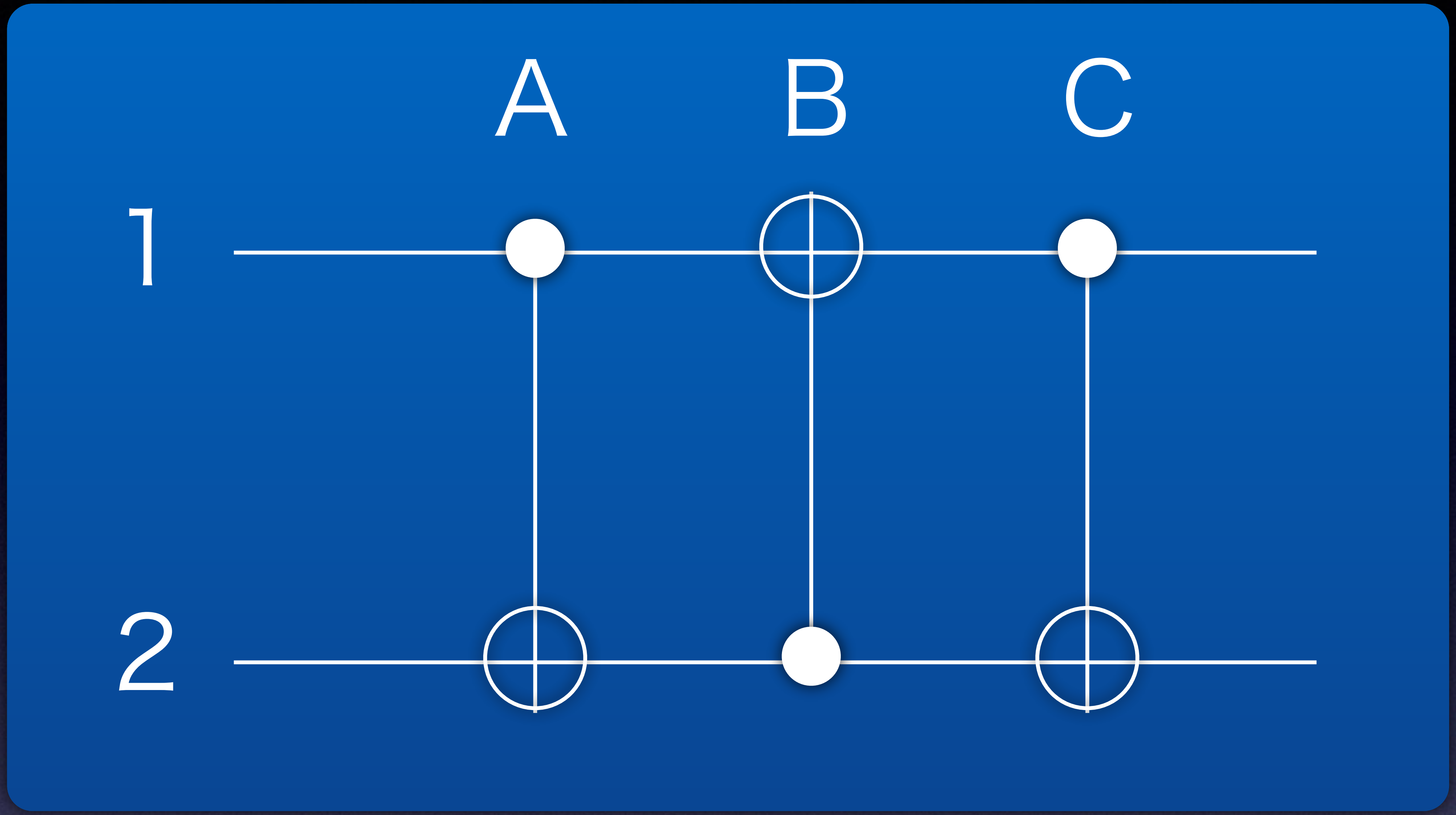
X_i , Z_i transformed by the gates

$\bar{X}_i$, $\bar{Z}_i$ $(i = 1,2)$ are generators of Pauli group for 2 qubits

Exercise

Follow the unitary evolution of the logical operators $\bar{X}_i$, $\bar{Z}_i$ $(i = 1,2)$ by the above circuit

