# BMS symmetry of gravity from Hamiltonian formulation(s)

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# Asymptotic symmetry analysis: the ‘usual’ way

BMS at $\mathcal{I}^+$: Bondi Approach [Bondi-van der Burg-Metzner-Sachs, 1962]

BMS group as the symmetry of gravity at null infinity for asymptotically flat spacetimes

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- Use Bondi coordinates $(u, r, x^A)$, Bondi gauge and impose boundary conditions as $r \to \infty$

$$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2\, e^{2\beta}\, du\, dr + g_{AB}(dx^A - U^A du)\,(dx^B - U^B du)$$

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- Look for diffeomorphisms that preserve the form of the metric

Poincaré in the bulk: $\xi^\mu(x) = \omega^\mu_\nu x^\nu + a^\mu$

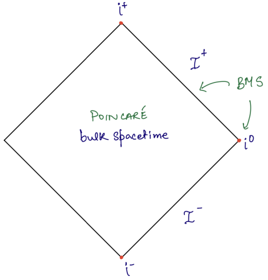
(6-dim Lorentz $\omega^\mu_\nu$ ) + (4 translations $a^\mu$) : 10 dim

Poincaré = Lorentz $\ltimes$ Translations

$\downarrow$

BMS at $\mathcal{I}^+$: $a^\mu$ replaced by a function $\alpha$ : $\infty$-dim

$BMS_4$ = Lorentz $\ltimes$ “Supertranslations”




- Define Noether charges and compute the asymptotic algebra

# All roads lead to BMS?

**Bondi approach:** [since 1960s]

- BMS as the asymptotic symmetry group at null infinity
- Further extensions to superrotations, *Diff*$(\mathbb{S}^2)$, Celestial Holography, $\mathcal{W}_{+\infty}$ algebras
- Links to soft theorems, Ward identities and memory effects, Strominger's IR triangle, ...

[**Bondi-van der Burg-Metzner-Sachs '62**, Barnich-Troessaert, Hawking-Perry-Strominger Compère, Campiglia, Detournay, Donnay, Freidel, Geiller, Grumiller, Laddha, Pasterski, Puhm, Raclariu, Sen, Sheikh-Jabbari, Zwikel and many more]

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**Conformal Carroll approach:** [since 2014]

- BMS group as conformal Carroll group
- Further extensions to other Carrollian structures
- Symmetries of null hypersurfaces, Carrollian field theory, Carrollian fluids

[**Duval-Gibbons-Hovarthy '14**, Campoleoni, Ciambelli, Donnay, Fiorucci, Freidel, Flanagan, Heffray, Leigh, Obers, Petropoulos, Ruzziconi and many more]

**Hamiltonian approach:** [since 2017]

- BMS symmetry at spatial infinity from the ADM Hamiltonian action
- Canonical realization of BMS in the ADM phase space
- Relevant for Initial value problem of GR on Cauchy/ Characteristics hypersurfaces, etc.

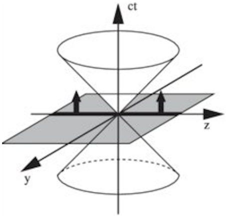
[**Henneaux-Troessaert '17**, Fuentealba, Guilini, SM, Matulich, Neogi, Riello, Tanzi, ...]

# Two lessons from Dirac

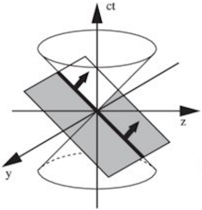
**Lesson I:** Forms of relativistic dynamics [Dirac 1959]

Three choices of "time" for describing Hamiltonian dynamics of relativistic systems

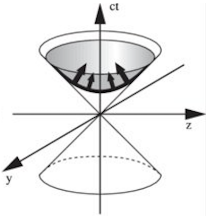
| Instant form: | Front form: : | Point form: |
| --- | --- | --- |
| $t = x^0$ | $x^+ = (x^0 + x^3)/\sqrt{2}$ | $\tau =$ proper time |




Spacelike foliations




Null foliations




Hyperbolic foliations

Poincaré algbera splits into

$\rightarrow$ Kinematical generators $\mathbb{K}$ that are "simple"

$\rightarrow$ Dynamical generators or "Hamiltonians" $\mathbb{D}$ that involve time derivatives

# Two lessons from Dirac

**Lesson II:** Constrained Hamiltonian systems [Bargmann 1959; Dirac 1959]

Gauge theories are constrained Hamiltonian systems

$$\mathcal{S}_H[\phi, \pi_\phi, \lambda_i] = \int dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H} - \lambda_i \mathcal{G}^i\right)$$

$\mathcal{G}^i \rightarrow$ gauge cosntraints, $\lambda_i \rightarrow$ Largrange multipliers

- algorithm for classifying gauge constraints (primary, first-class, ...)
- symmetries generated by first-class constraints

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Many successes: BRST quantization, Duality-invariant actions, *Asymptotic symmetries at* $i^0$, ...

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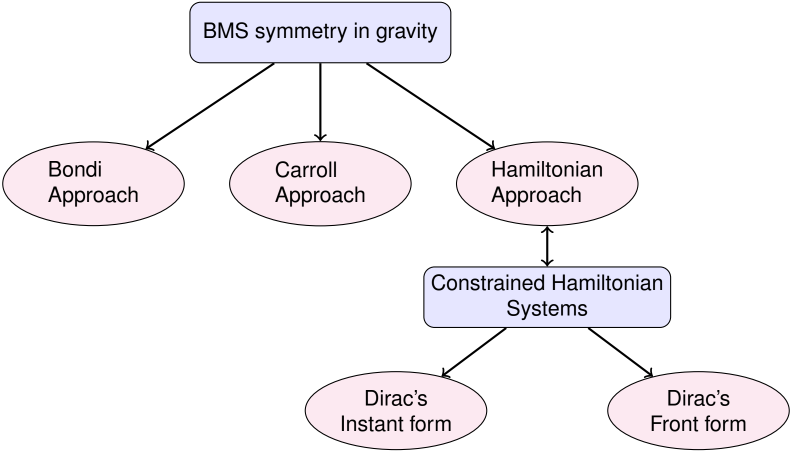
**The usual route:** Instant form + Constrained Hamiltonian systems

Many successes: BRST quantization, Duality-invariant actions, *Asymptotic symmetries at* $i^0$, . . .

**An alternative route:** Front form + Constrained Hamiltonian systems

- Gauge constraint in the front form are often solvable
- Provides a Hamiltonian framework for symmetries of null hypersurfaces
- Many successes: Discrete light-cone quantization (DLCQ), Light-cone quantization of strings, UV finiteness of $\mathcal{N} = 4$ SYM, Higher-spin cubic action, etc.

