Geometric Transformations on Digital Images

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Summer school on **Geometry and data**

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Motivation

Geometric transformations on digital images

Given a **source image S**, we generate a **target image S** depending on the chosen transformation, for example:

- translation, rotation (and its combination, called rigid motions)
- affine transformation (scaling, symmetries and rigid motion)
- projective transformation, ...

S → **Geometric transformation** → **T**

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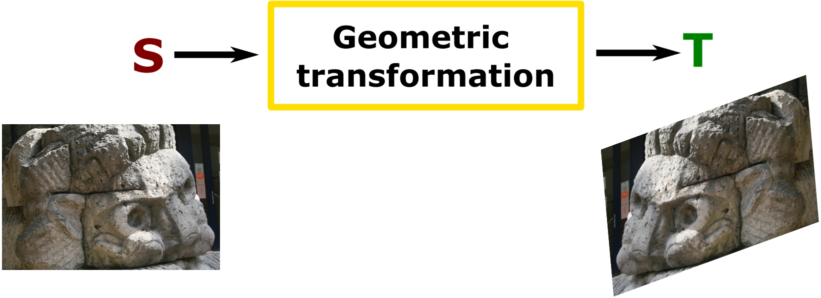

Applications

- 2D: Image registration, image warping, data augmentation ...
- 3D: Medical imagery, deformable models, 3D reconstruction ...

Image registration on satellite imagery [Sommervold et al., 2023]

Applications

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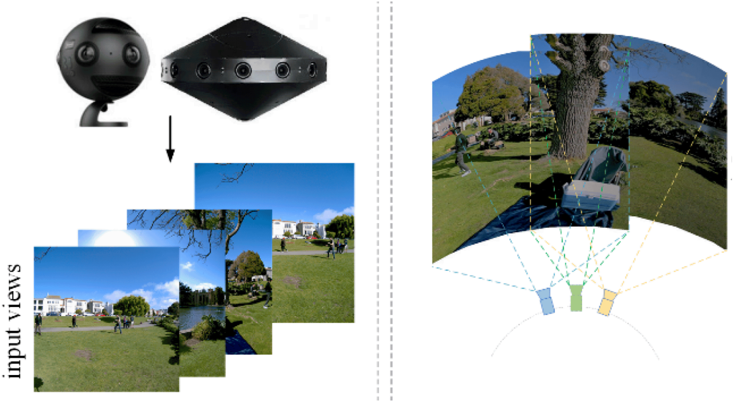

Image registration for panorama [Zhang et al., 2022]

Applications

- 2D: Image registration, image warping, data augmentation . . .
- 3D: Medical imagery, deformable models, 3D reconstruction . . .

Image registration for object detection [Rodríguez et al., 2023]

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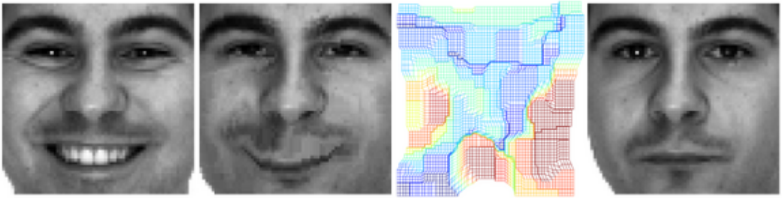

Image Warping For Face Recognition [Pishchulin et al., 2011]

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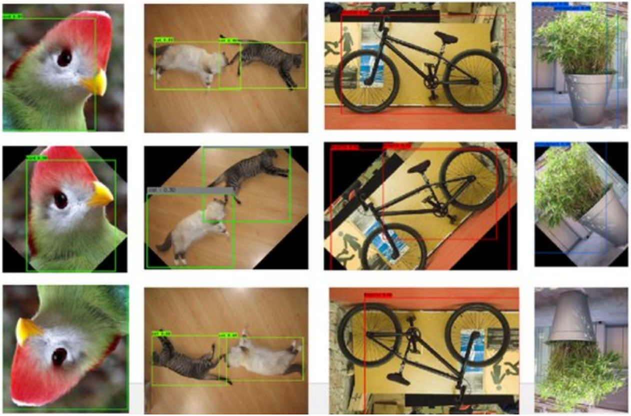

Image transformation for data augmentation [Shorten and Khoshgoftaar, 2019]