Application of the Heisenber picture I

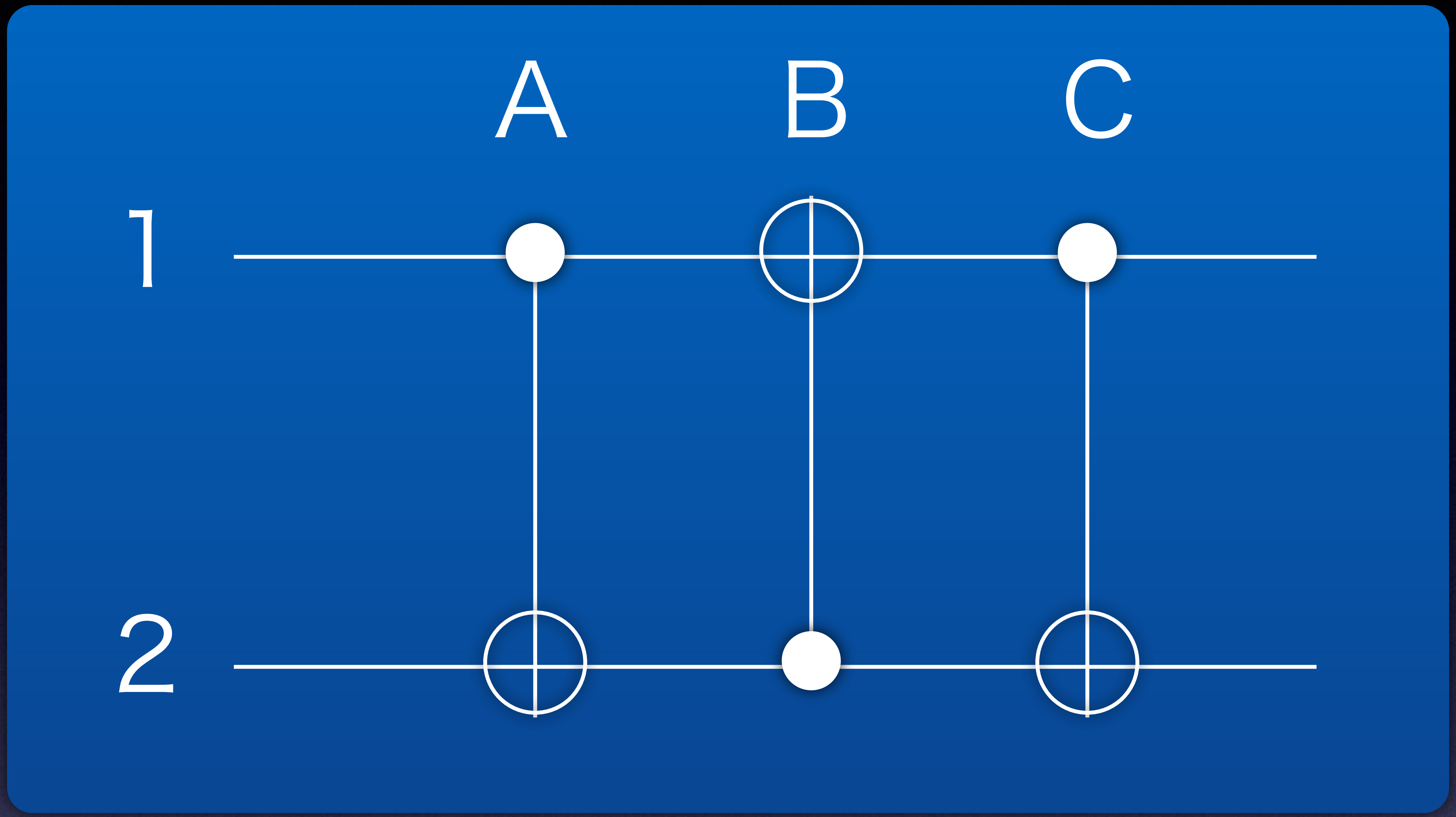


$\bar{X}_i$, $\bar{Z}_i$: logical operators

X_i , Z_i transformed by the gates

	A: CNOT(1->2)	B: CNOT(2->1)	C: CNOT(1->2)
$\bar{X}_1$	$X \otimes X$	$I \otimes X$	$I \otimes X$
$\bar{X}_2$	$I \otimes X$	$X \otimes X$	$X \otimes I$
$\bar{Z}_1$	$Z \otimes I$	$Z \otimes Z$	$I \otimes Z$
$\bar{Z}_2$	$Z \otimes Z$	$Z \otimes I$	$Z \otimes I$

Application of the Heisenber picture I



$\bar{X}_i$, $\bar{Z}_i$: logical operators

X_i , Z_i transformed by the gates

focus on just the input and output of the circuit

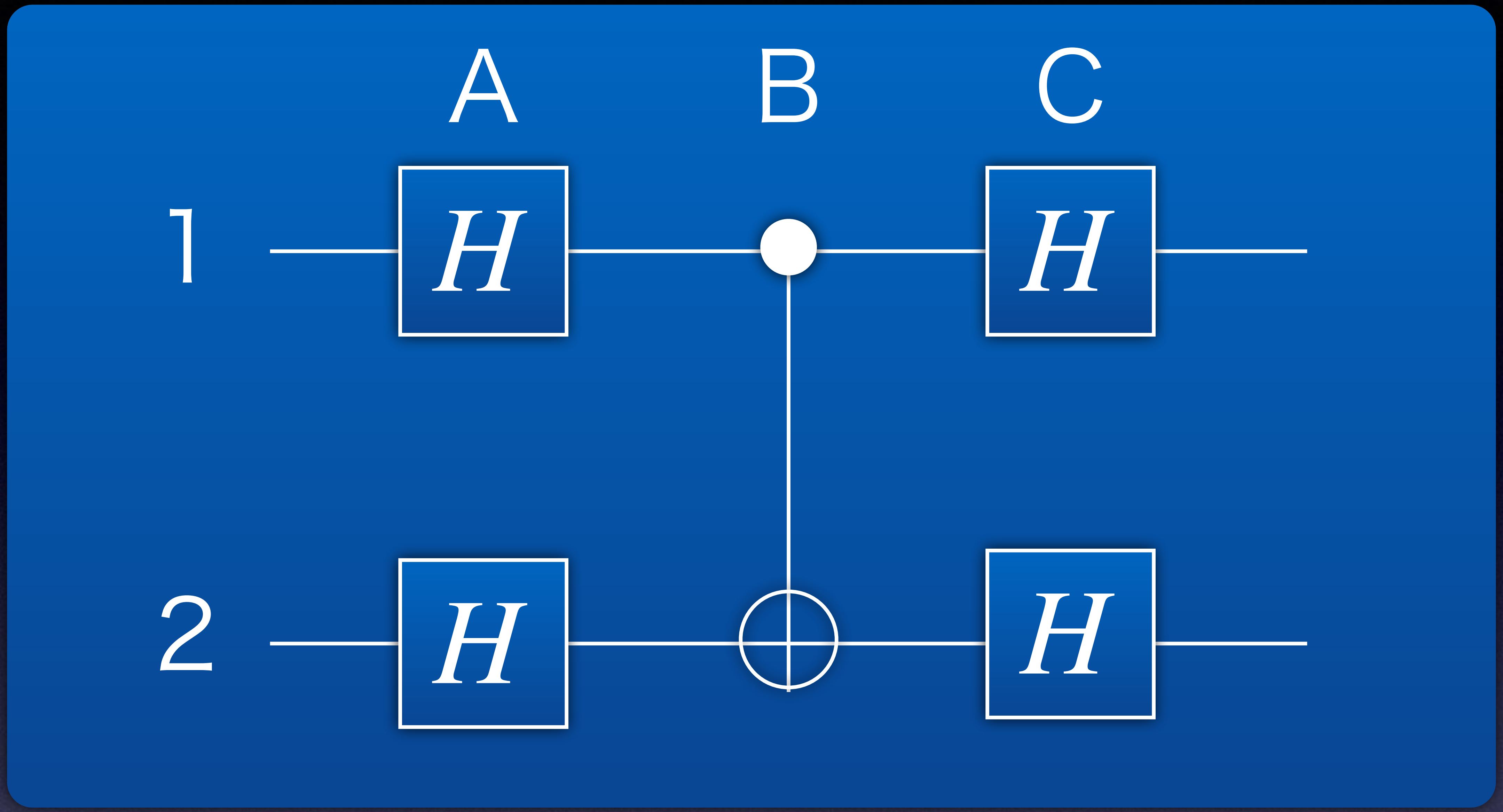
$X \otimes I \rightarrow I \otimes X$ $\quad I \otimes X \rightarrow X \otimes I$

$Z \otimes I \rightarrow I \otimes Z$ $\quad I \otimes Z \rightarrow Z \otimes I$