# BMS symmetry at a glance




Focus of this talk:

BMS-like symmetries (infinite-dimensional extension of Poincaré) using Hamiltonian methods

# BMS in Hamiltonian formulations: (3+1) and (2+2)[1]

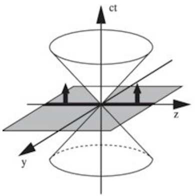
## Part 1: Instant form

(3+1): Hamiltonian dynamics on a spatial hypersurface

$\rightarrow$ BMS symmetry from ADM action

Work done with Oscar **Feuntealba**, Marc **Henneaux**, Javier **Matulich** and Cedric **Troessaert**

[ArXiv:1904.04495 and ArXiv:2007.12721]



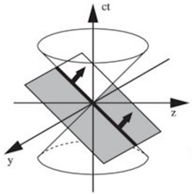

## Part 2: Front form

(2+2): Hamiltonian dynamics on a null hypersurface

$\rightarrow$ BMS symmetry from light-cone action

Work done with Sudarshan **Ananth** and Lars **Brink**

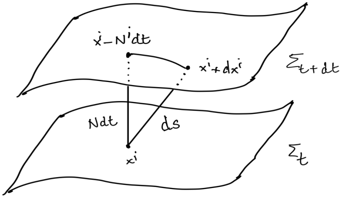
[ArXiv:2012.07880 and ArXiv:2101.00019]




[1] split of four dimensional spacetime into 2 null + 2 transverse spatial coordinates

# Hamiltonian formulation of GR à la Dirac and ADM

- 3+1 foliation of spacetime by a family of spacelike surfaces $\Sigma_t$
- ADM decomposition: $\quad {}^{(4)}g_{00} = -N^2 + N^i N_i\,, \quad {}^{(4)}g_{0i} = N_i\,, \quad {}^{(4)}g_{ij} = g_{ij}$




Dynamical variables:

$g_{ij}$ = metric on $\Sigma_t$

$\pi^{ij}$ = conjugate momenta

ADM action for gravity [Dirac '58, Arnowitt-Deser-Misner '62]

$$\mathcal{S}_{ADM}[g_{ij}, \pi^{ij}, N, N^i] \;=\; \int dt \{ \int d^3x (\pi^{ij}\dot{g}_{ij} - N\mathcal{H} - N^i\mathcal{H}_i) - \oint B_\infty \}$$

Boundary terms $B_\infty$ ensure a good variational principle [Regge-Teitelboim '74]

# Hamiltonian formulation of GR à la Dirac and ADM

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- Lagrange multipliers, $N$ and $N^i$ implement the constraints

$$\mathcal{H} \;=\; -\sqrt{g}R + \frac{1}{\sqrt{g}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2)\,, \quad \mathcal{H}_i \;=\; -2\,\nabla_j \pi^j_i$$

Constraints generate gauge symmetries

- Symplectic form

$$\Omega \;=\; \int d^3x\; d_V \pi^{ij} \wedge d_V g_{ij}\;, \quad d_V \equiv \text{exterior derivative in field space}$$

Phase space : $\{g_{ij}, \pi^{ij}\}$

Poisson bracket: $\{g_{ij}(x), \pi^{kl}(x')\} = \delta^{(k}_{(i}\delta^{l)}_{j)}\delta^{(3)}(x - x')$

# Symmetries of the ADM action

Symmetries $\equiv$ *Strict* invariance of the symplectic form

$$\Omega \;=\; \int d^3x \; d_V \pi^{ij} \wedge d_V g_{ij} \;,$$

$\xi$ generates a canonical transformation if

$$\boxed{\mathcal{L}_\xi \Omega = \; d_V(\iota_\xi \Omega) = 0 \quad \Rightarrow \quad \iota_\xi \Omega = -d_V G_\xi}$$

$G_\xi$ is the generator associated with this canonical transformation.

- Diffeomorphisms:

$$\begin{aligned}
\delta_\xi g_{ij} \quad &= \quad \frac{2\xi}{\sqrt{g}} \left( \pi_{ij} - \frac{1}{2} g_{ij} \pi \right) + \mathcal{L}_\xi g_{ij} \;, \\
\delta_\xi \pi^{ij} \quad &= \quad -\xi \sqrt{g} \left( R^{ij} - \frac{1}{2} g^{ij} R \right) + \frac{1}{2} \xi \sqrt{g} \left( \pi_{mn} \pi^{mn} - \frac{1}{2} \pi^2 \right) \\
&\quad -2\xi \sqrt{g} \left( \pi^{im} \pi^j_m - \frac{1}{2} \pi^{ij} \pi \right) + \sqrt{g} \left( \xi^{|ij} - g^{ij} \xi^{|m}{}_{|m} \right) + \mathcal{L}_\xi \pi^{ij}
\end{aligned}$$

- Canonical generator for *all* symmetries

$$G_{\xi,\xi^i} \;=\; \int d^3x \left( \xi \mathcal{H} + \xi^i \mathcal{H}_i \right) + Q_{\xi,\xi^i} \;,$$

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Proper gauge transformations do not affect the physical states

b) True symmetry: $Q_{\xi,\xi^i} \neq 0$

Improper gauge transformations affect the physical states

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E.g. Poincaré symmetry

$$\xi = b_i x^i + a^0,$$
$$\xi^i = \omega^i{}_j x^j + a^i$$

$b^i$ boosts, $\omega^i_j$ rotations, $a^0$ time translation, $a^i$ spatial translations

$Q_{\text{Poincaré}} \neq 0$ but no $\infty$-dimensional BMS at spatial infinity [Regge-Teitelboim' 74]

How to recover the BMS group at spatial infinity?

# How to ‘see’ BMS symmetry in the ADM formulation?

Hamiltonian action with standard boundary conditions

$\downarrow$

Carefully relax the boundary conditions

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Ensure finiteness of the ADM action and the symplectic form

$\downarrow$

Check that all Poincaré charges are *still* canonical

$\downarrow$

Define canonical generators and compute the asymptotic symmetry algebra

# Asymptotic conditions I

First ingredient: fall-off conditions

We use spherical coordinates $(r, x^A)$ where $x^A$ are coordinates on the sphere at $i^0$

- Asymptotically flat spacetimes: metric approaches Minkowski as $r \to \infty$

$$
\begin{aligned}
g_{rr} &= 1 + \frac{1}{r}\bar{h}_{rr} + \mathcal{O}(r^{-2}) \,, \\
g_{rA} &= \bar{\lambda}_A + \frac{1}{r} h^{(2)}_{rA} + \mathcal{O}(r^{-2}) \,, \\
g_{AB} &= r^2 \bar{g}_{AB} + r\bar{h}_{AB} + h^{(2)}_{AB} + \mathcal{O}(r^{-1}) \,.
\end{aligned}
$$

Barred quantities (e.g., $\bar{h}_{ij}$, $\bar{\pi}^{ij}$) are functions on the 2-sphere