Applications

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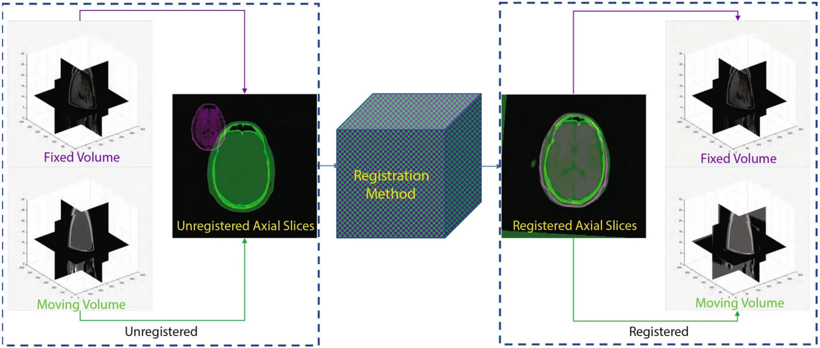

Registration of 3D multi-modal medical images [Islam et al., 2021]

Applications

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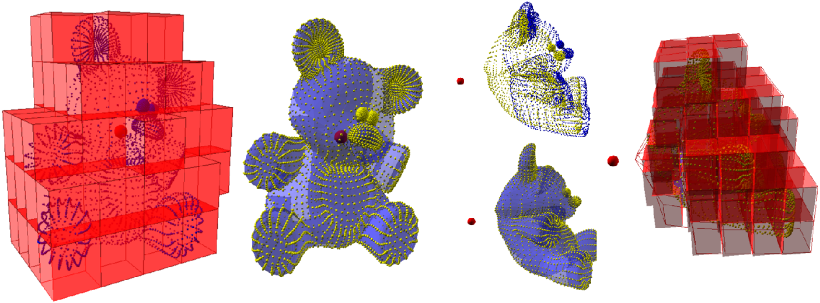

Voxel Free-Form Deformations [Kenwright, 2013]

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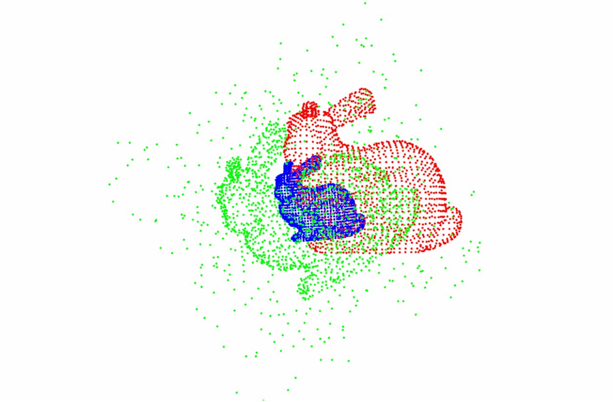

Point set registration with probablistic model [Kenta-Tanaka et al., 2019]

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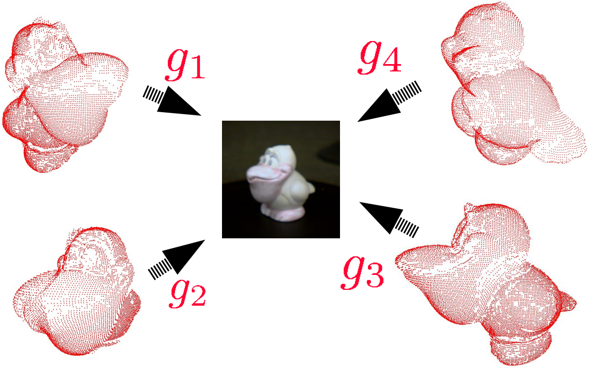


3D object reconstruction from laser point cloud data [Nguyen et al., 2012]

Applications

- 2D: Image registration, image warping, data augmentation . . .
- 3D: Medical imagery, deformable models, 3D reconstruction . . .

Content

In this course, we are interested in

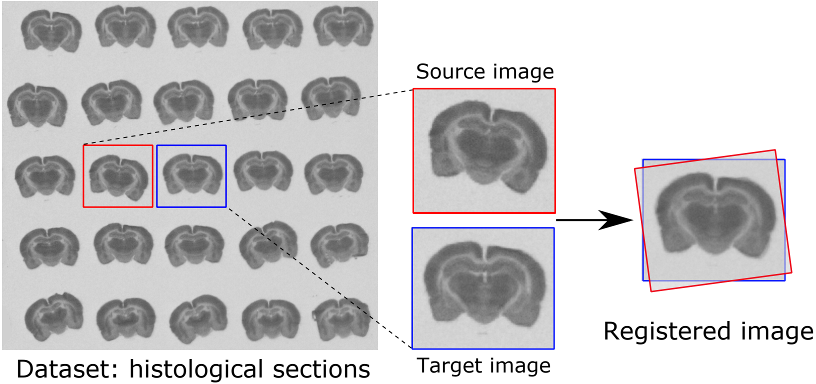
- Discrete data: Digital images and discrete points of $\mathbb{Z}^2$ / $\mathbb{Z}^3$
- Classes of transformation: Rigid motion and affine transformation
- Topic: Geometric and topological properties of such transformations in the discrete space of $\mathbb{Z}^2$ / $\mathbb{Z}^3$
- Applications: Digital image processing and analysis