exchanges $X_1 \leftrightarrow X_2$ and $Z_1 \leftrightarrow Z_2$

→ the circuit acts as SWAP gate

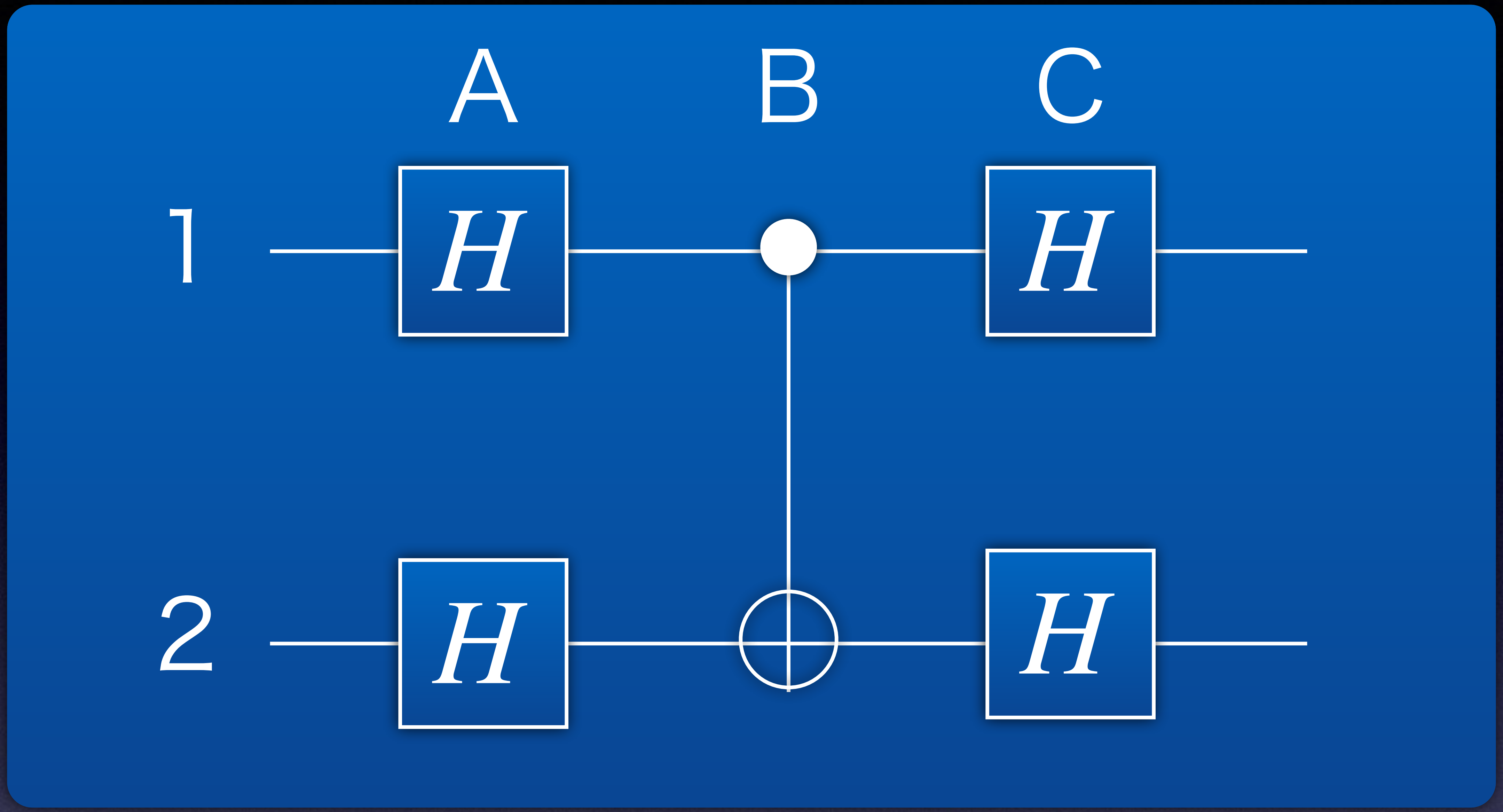
Application of the Heisenber picture II



Exercise

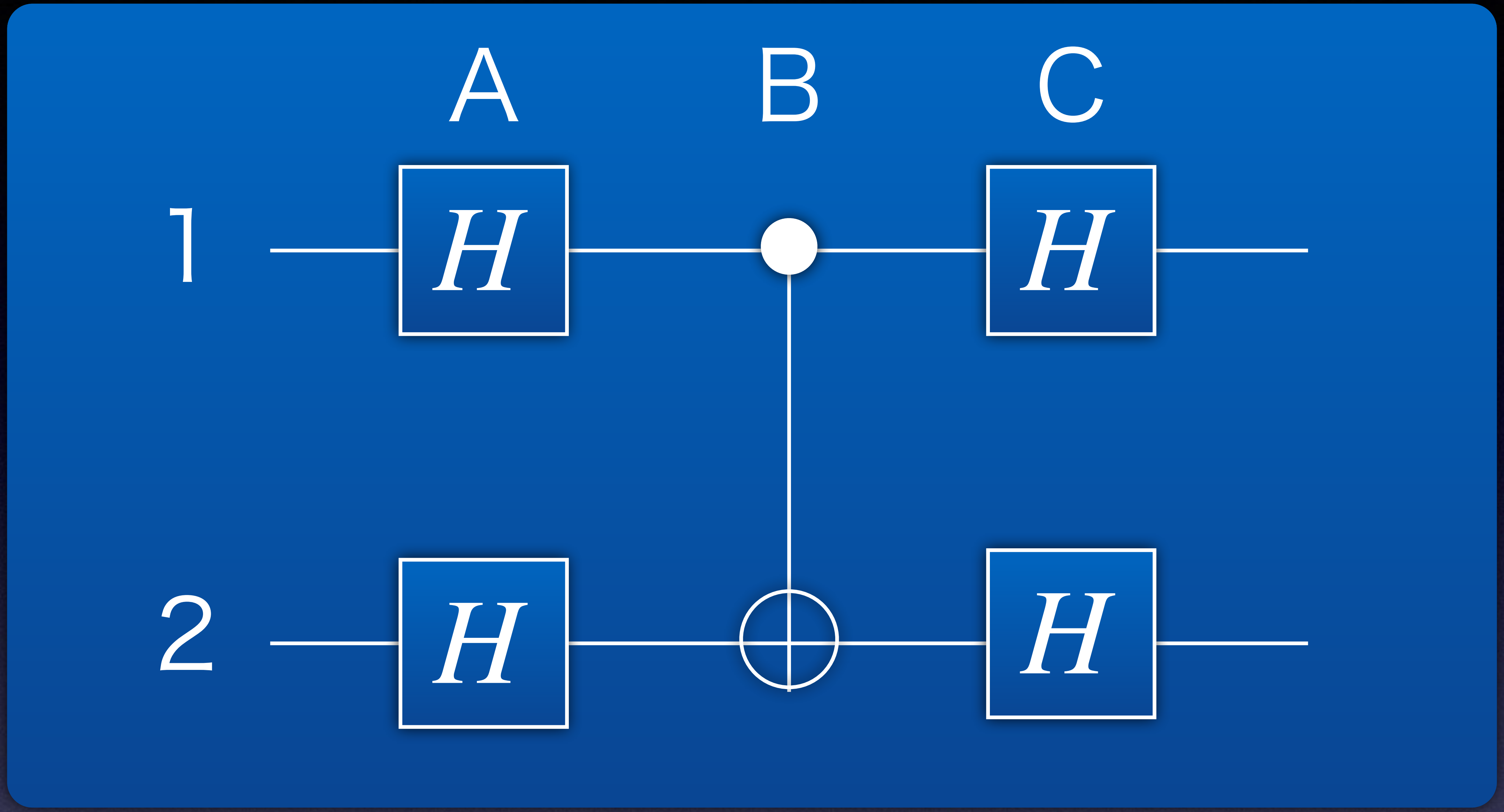
Follow the unitary evolution of the logical operators $\bar{X}_i$, $\bar{Z}_i$ $(i = 1,2)$ by the above circuit

Application of the Heisenber picture II



	A: $H(1)H(2)$	B: CNOT(1->2)	C: $H(1)H(2)$
$\bar{X}_1$	$Z \otimes I$	$Z \otimes I$	$X \otimes I$
$\bar{X}_2$	$I \otimes Z$	$Z \otimes Z$	$X \otimes X$
$\bar{Z}_1$	$X \otimes I$	$X \otimes X$	$Z \otimes Z$
$\bar{Z}_2$	$I \otimes X$	$I \otimes X$	$I \otimes Z$