- Conjugate momenta

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**This part is the same as that of Regge-Teitelboim**

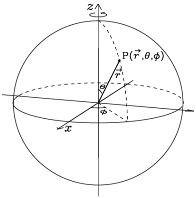
# Asymptotic conditions II

Second ingredient: parity conditions on leading terms

"Gauge-twisted" parity conditions: [Henneaux-Troessaert '18]

Parity: $(r, \theta, \phi) \to (r, \pi - \theta, \phi + 2\pi)$

$$\begin{aligned}
\overline{h}_{rr} &= \text{even}\,, \\
\overline{\lambda}_A &= (\overline{\lambda}_A)^{\text{odd}} + \overline{D}_A \zeta_r - \overline{\zeta}_A\,, \\
\overline{h}_{AB} &= (\overline{h}_{AB})^{\text{even}} + \overline{D}_A \overline{\zeta}_B + \overline{D}_B \overline{\zeta}_A + 2\overline{g}_{AB}\zeta_r \\
\overline{\pi}^{rr} &= (\overline{\pi}^{rr})^{\text{odd}} - \sqrt{\overline{g}}\,\overline{\triangle} V\,, \\
\overline{\pi}^{rA} &= (\overline{\pi}^{rA})^{\text{even}} - \sqrt{\overline{g}}\,\overline{D}^A V\,, \\
\overline{\pi}^{AB} &= (\overline{\pi}^{AB})^{\text{odd}} + \sqrt{\overline{g}}(\overline{D}^A \overline{D}^B V - \overline{g}^{AB}\overline{\triangle} V)\,,
\end{aligned}$$




Recall:

$$\oint (\text{odd function}) = 0 \quad \text{on the sphere}$$

With these parity conditions, Hamiltonian action and symplectic form are finite as $r \to \infty$

**Generalization of Regge-Teitelboim strict parity conditions**

# Asymptotic conditions III

- Third ingredient: stronger fall-off of the constraints

$$\mathcal{H} \sim \mathcal{O}(r^{-3}), \qquad \mathcal{H}_i \sim \mathcal{O}(r^{-3})$$

  to remove divergent conributions to Poincaré charges

- Fourth ingredient: Involves the mixed radial-angular component, $\bar{h}_{rA} \rightarrow$ more on this later

- Fall-off of the Poincaré $(\xi, \xi^i)$:

$$\begin{aligned} \xi &= br + a^0, \\ \xi^r &= w_1 , \\ \xi^A &= Y^A + \frac{1}{r}\bar{D}^A w_1 , \end{aligned}$$

$Y^A$ rotations, $b$ boosts, $a^0$ time translations, $w_1$ spatial translations

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Next step: Check if Poincaré generators are still canonical

# Canonical realization of Poincare generators

*Strict* invariance of the symplectic form

$$\Omega \;=\; \int d^3x \; d_V\pi^{ij} \wedge d_V g_{ij} \;,$$

$\xi$ generates a canonical transformation if

$$\boxed{\mathcal{L}_\xi \Omega = \; d_V(\iota_\xi \Omega) = 0 \quad \Rightarrow \quad \iota_\xi \Omega = -d_V G_\xi}$$

$G_\xi$ is the generator associated with this canonical transformation.

- Under Lorentz rotations $Y^A$ and spacetime translations $(a^0, a^i)$,

$$\mathcal{L}_{(Y^A, a^0, a^i)} \Omega = 0 \quad \Rightarrow \quad \text{Canonical generators well-defined}$$

- Under Lorentz boosts $b$ (in spherical coordinates)

$$d_V\left(\iota_b \Omega\right) \;=\; -\int d\theta d\varphi \sqrt{\bar{g}} \Big[ b\, d_V \bar{h} d_V \left( \bar{h}_{rr} + \bar{D}_A \bar{\lambda}^A \right) - \bar{D}_A b\, d_V \bar{\lambda}^A d_V \bar{h} + b \bar{D}^A d_V \bar{\lambda}^B d_V \bar{h}_{AB} \Big) \Big] \neq 0$$

How to make the symplectic form invariant under boosts?

## Non-integrability of the boost generators: Resolution

- Perform a gauge transformation

$$\epsilon_{(b)} \equiv bF, \quad \text{F is field-dependent}$$

$$d_V\left(\iota_b\Omega\right) + d_V\left(\iota_{\epsilon(b)}\Omega\right) = -\int d\theta d\varphi\sqrt{\bar{g}}\Big[2b\left(d_V F + \frac{1}{2}d_V\bar{h}\right)d_V\left(\bar{h}_{rr} + \bar{D}_A\bar{\lambda}^A\right) - \bar{D}_A b\, d_V\bar{\lambda}^A d_V\bar{h} + b\bar{D}^A d_V\bar{\lambda}^B d_V\bar{h}_{AB}\Big]$$

$$\boxed{\text{Set } F = -\frac{1}{2}\bar{h}}$$

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$$\boxed{\text{Set } F = -\frac{1}{2}\bar{h}}$$

- Fourth ingredient of asymptotic conditions (Recall: $h_{rA} = \bar{\lambda}_A + \mathcal{O}(r^{-1})$)

$$\int d\theta d\varphi\sqrt{\bar{g}}\left[\bar{D}_A b\, d_V\bar{\lambda}^A d_V\bar{h} - b\bar{D}^A d_V\bar{\lambda}^B d_V\bar{h}_{AB}\right]$$

$$\boxed{\text{Set } \bar{\lambda}_A = 0}$$

$$d_V(\iota_b\Omega) = 0 \quad \Rightarrow \quad \iota_b\Omega = -d_V G_b \quad \rightarrow \quad \text{Boosts are canonical again!}$$

No need for *an extra boundary field* in order to define canonical generators: more on this later

# Finally, the new boundary conditions read

$$
\begin{aligned}
g_{rr} &= 1 + \frac{1}{r}\bar{h}_{rr} + \dots\,, \\
g_{rA} &= \cancel{\bar{\lambda}_A} + \frac{1}{r} h^{(2)}_{rA} + \dots\,, \\
g_{AB} &= r^2 \bar{g}_{AB} + r\bar{h}_{AB} + h^{(2)}_{AB} + \dots\,.
\end{aligned}
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\end{aligned}
$$