Application of the Heisenber picture II



focus on just the input and output of the circuit

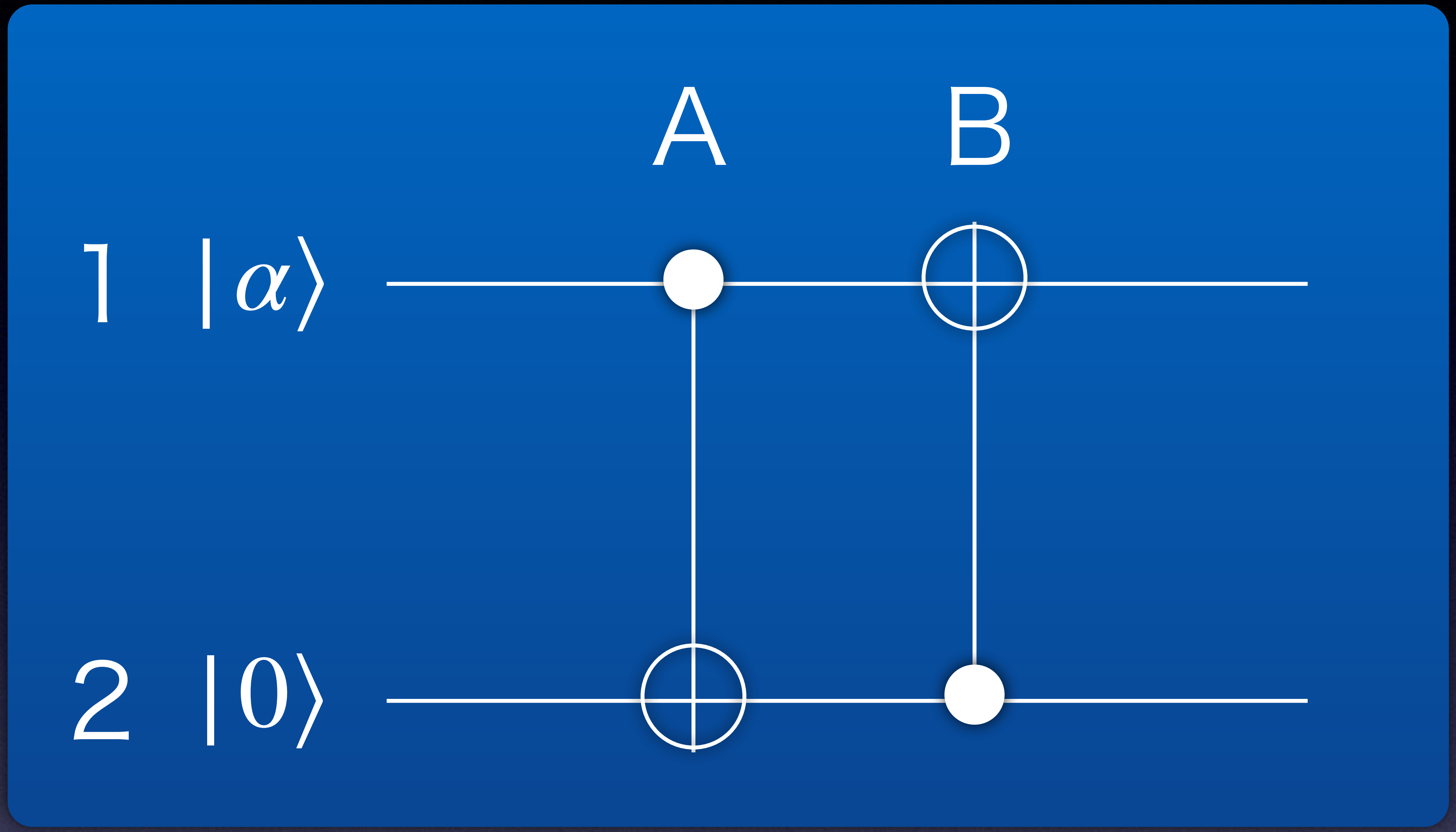
$X \otimes I \rightarrow X \otimes I \qquad I \otimes X \rightarrow X \otimes X$

$Z \otimes I \rightarrow Z \otimes Z \qquad I \otimes Z \rightarrow I \otimes Z$

same evolution as CNOT(2->1)

→ the circuit acts as CNOT(2->1)

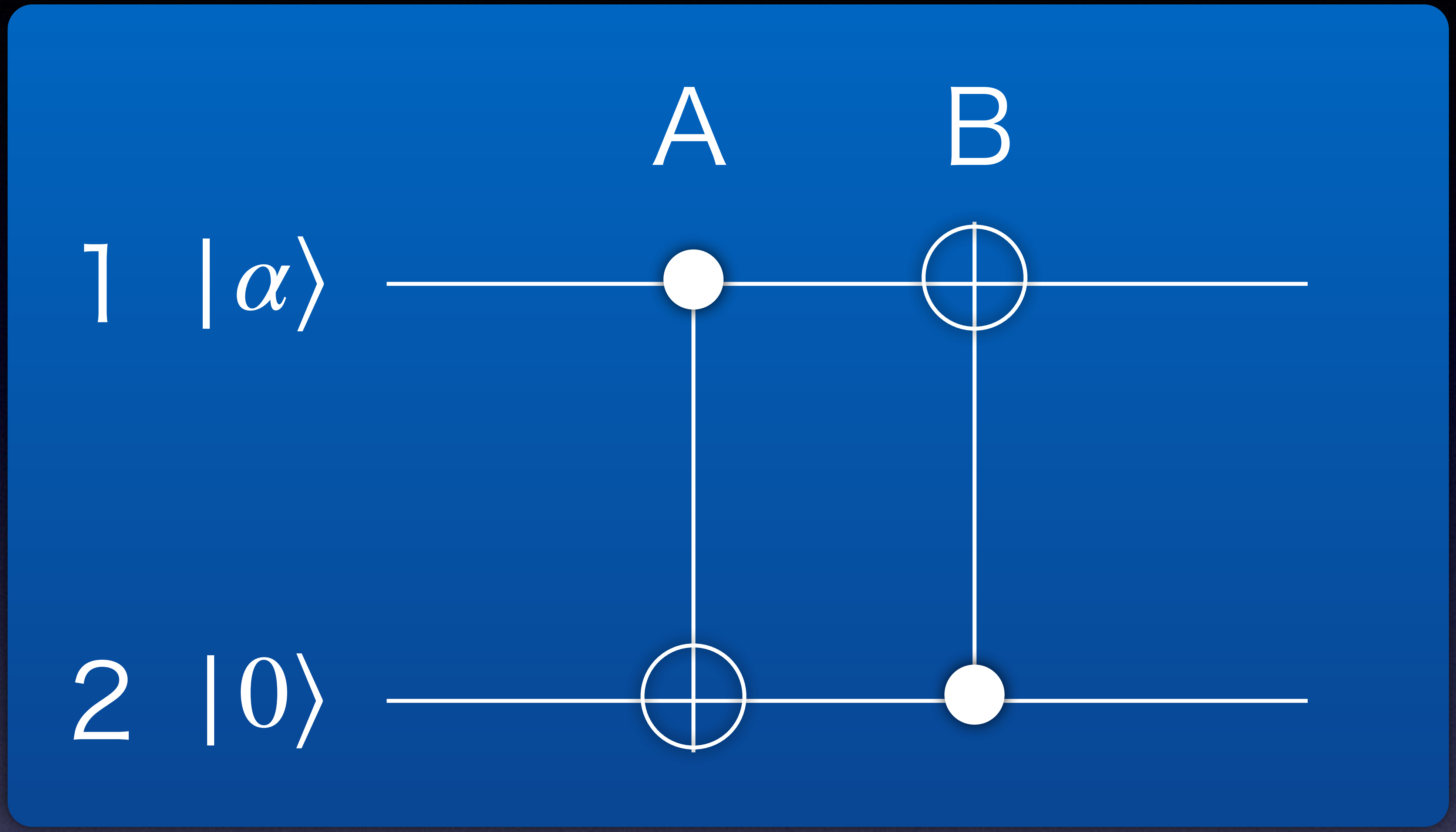
Stabilizers



Q. when $|0\rangle$ is input in qubit 2, what is the function of the circuit?

- since the input state $|0\rangle_2$ is +1 eigenstate of Z_2, the state evolved by each gate is always +1 eigenstate of $\bar{Z}_2$
- $\bar{Z}_2$ fixes the state to its +1 eigenstate → $\bar{Z}_2$: stabilizer

Stabilizers

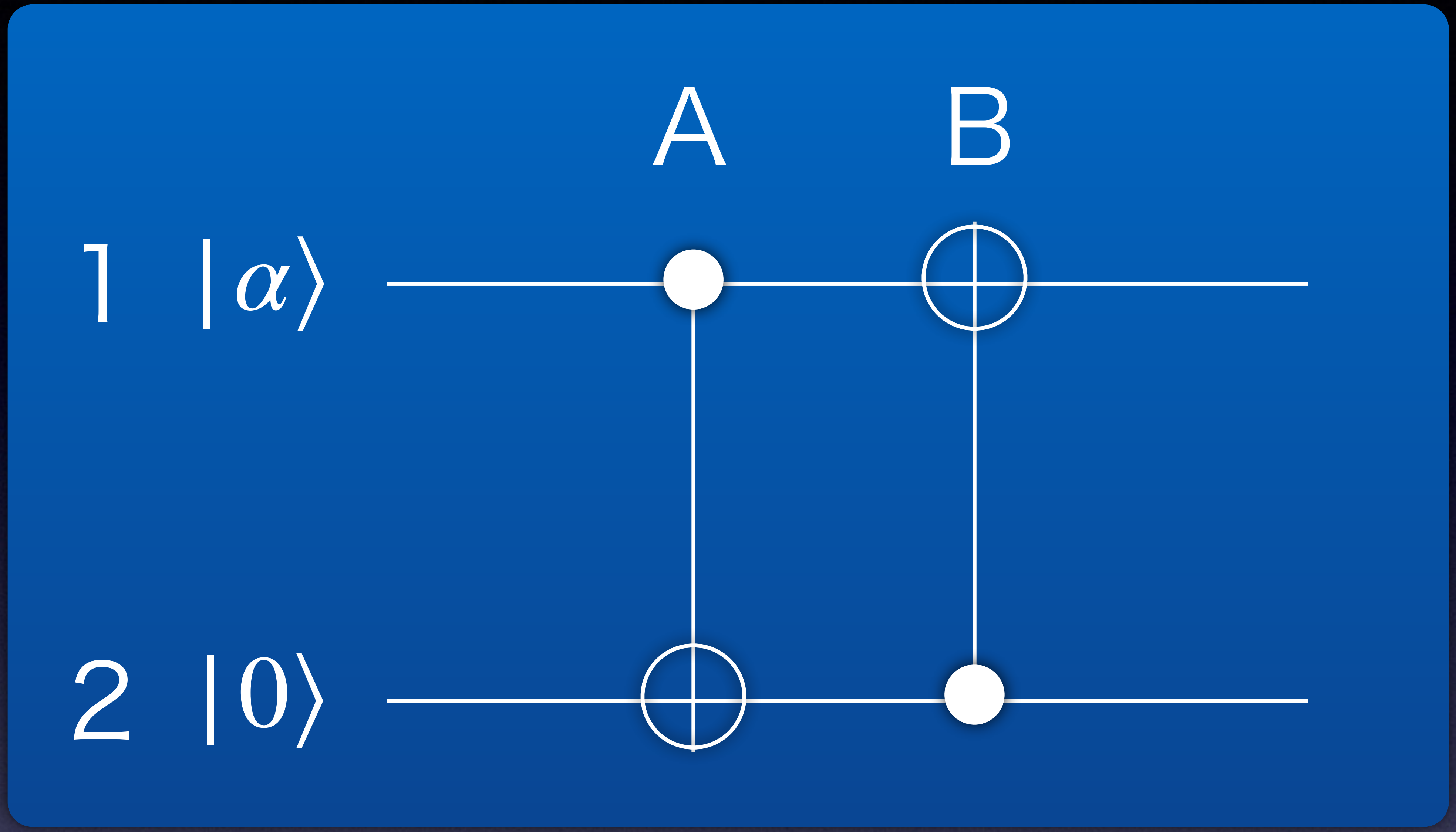


Q. when $|0\rangle$ is input in qubit 2, what is the function of the circuit?

no need to follow $\bar{X}_2$, as it moves the state to -1 eigenstate of $\bar{Z}_2$

		A: CNOT(1->2)	B: CNOT(1->2)
stabilizer	$\bar{Z}_2$	$Z \otimes Z$	$Z \otimes I$
logical operators	$\bar{X}_1$	$X \otimes X$	$I \otimes X$
	$\bar{Z}_1$	$Z \otimes I$	$Z \otimes Z$

Stabilizers



Q. when $|0\rangle$ is input in qubit 2,

what is the function of the circuit?

A. qubit 1 is migrated
to qubit 2

$\bar{Z}_2$ is identity

		A: CNOT(1->2)	B: CNOT(1->2)	
stabilizer		$Z \otimes Z$	$Z \otimes I$	$\longrightarrow |0\rangle_1$
logical operators	$\bar{X}$	$X \otimes X$	$I \otimes X$	
	$\bar{Z}$	$Z \otimes I$	$Z \otimes Z$	$\xrightarrow{Z \otimes I} I \otimes Z$

Stabilizer group

- stabilizer operators: Pauli operators for which the input state $|\psi\rangle$ is a +1 eigenstate
- stabilizer operators form a group stabilizer group $\mathcal{S}$

$$S_1, S_2 \in \mathcal{S}$$

$$S_1|\psi\rangle = S_2|\psi\rangle = |\psi\rangle$$

$$S_1 S_2|\psi\rangle = S_2 S_1|\psi\rangle = |\psi\rangle$$

→

$$S_1^2 = S_2^2 = I$$

$$[S_1, S_2] = 0$$

S_1 and S_2 are Pauli operators, so they either commute or anticommute

$\mathcal{S}$ is an Abelian subgroup of the Pauli group

Stabilizer group

- stabilizer operators: Pauli operators for which the input state $|\psi\rangle$ is a +1 eigenstate