With gauge-twisted parity conditions

$$
\begin{aligned}
\bar{h}_{rr} &= \text{even}\,, \\
\bar{\lambda}_A &= (\bar{\lambda}_A)^{\text{odd}} + \bar{D}_A \zeta_r - \bar{\zeta}_A = 0 \qquad \Rightarrow \quad \bar{\zeta}_A = \bar{D}_A \zeta_r = \bar{D}_A U\,, \\
\bar{h}_{AB} &= (\bar{h}_{AB})^{\text{even}} + 2(\bar{D}_A \bar{D}_B U + \bar{g}_{AB} U)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\pi}^{rr} &= (\bar{\pi}^{rr})^{\text{odd}} - \sqrt{\bar{g}}\,\bar{\triangle} V\,, \\
\bar{\pi}^{rA} &= (\bar{\pi}^{rA})^{\text{even}} - \sqrt{\bar{g}}\,\bar{D}^A V\,, \\
\bar{\pi}^{AB} &= (\bar{\pi}^{AB})^{\text{odd}} + \sqrt{\bar{g}}(\bar{D}^A \bar{D}^B V - \bar{g}^{AB}\bar{\triangle} V)\,,
\end{aligned}
$$

Regge-Teitelboim parity conditions relaxed with two functions: $U$ odd and $V$ even

# Are there more symmetries?

Yes, diffeos $x^\mu \to x^\mu + \epsilon^\mu(x)$ with parameters

$$\epsilon^0(\theta,\phi) = T^{\text{even}}\,, \quad \epsilon^i(\theta,\phi) = \partial_i W^{\text{odd}} \quad \to \quad \text{one single arbitrary function } \boxed{\mathcal{T}(\theta,\phi)}$$

- Time component of gauge parameter

$$\epsilon^0 = T^{\text{even}} = T_0 + T_2 + T_4 + T_6 + \cdots$$

- Spatial components

$$\epsilon^i = \bar{\epsilon}^i + \mathcal{O}(r^{-1})\,, \quad \bar{\epsilon}^i = D^i(rW)\,,$$

$$W^r = W = W_1 + W_3 + W_5 + W_7 + \cdots\,, \qquad W^A = \frac{1}{r}\overline{D}^A W\,,$$

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- Where are the spacetime translations? Expand $T(\theta,\phi)$ in spherical harmonics

$$\mathcal{T}(\theta,\phi) \;=\; T_{0,0}Y_{0,0} \;+\; \sum_{m=-1}^{1} W_{1,m}Y_{1,m} \;+\; \underbrace{\frac{1}{4}\sum_{m=-2}^{2} T_{2,m}Y_{2,m} \;+\; \cdots}_{\text{supertranslations}}$$

time translations $a^0$ — spatial translations $a^i$ — supertranslations

[Henneaux-Troessaert 2018; Henneaux-Fuentealba-SM-Matulich-Troessaert 2020]

# Asymptotic symmetries at spatial infinity

- Canonical generator for BMS

$$G_{\xi,\xi^i} = \int d^3x \left(\xi\mathcal{H} + \xi^i\mathcal{H}_i\right) + Q_{\xi,\xi^i}\,,$$

$$Q_{\xi,\xi^i} = \int d\theta d\varphi \Big\{ b\left[\sqrt{\bar{g}}(-\frac{1}{2}\bar{h}\bar{h}_{rr} + \frac{1}{4}\bar{h}^2 - \frac{3}{4}\bar{h}_{AB}\bar{h}^{AB}) + \frac{2}{\sqrt{\bar{g}}}\bar{\pi}^r_A\bar{\pi}^{rA}\right] + 2Y_A\bar{\pi}^{rB}\bar{h}^A_B$$
$$+2\sqrt{\bar{g}}T\underbrace{\bar{h}_{rr}}_{\text{even}} + 2W\underbrace{(\bar{\pi}^{rr} - \bar{\pi}^A_A)}_{\text{odd}}\Big\}$$

$(T_{odd}, W_{even})$ $\rightarrow$ proper gauge transformations

$(T_{even}, W_{odd})$ $\rightarrow$ improper gauge transformations : Supertranslations

- Poisson bracket algebra

$$\left\{G_{\xi_1,\xi_1^i}, G_{\xi_2,\xi_2^i}\right\} = \hat{G}_{\hat{\xi},\hat{\xi}^i}\,,$$

**Asymptotic symmetry algebra of gravity at spatial infinity**

$$BMS_4 = SO(3,1) \ltimes \text{supertranslations}$$

# Asymptotic symmetry algebra

- Poisson bracket algebra

$$\left\{ G_{\xi_1,\xi_1^i}, G_{\xi_2,\xi_2^i} \right\} = \hat{G}_{\hat{\xi},\hat{\xi}^i} ,$$

with the parameters

$$\begin{aligned}
\hat{Y}^A &= Y_1^B \partial_B Y_2^A + \overline{\gamma}^{AB} b_1 \partial_B b_2 - (1 \leftrightarrow 2) , \\
\hat{b} &= Y_1^B \partial_B b_2 - (1 \leftrightarrow 2) , \\
\hat{T} &= Y_1^A \partial_A T_2 - 3 b_1 W_2 - \partial_A b_1 \overline{D}^A - 2W - b_1 \overline{D}_A \overline{D}^A W_2 - (1 \leftrightarrow 2) , \\
\hat{W} &= Y_1^A \partial_A W_2 - b_1 T_2 - (1 \leftrightarrow 2)
\end{aligned}$$

- BMS as the infinite-dimensional enhancement of Poincaré, $G_{\xi,\xi^i} = G_{Lorentz} + G_{T,W}$

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- BMS as the infinite-dimensional enhancement of Poincaré, $G_{\xi,\xi^i} = G_{Lorentz} + G_{T,W}$

$$\left\{G_{Lorentz}, G_{Lorentz}\right\} = G_{Lorentz}$$

$$\left\{G_{Lorentz}, G_{a,a^i}\right\} = \hat{G}_{(\hat{a},\hat{a}^i)}$$

$$\left\{G_{a,a^i}, G_{a,a^i}\right\} = 0$$

$$G_{a,a^i} \to G_{T,W}$$

$$\left\{G_{Lorentz}, G_{Lorentz}\right\} = G_{Lorentz}$$

$$\left\{G_{Lorentz}, G_{T,W}\right\} = \hat{G}_{\hat{T},\hat{W}}$$

$$\left\{G_{T,W}, G_{T,W}\right\} = 0$$

[Henneaux-Troessaert 2018; Henneaux-Fuentealba-SM-Matulich-Troessaert 2020]

# BMS in Hamiltonian formulations: (3+1) and (2+2)

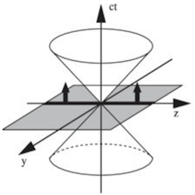
## Part 1: Instant form

(3+1): Hamiltonian dynamics on a spatial hypersurface

→ BMS symmetry from ADM action

Work done with Oscar **Feuntealba**, Marc **Henneaux**, Javier **Matulich** and Cedric **Troessaert**

[ArXiv:1904.04495 and ArXiv:2007.12721]



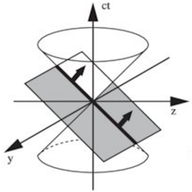

## Part 2: Front form

(2+2): Hamiltonian dynamics on a null hypersurface

→ BMS symmetry from light-cone action

Work done with Sudarshan **Ananth** and Lars **Brink**

[ArXiv:2012.07880 and ArXiv:2101.00019]