- stabilizer operators form a group stabilizer group $\mathcal{S}$
  - example

0) $|\psi\rangle = |0\rangle_1$ $\mathcal{S} = \{II, ZI\}$

1) $|\psi\rangle = |+0\rangle$ $\mathcal{S} = \{II, XI, IZ, XZ\}$

2) $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$

$\mathcal{S} = \{II, XX, ZZ, -YY\}$

Stabilizer group

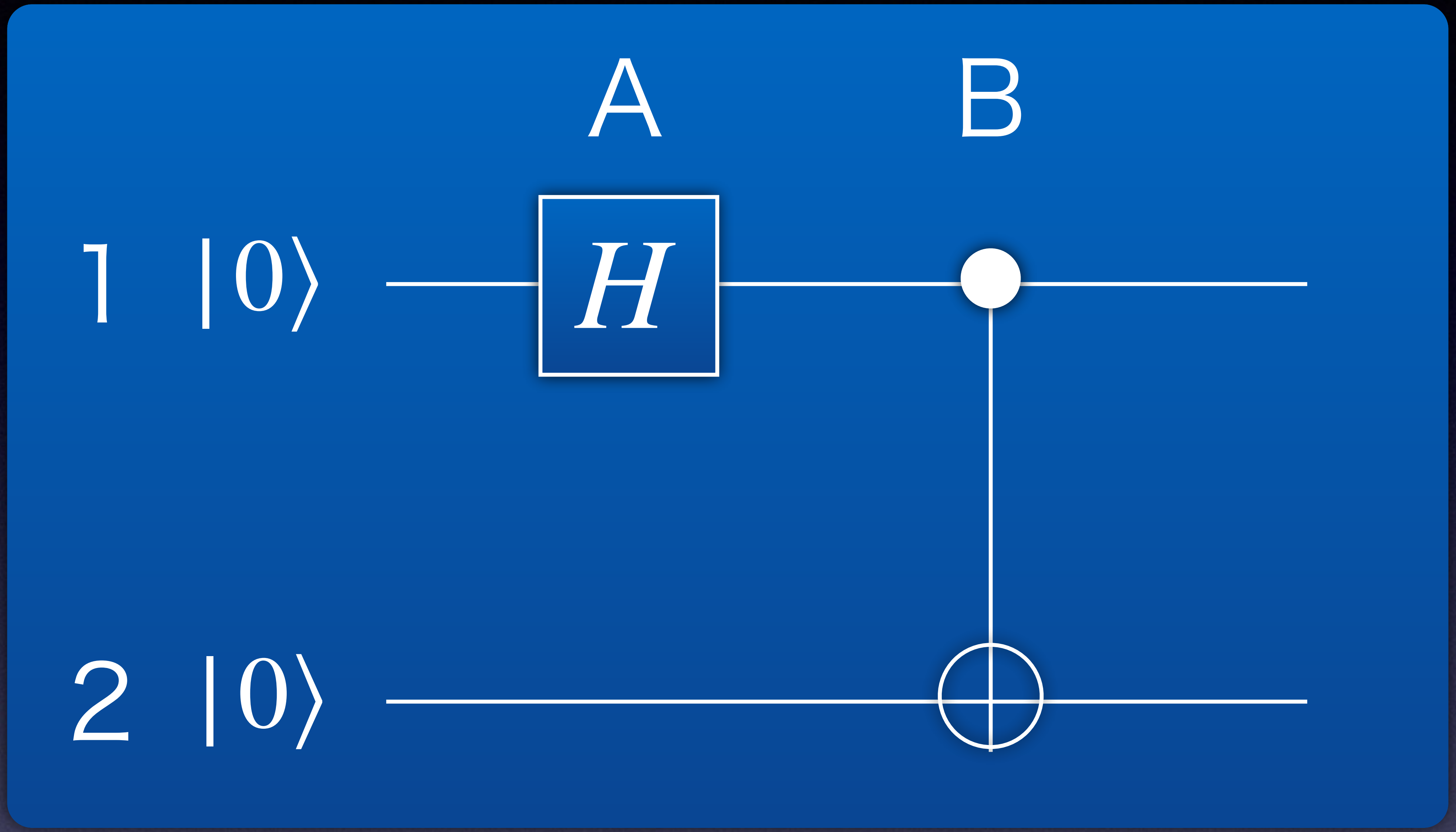
- stabilizer operators: Pauli operators for which the input state $|\psi\rangle$ is a +1 eigenstate
- stabilizer operators form a group stabilizer group $\mathcal{S}$
- stabilizer generators $\mathcal{G}$: maximum independent elements of $\mathcal{S}$

· example

1) $\mathcal{S} = \{II, XI, IZ, XZ\}$ ← $\mathcal{G} = \langle XI, IZ \rangle$

2) $\mathcal{S} = \{II, XX, ZZ, -YY\}$ ← $\mathcal{G} = \langle XX, ZZ \rangle$

Stabilizer state

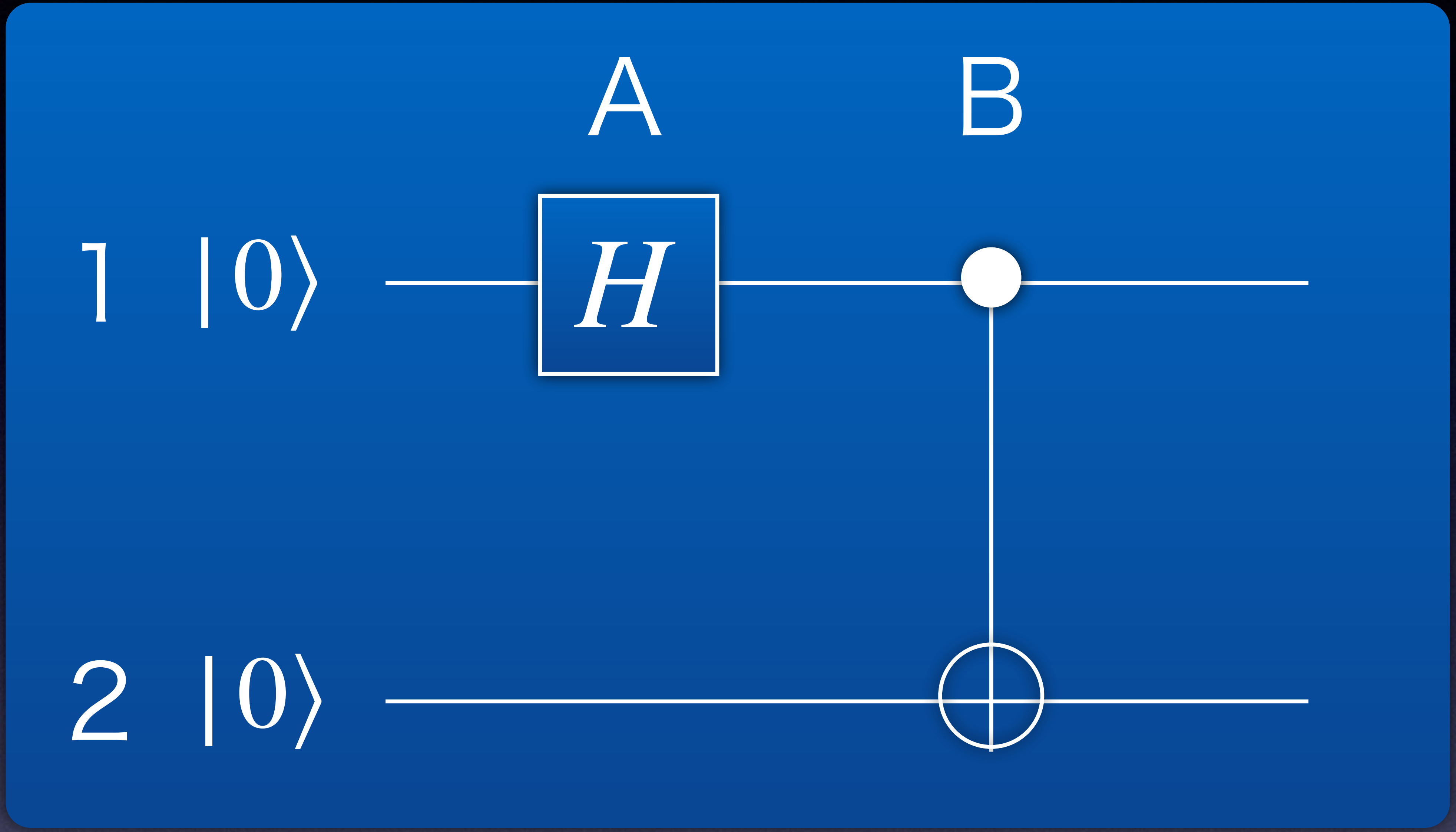


start	A: $H(1)$	B: CNOT(1->2)
$Z \otimes I$	$X \otimes I$	$X \otimes X$
$I \otimes Z$	$I \otimes Z$	$Z \otimes Z$

stabilizers

- the stabilizer operator for the input state of each qubit is a generator of the stabilizer group