# Poincaré algebra in Dirac's front form

- Light-cone coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}\,, \quad x^- = \frac{x^0 - x^3}{\sqrt{2}}\,, \quad x^i \quad (i = 1, 2)$$

$x^+$ Light-cone time $\Rightarrow$ $P_+ = i\partial_+ = -P^-$ Hamiltonian

- The three "Hamiltonians" in the front form

Poincaré generators in the instant form: $(P_\mu, M_{\mu\nu})$

$$[P, P] \sim 0\,, \quad [P, M] \sim P\,, \quad [M, M] \sim M$$

$(P_0, M_{0i}) \rightarrow$ four dynamical generators or "Hamiltonians"

Poincaré generators in front form

Kinematical $\mathbb{K} = \{P^i, P^+, M^{ij}, M^{+i}, M^{+-}\}\,, \quad (i = 1, 2)$

Dynamical $\mathbb{D} = \{P^-, M^{i-} \equiv \underbrace{J^-, \bar{J}^-}_{\text{2 boosts}}\} \rightarrow$ three "Hamiltonians" in the front form

$$[\,\mathbb{K}, \mathbb{K}\,] = \mathbb{K}\,, \quad [\,\mathbb{K}, \mathbb{D}\,] = \mathbb{D}\,, \quad [\,\mathbb{D}, \mathbb{D}\,] = 0$$

Poincaré algebra in front form has a Carrollian structure – isometry of null hypersurfaces

# Null-front Hamiltonian formulation of gravity

*"Forms of relativistic dynamics"* [Dirac '49] $\rightarrow$ Use a null time parameter to study dynamics

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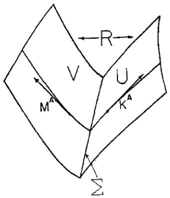
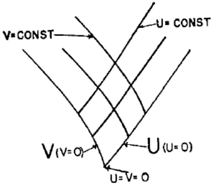
*"Forms of relativistic dynamics"* [Dirac '49] → Use a null time parameter to study dynamics

*"On the characteristic initial value problem in gravitational theory"* [R. K. Sachs '62]

*"Covariant 2+2 formulation of the initial-value problem in general relativity"*
[d'Inverno and Smallwood '79] [Gambini-Restuccia, C. Torre, M. Kaku,...]




- Spacelike foliation of codim 2 (instead of 1)
- Unconstrained Hamiltonian systems: constraint equations often become solvable
- Gravitational d.o.f. identified with the "conformal two-metric"

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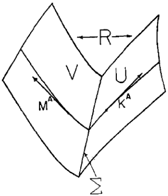


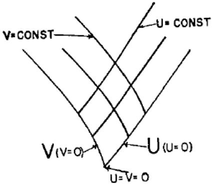


- Spacelike foliation of codim 2 (instead of 1)
- Unconstrained Hamiltonian systems: constraint equations often become solvable
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Our focus:

- Set up a particular example of the 2+2 formulation: $lc_2$ gravity [Scherk-Schwarz' 75]
- Study the BMS symmetry from residual gauge invariance

# Gravity in the light-cone gauge

**"$lc_2$ formalism"** [Scherk-Schwarz, Schwarz-Goroff, Bengtsson-Cederwall-Lindgren]

- Light-cone gauge: Set the "minus" components to zero

$$g_{--} = g_{-i} = 0, \quad (i = 1, 2)$$

$$\boxed{10 - 3 = 7}$$

Parametrization

$$g_{+-} = -e^{\phi}, \quad g_{ij} = e^{\psi}\gamma_{ij}$$

$\phi$,$\psi$,$\gamma_{ij}$ are real and det $\gamma_{ij} = 1$

Light-cone metric

$$dS^2_{LC} = g_{\mu\nu}dx^{\mu}dx^{\nu} = -2e^{\phi}dx^{+}dx^{+} + g_{++}(dx^{+})^2 + g_{+i}dx^{+}dx^{i} + e^{\psi}\gamma_{ij}dx^{i}dx^{j}$$

given in terms of 7 functions $\{\phi, \psi, \gamma_{ij}, g_{++}, g_{+i}\}$

- "2+2" split of the Einstein field equations $R_{\mu\nu} = 0$ [Sachs, d'Inverno-Smallwood, ...]

  Dynamical equations: $R_{ij} = 0$

  Constraint equations: $R_{--} = R_{-i} = 0$

  Subsidiary equations: $R_{++} = R_{+i} = 0$

  Trivial equations: $R_{+-} = 0$

# Gravity in the light-cone gauge

Can we solve the constraint equations? Subject to choice of coordinates, gauge conditions, etc.

- Constraint equation $R_{--} \;=\; 0$

$$2\,\partial_-\phi\,\partial_-\psi \;-\; (\partial_-\psi)^2 \;-\; 2\partial_-{}^2\psi \;+\; \frac{1}{2}\,\partial_-\gamma^{ij}\,\partial_-\gamma_{ij} \;=\; 0\,.$$

*Fourth* gauge choice : [Scherk-Schwarz]

$$\phi = \frac{\psi}{2} \qquad \boxed{7-1=6}$$

allows us to integrate [2] out $\psi$

$$\psi \;=\; \frac{1}{4}\,\frac{1}{\partial_-{}^2}(\partial_-\gamma^{ij}\,\partial_-\gamma^{ij}) \qquad \boxed{6-1=5}$$

- The constraint $R_{-i} = 0$ eliminates $g_{+i}$ $\boxed{5-2=3}$
- $R_{-+} \;=\; 0$ allows us to eliminates $g_{++}$ $\boxed{3-1=2}$

---

[2] Inverse derivative defined as

$$f(x^-) = \frac{1}{\partial_-}g(x^-) = -\int \epsilon(x^- - y^-)\,g(y^-)\,dy^- + \text{“}\,constant\,\text{”}$$

Integration constants set to zero for asymptotically Minkowski spacetimes

# Light-cone action for gravity

- Closed form expression

$$S[\gamma_{ij}] = \frac{1}{2\kappa^2}\int d^4x\, e^{\psi}\left(2\,\partial_+\partial_-\phi + \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) - \frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\,,$$
$$-e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)$$

where

$$R_i \equiv e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})$$

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$$-e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)$$

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- Perturbative expansion

$$\gamma_{ij} \;=\; (e^{\kappa H})_{ij}\;,\quad H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix}$$

Complexify the fields (and $x^i$)

$$h \;=\; \frac{1}{\sqrt{2}}\,(h_{11} + i\,h_{12})\;,\quad \bar{h} \;=\; \frac{1}{\sqrt{2}}\,(h_{11} - i\,h_{12})$$