Stabilizer state



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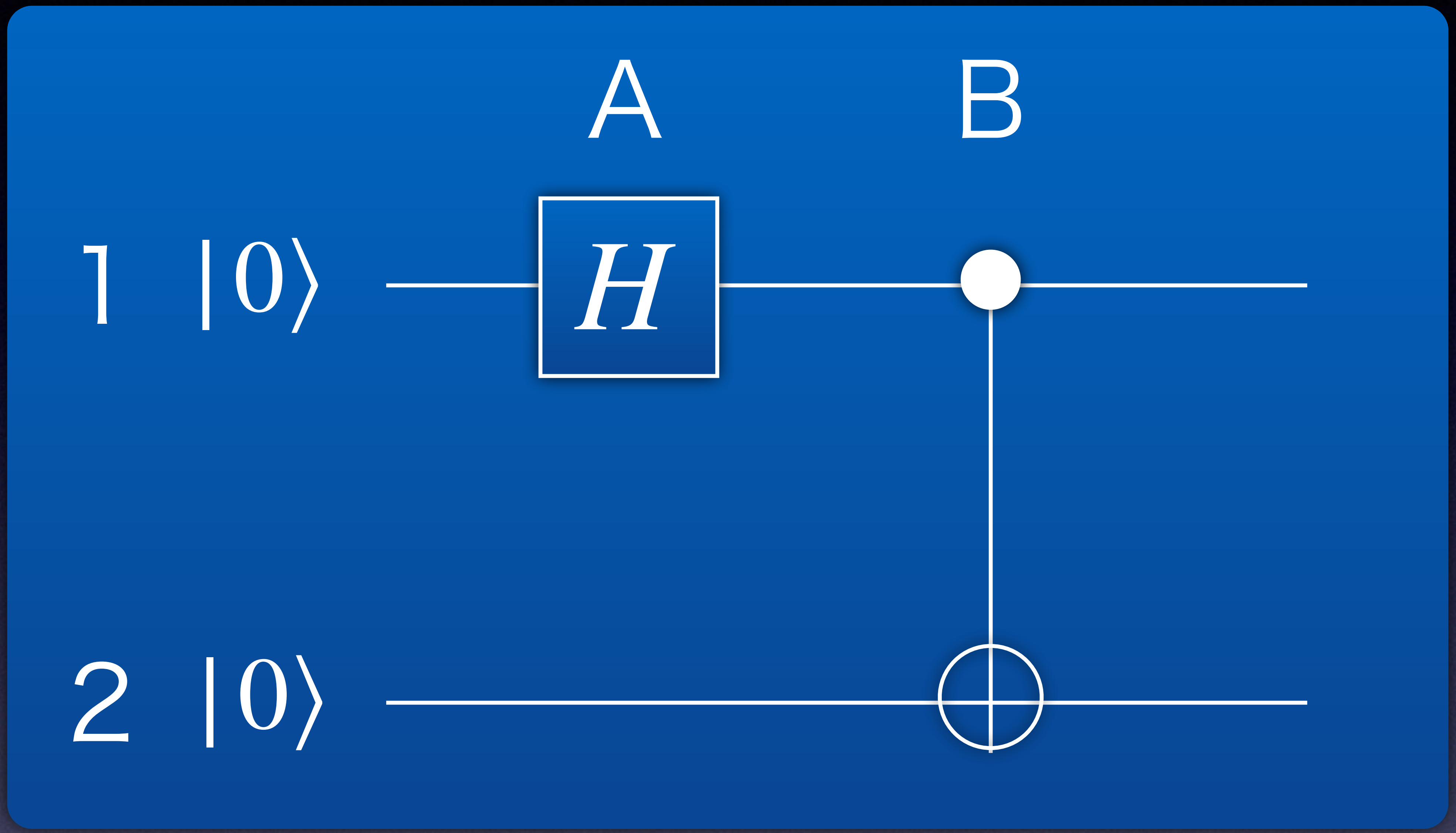
$Z \otimes I$	$X \otimes I$	$X \otimes X$
$I \otimes Z$	$I \otimes Z$	$Z \otimes Z$

stabilizers

- fix k qubits -> k generators -> $\mathcal{S}$ has 2^k elements

$$S_l = \prod_{i=1}^{k} g_i^{l_i} \qquad l_i \in \{0,1\} \qquad l = l_1 l_2 \cdots l_k \in \{0,1\}^k$$

Stabilizer state



start	A: $H(1)$	B: CNOT(1->2)
$Z \otimes I$	$X \otimes I$	$X \otimes X$
$I \otimes Z$	$I \otimes Z$	$Z \otimes Z$