$h$ and $\bar{h}$ have helicity +2 and -2 respectively $\rightarrow$ gravitational d.o.f. identified with $\gamma_{ij}$

- Light-cone Lagrangian (perturbative)

$$\mathcal{L} = \frac{1}{2}\bar{h}\,\Box\,h \;+\; 2\kappa\,\bar{h}\,\partial_-{}^2\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right) \;+ c.c. \;+\; \text{higher order terms}$$

# Light-cone Hamiltonian for gravity

- Conjugate momenta (recall: $x^+$ is time)

$$\mathcal{L} = \bar{h}\,(\partial_-\partial_+ - \partial\,\bar{\partial})\,h \;+\; 2\kappa\,\bar{h}\,\partial_-{}^2\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right) \;+\cdots$$

$$\pi \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ h)} = -\,\partial_-\,\bar{h}\,, \quad \bar{\pi} \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+\bar{h})} = -\,\partial_-\,h$$

$(\pi, \bar{\pi})$ are primary constraints $\Rightarrow$ Half the d.o.f than in the 3+1 formalism

$\rightarrow$ a feature of *all* null-front Hamiltonian systems

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- Light-cone Hamiltonian for gravity

$$\mathcal{H} \;\;=\;\; \partial\bar{h}\,\bar{\partial}h \;+\; 2\,\kappa\,\partial_-^2\,\bar{h}\left(h\,\frac{\bar{\partial}^2}{\partial_-^2}h \;-\; \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right) + c.c. \;+\; \mathcal{O}(\kappa^2)$$

- Poisson brackets

$$[h(x), \pi(y)] \;=\; \delta(x^- - y^-)\delta^{(2)}(x-y) \;\Rightarrow\; [h(x), \bar{h}(y)] \;=\; \epsilon(x^- - y^-)\,\delta^{(2)}(x-y)\,,$$

$$[\,h(x), h(y)\,] \;=\; [\,\bar{h}(x), \bar{h}(y)\,] \;=\; 0\,.$$

[Scherk-Schwarz' 75, Bengtsson-Cederwall-Lindgren '83]

# Symmetries of light-cone gravity

## Notion of symmetry

A canonical transformation $(h, \bar{h}) \xrightarrow{\delta_X} (\tilde{h}, \tilde{\bar{h}})$

which leaves the action invariant

$$\delta_X \mathcal{S}[h, \bar{h}] = 0$$

Transformation laws = P.B. with the generator $G_X[h, \bar{h}]$, e.g.

$$\delta_X h = \{\, G_x, h \,\}_{PB}$$

For instance,

Poincaré generators in terms of the fields $h$ and $\bar{h}$ [Bengtsson-Bengtsson-Brink, 1983]

$$H = P_+ = \int d^3x\, \mathcal{H}(h, \bar{h})\,, \quad P = \int d^3x \partial_- \bar{h}\, \partial h\,, \quad P_- = d^3x \partial_- \bar{h} \partial_- h\,, \quad \cdots$$

$$J = i \int d^3x \partial_- \bar{h}\, (x\bar{\partial} - \bar{x}\partial - 2) h\,,$$

$$J^- = \int d^3x [x \mathcal{H}(h, \bar{h}) + \partial_- \bar{h}\, (x^- \partial - 2 \frac{\partial}{\partial_-}) h + \mathcal{S}]\,, \quad \cdots$$

→ canonical realization of Poincarè algebra in light-cone gravity

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→ canonical realization of Poincarè algebra in light-cone gravity

Is there any residual reparameterization freedom left?

# BMS symmetry from residual gauge invariance

- First gauge condition $g_{--} = 0$

$$\Rightarrow \quad \partial_- \xi^+ = 0 \quad \Rightarrow \quad \xi^+ = f(x^+, x^j)$$

  Second gauge condition $g_{-i} = 0$ yields

$$\partial_- \xi^j \, g_{ij} \; + \; \partial_i \xi^+ \, g_{+-} = 0$$

  Fourth gauge condition fixes $x^+$ dependence of $f(x^+, x^j)$, etc.

- Residual diffeomorphisms (expressed in $x, \bar{x}$ basis)

$$\begin{aligned}
\xi^+ &= f(x^+, x, \bar{x}) \;=\; \frac{1}{2} x^+ (\partial \overline{Y} + \bar{\partial} Y) + T(x, \bar{x}) \\
\xi &= -\partial f \, x^- + \kappa \, \bar{\partial} f \frac{1}{\partial_-} h + Y(x, \bar{x}) + \mathcal{O}(\kappa^2) \,, \quad \bar{\xi} = (\xi)^* \\
\xi^- &= -(\partial \overline{Y} + \bar{\partial} Y) x^- + (\partial_+ \xi) x + (\partial_+ \bar{\xi}) \bar{x}
\end{aligned}$$

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- Is this a symmetry of the light-cone action? Yes,

$$\delta_\xi \mathcal{S}[\, h, \bar{h}\,] = 0 \quad \textbf{iff} \quad \partial^2 Y = 0 = \bar{\partial}^2 \overline{Y}$$

$Y, \overline{Y}$ at most linear in $x, \bar{x}$ $\quad \rightarrow \quad$ only Lorentz rotations, no superrotations :(

Poincaré symmetry enhanced by one arbitrary constant: $T(x, \bar{x})$

[Ananth, Brink and SM]

# BMS algebra in light-cone gravity

- Transformation law (on $x^+ = 0$ surface),

$$\begin{aligned}\delta_{Y,\overline{Y},T}\, h \;\; = \;\; & Y(x)\,\bar\partial h + \overline{Y}(\bar x)\,\partial h \; + (\partial \overline{Y} - \bar\partial Y)\, h + T\,\frac{\partial\bar\partial}{\partial_-}\,h - \partial T\,\frac{\bar\partial}{\partial_-}h - \bar\partial T\,\frac{\partial}{\partial_-}h \\ & - 2\,\kappa\, T\,\partial_-\,\left(h\,\frac{\bar\partial^2}{\partial_-^2}h \;-\; \frac{\bar\partial}{\partial_-}h\,\frac{\bar\partial}{\partial_-}h\right) \\ & - 2\,\kappa\, T\,\frac{\partial^2}{\partial_-^3}\,(\bar h\,\partial_-^2\, h) \;+\; 4\,\kappa\, T\,\frac{\partial}{\partial_-^2}\,\left(\frac{\partial}{\partial_-}\,\bar h\,\partial_-^2\,h\right) \;+\; \mathcal{O}(\kappa^2)\end{aligned}$$