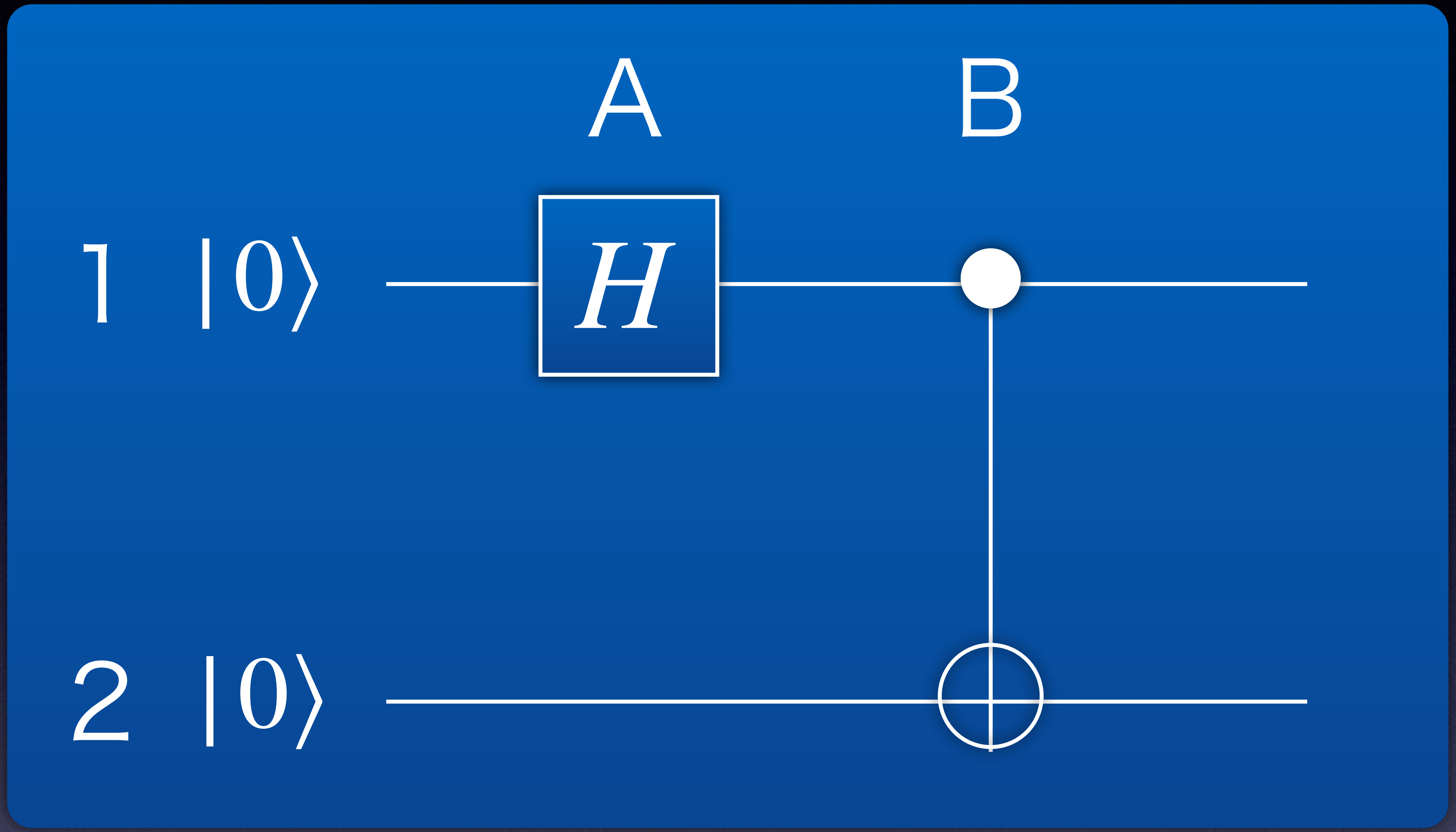
stabilizers

- stabilizer state: a n-qubit state uniquely specified by n generators

the 2^n-dimensional Hilbert space is divided into the subspaces with eigenvalue ± 1 by each of the generators

ZZ \ XX	+1	-1
+1	$(\lvert 00\rangle + \lvert 11\rangle)/\sqrt{2}$	$(\lvert 00\rangle - \lvert 11\rangle)/\sqrt{2}$
-1	$(\lvert 01\rangle + \lvert 10\rangle)/\sqrt{2}$	$(\lvert 01\rangle - \lvert 10\rangle)/\sqrt{2}$

Stabilizer state



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$Z \otimes I$	$X \otimes I$	$X \otimes X$
$I \otimes Z$	$I \otimes Z$	$Z \otimes Z$

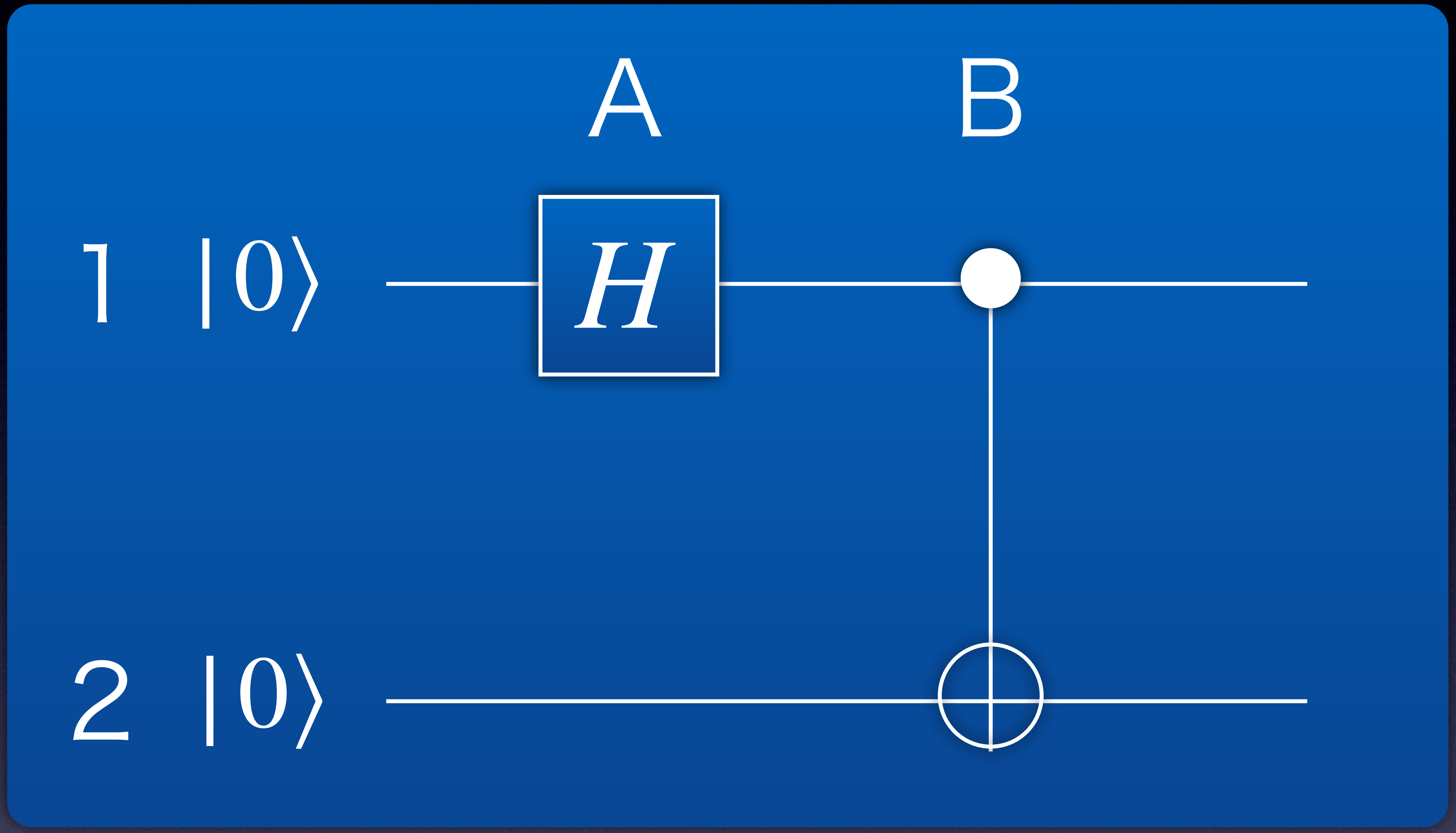
stabilizers

- stabilizer state: a n-qubit state uniquely specified by n generators

$\frac{I_i \pm g_i}{2}$ projection operator for ± 1 eigenvalue

$$|\psi\rangle = \prod_{i=1}^{n} \frac{I_i + g_i}{2} |0\rangle^{\otimes n} = \frac{1}{2^n} \sum_{l=1}^{2^n} S_l |0\rangle^{\otimes n}$$

Stabilizer state



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$Z \otimes I$	$X \otimes I$	$X \otimes X$
$I \otimes Z$	$I \otimes Z$	$Z \otimes Z$

stabilizers

- stabilizer state: a n-qubit state uniquely specified by n generators

Exercise Show $g_1 = X \otimes X$ $g_2 = Z \otimes Z$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{I+g_1}{2}\frac{I+g_2}{2}|00\rangle = (II + XX + ZZ - YY)|00\rangle$$