- BMS algebra in the phase space of $(h, \bar h)$

$$\Big[\,\delta(Y_1, \overline{Y}_1, T_1)\,,\; \delta(Y_2, \overline{Y}_2, T_2)\Big]\; h \quad = \quad \delta(Y_{12}, \overline{Y}_{12}, T_{12})\, h\,,$$

with parameters

$$\begin{aligned} Y_{12} \;\; &\equiv \;\; Y_2\,\bar\partial\, Y_1 \;-\; Y_1\,\bar\partial\, Y_2 \\ \overline{Y}_{12} \;\; &\equiv \;\; \overline{Y}_2\,\partial\,\overline{Y}_1 \;-\; \overline{Y}_1\,\partial\,\overline{Y}_2 \\ T_{12} \;\; &\equiv \;\; [Y_2\,\bar\partial\, T_1 \;+\; \overline{Y_2}\,\partial\, T_1 \;+ \frac{1}{2}\, T_2(\bar\partial Y_1 \;+\; \partial \overline{Y}_1)] \;-\; (1 \leftrightarrow 2)\,. \end{aligned}$$

- Canonical generator for supertranslations

$$G_T \;\; = \;\; \int d^3x\,\partial_-\bar h\,(\delta_T h) \;=\; \int\, d^3x\,\partial_-\bar h\,\left\{T\,\frac{\partial\bar\partial}{\partial_-}\,h - \partial T\frac{\bar\partial}{\partial_-}h - \bar\partial T\frac{\partial}{\partial_-}h\right\} \;+\; \mathcal{O}(\kappa)\,,$$

$$\delta_T h \;=\; [G_T, h]\,, \quad \delta_T \bar h \;=\; [G_T, \bar h]\,.$$

# Light-cone representation of the BMS algebra

- Light-cone Poincaré algebra

$$\mathbb{K}: \quad \{P, \bar{P}, P^+, J, J^+, \bar{J}^+, J^{+-}\}$$
$$\mathbb{D}: \quad \{P^- \equiv H, J^-, \bar{J}^-\}$$

$$[\,\mathbb{K}, \mathbb{K}\,] = \mathbb{K}\,, \quad [\,\mathbb{K}, \mathbb{D}\,] = \mathbb{D}\,, \quad [\,\mathbb{D}, \mathbb{D}\,] = 0\,.$$

- Light-cone BMS algebra

$$\mathbb{K} \to \mathbb{K}\,,$$
$$\mathbb{D} \to \mathbb{D}(T)\,,$$

$$[\,\mathbb{K}, \mathbb{K}\,] = \mathbb{K}\,, \quad [\,\mathbb{K}, \mathbb{D}(T)\,] = \mathbb{D}(T)\,, \quad [\,\mathbb{D}(T), \mathbb{D}(T)\,] = 0\,.$$

Dynamical part enhanced to infinite-dim supertranslations labeled by a single parameter

$$T(x, \bar{x}) = \sum_{m,n=0}^{\infty} c_{m,n} x^m \bar{x}^n = c_{0,0} + c_{1,0}\,x + c_{0,1}\,\bar{x} + \ldots$$

- Poincaré part of the BMS

$$\partial^2 T = \bar{\partial}^2 T = 0$$

$\Rightarrow \mathbb{D}(T)$ reduces to $\mathbb{D}: \ \{H, J^-, \bar{J}^-\} \quad \to \quad$ the three “Hamiltonians” of Dirac

[Ananth, Brink and SM]

# Light-cone BMS versus BMS at spatial infinity

## BMS in front form

- Light-cone Poincaré algebra

$$\mathbb{K}: \quad \{P, \bar{P}, P^{+}, J^{12}, J^{+}, \bar{J}^{+}, J^{+-}\}$$
$$\mathbb{D}: \quad \{P^{-} \equiv H, J^{-}, \bar{J}^{-}\}$$

$$[\mathbb{K}, \mathbb{K}] = \mathbb{K}\,, [\mathbb{K}, \mathbb{D}] = \mathbb{D}\,, [\mathbb{D}, \mathbb{D}] = 0$$

- Going from Poincaré to BMS

$$\mathbb{K} \to \mathbb{K}\,, \quad \mathbb{D} \to \mathbb{D}(T)\,,$$
$$[\mathbb{D}(T), \mathbb{D}(T)] = 0$$

labelled by

$$T(x, \bar{x}) = c_{0,0} + c_{1,0}\,x + c_{0,1}\,\bar{x} + \dots$$

- Poincaré subgroup

$$\partial^2 T = \bar{\partial}^2 T = 0$$

$$\mathbb{D}(T) \to \{H, J^{-}, \bar{J}^{-}\}$$

$\to$ 3 “Hamiltonians” of Dirac’s front form

## BMS in Instant form

- Poincaré algebra

in spherical coordinates : $x^{\mu} = (t, r, \theta, \varphi)$

$$\{\text{Lorentz } M^{\mu\nu}, \text{Translations } P^{\mu}\}$$

$$[M, M] = M\,, [P, M] = P\,, [P, P] = 0$$

- Going from Poincaré to BMS

$$M^{\mu\nu} \to M^{\mu\nu}\,, \quad P^{\mu} \to \mathcal{ST}\,,$$
$$[\mathcal{ST}, \mathcal{ST}] = 0$$

labelled by

$$\mathcal{T}(\theta, \varphi) = a_{0,0}\, Y_{0,0} + \sum_{m=-1}^{1} a_{1,m}\, \underbrace{Y_{1,m}}_{\text{spherical harmonics}} + \dots$$

- Poincaré subgroup

$$\partial_A \mathcal{T}(\theta, \varphi) = 0\,, \quad x^A = \{\theta, \varphi\}$$

$$\mathcal{ST} \to \{P^0, P^r, P^{\theta}, P^{\varphi}\} : \text{Abelian ideal}$$

[Ananth and SM; arXiv:2305.09735]

# Summary: Does (3+1) equal (2+2)?

(3+1): Asymptotic symmetries at spatial infinity

- Symmetry $\equiv$ invariance of symplectic form or Hamiltonian action
- Boundary value problem on a Cauchy hypersurface
- Integrability of boost charges is a subtle issue
- Spin 1: Must include a surface dof $\overline{\Psi}$ to obtain full U(1) gauge symmetries
  Setting $\overline{\Psi}$ to zero amounts to improper gauge fixing
- Spin 2: Supertranslations obtained without any extra surface degrees of freedom
- Superrotations could not be canonically realized (for asymptotically flat BCs)

(2+2): Residual gauge symmetries in light-cone formulation

- Symmetry $\equiv$ invariance of light-cone action
- Characteristic initial value problem on a null hypersurface
- Integrability of boost charges is a subtle issue
- Spin 1: Must include a zero mode $\alpha$ to obtain *all* residual gauge symmetries
  Setting $\alpha$ to zero amounts to residual gauge fixing
- Spin 2: Supertranslations obtained without introducing any zero modes
- Superrotations could not be canonically realized (on Mink background)

# Some concluding remarks...

## How to connect to null infinity?

- Celestial and Carrollian holography, scattering amplitudes, ...
- Superrotations, $\mathit{Diff}(\mathbb{S}^2)$ and other extensions:
  Do we need to extend the phase space?
  Why do we need boundary d.o.f. in some cases, such as spin 1 and spin 3/2?
- Decoupling of gauge algebra (‘pure supertranslations’) from Poincaré using supertranslation-inv Lorentz charges $\rightarrow$ Can we see this at $\mathcal{I}^+$ or in the front form?

[Oscar Fuentealba, Marc Henneaux,and Cédric Troessaert]

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## Connections with amplitudes, (Anti) self-dual and all that

- Light-cone action in a basis of helicity states - well suited for on-shell physics
- Various applications- MHV Lagrangians , KLT relations , Double copy methods

[Gorsky-Rosly, Ananth-Theisen, Ananth-Kovacs-Parikh, ...]

- Double copy construction for SD sectors [Campiglia-Nagy '21]
- Double copy for BMS symmetries, Newmann-Penrose formalism, Weyl double copy, ...

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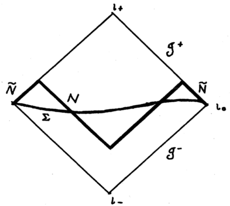
## Formal aspects of null-front Hamiltonian analysis

- Role of boundary degrees of freedom, zero modes, etc.
- Dictionary between residual gauge symmetries in (2+2) with asymptotic symmetries at $\mathcal{I}^+$ and $i^0$
- Comparison with the initial value problem in the instant form, Equivalent of Cauchy hypersurfaces in the front form?




[Nagarajan-Goldberg '85]

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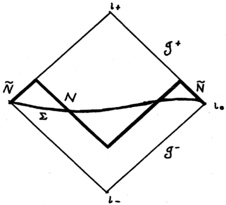
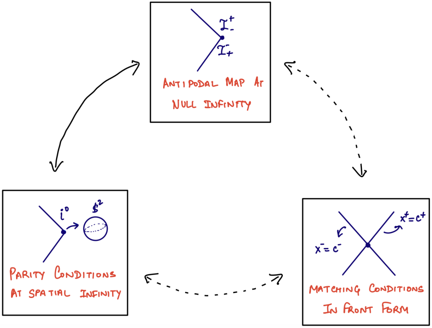


[Nagarajan-Goldberg '85]

Parity conditions at $i^0 \longleftrightarrow$ Antimopal map at $\mathcal{I}^+ \longleftrightarrow$ Null matching conditions in Front form



“I feel that there will always be something missing from them [non-Hamiltonian methods], which we can only get by working from a Hamiltonian”

-*P.A.M. Dirac*,

Lectures on Quantum Mechanics (1964)

THANK YOU!

APPENDIX

# Light-cone Poincaré algebra in $d = 4$

- Non-vanishing commutators of the Poincaré algebra

$$J^+ = \frac{J^{+1} + iJ^{+2}}{\sqrt{2}}\,, \quad \bar{J}^+ = \frac{J^{+1} - iJ^{+2}}{\sqrt{2}}\,, \quad J = J^{12}\,, \quad H = P_+ = -P^-\,.$$

$$\begin{aligned}
&[H, J^{+-}] = -iH\,, && [H, J^+] = -iP\,, && [H, \bar{J}^+] = -i\bar{P}\\
&[P^+, J^{+-}] = iP^+\,, && [P^+, J^-] = -iP\,, && [P^+, \bar{J}^-] = -i\bar{P}\\
&[P, \bar{J}^-] = -iH\,, && [P, \bar{J}^+] = -iP^+\,, && [P, J] = P
\end{aligned}$$

... and many more

- Underlying Carrollian algebra

Rotation $\mathbb{J} = \{J^{12}, J^{+-}, J^+, \bar{J}^+\}$ , Boosts $\mathbb{K} = \{J^-, \bar{J}^-\}$

Translations $\mathbb{P} = \{P, \bar{P}, P_-\}$ , Hamiltonian $\mathbb{H} = P_+$

$$\begin{aligned}
&[\mathbb{J}, \mathbb{J}] = \mathbb{J}\,, && [\mathbb{J}, \mathbb{P}] = \mathbb{P}\,, && [\mathbb{J}, \mathbb{K}] = \mathbb{K}\\
&[\mathbb{J}, \mathbb{H}] = 0\,, && [\mathbb{H}, \mathbb{P}] = 0\,, && [\mathbb{H}, \mathbb{K}] = 0\\
&[\mathbb{P}, \mathbb{P}] = 0\,, && [\mathbb{K}, \mathbb{K}] = 0\,, && [\mathbb{P}, \mathbb{K}] = \mathbb{H}
\end{aligned}$$

- In terms of the Kinematical-Dynamical split

$$\mathbb{K} = \{P_i, P_-, M_{ij}, M_{-i}, M_{+-}\}\,, \quad \mathbb{D} = \{P_+, M_{i+}\}$$

$$[\mathbb{K}, \mathbb{K}] = \mathbb{K}\,, \quad [\mathbb{K}, \mathbb{D}] = \mathbb{D}\,, \quad [\mathbb{D}, \mathbb{D}] = 0$$

# Decoupling of gauge algebra from Poincaré at $i^0$

Recent developments in the asymptotic symmetry analysis at spatial infinity

[Oscar Fuentealba, Marc Henneaux,and Cédric Troessaert]

- Spin 1: Large gauge transformations

$$A_\mu = \partial_\mu \epsilon \,, \qquad \epsilon \sim a(\theta,\varphi) r + b(\theta,\varphi) \ln r + c(\theta,\varphi) + \dots$$

Asymptotic algebra

$$[G_{\text{Poincaré}}\,, G_{\text{Poincaré}}] = G_{\text{Poincaré}} \,, \quad [G_{\text{Gauge}}\,, G_{\text{Poincaré}}] = G_{\text{Gauge}} \,, \quad [G_{\text{Gauge}}\,, G_{\text{Gauge}}] = 0 \,,$$

$$\downarrow$$

$$[G_{\text{Gauge}}\,, G_{\text{Gauge}}] = \text{C} \,,$$

Central charge $C$ allows a definition of ‘gauge-invariant’ Lorentz generators such that

$$[G_{\text{Poincaré}}\,, G_{\text{Poincaré}}] = G_{\text{Poincaré}} \,, \quad [G_{\text{Gauge}}\,, G_{\text{Poincaré}}] = 0 \,,$$

$\rightarrow$ Gauge algebra completely decoupled form Poincaré

[arXiv: 2301.05989]

- Further extended to spin-2: $BMS_4$, $BMS_5$, super-$BMS_4$ etc.

[arXiv: 2211.10941 and arXiv:2305.05436]

- Can we find similar decoupling at $\mathcal{I}^+$ or in the front form? Does this happen only at $i^0$?