Topological issue of rigid motion on digital images

Dataset: histological sections

(Laboratoire ICube - Strasbourg)

Topological issue of rigid motion on digital images



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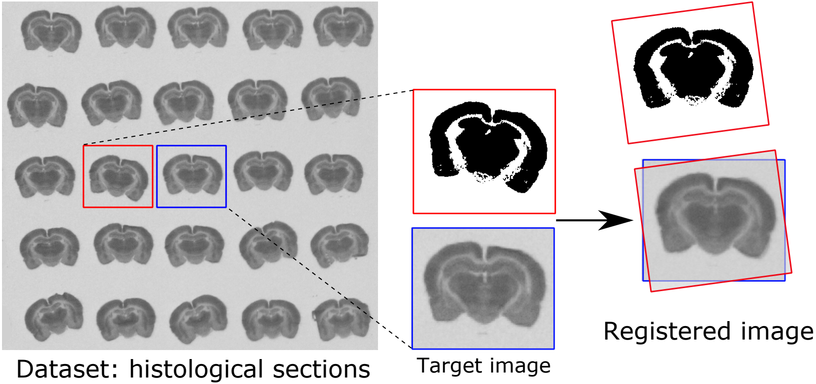

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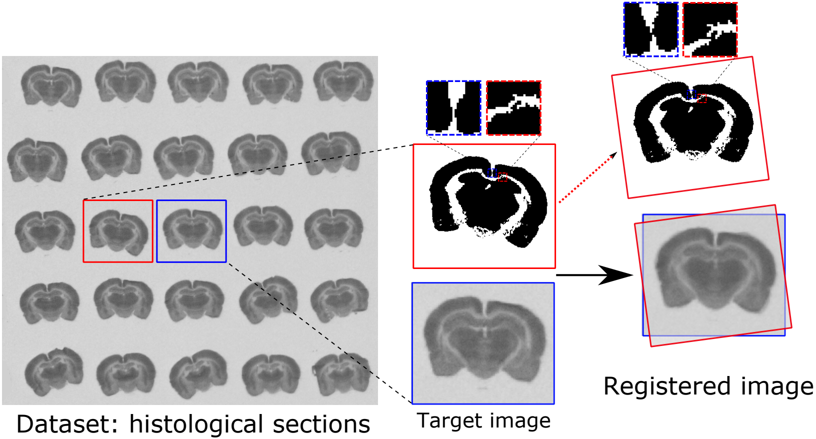

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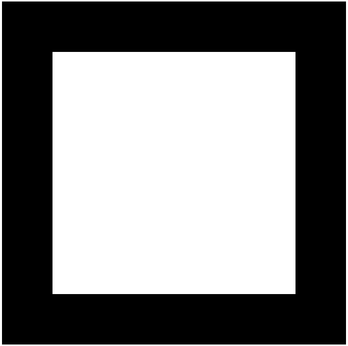
Information loss of digitized rotation on digital images

Discretization of isometries [Guihéneuf, 2016]

All the information of a numerical image will be **lost** by applying many times a naive algorithm of rotation.



Discretization of rotations on a white-pixel image of size 50×50 pixels

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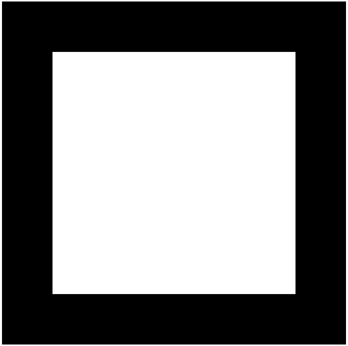


Successive random rotations on an image of size 50 × 50 pixels

Contents

Digitized rigid motion

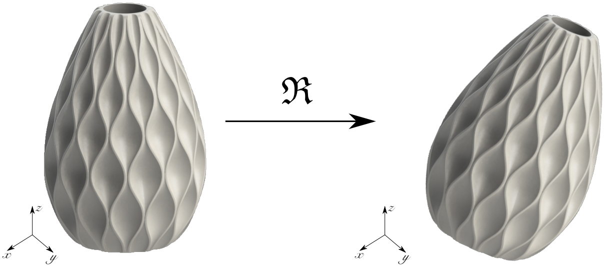
Rigid motion on $\mathbb{R}^d$

Definition

A **rigid motion** is a bijection defined for $\mathrm{x} \in \mathbb{R}^d \in \mathbb{R}^2$, as

$$\begin{array}{rcccl} \mathfrak{R} & : & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & & \mathrm{x} & \longmapsto & R\mathrm{x} + t \end{array}$$

with R a rotation matrix et $t \in \mathbb{R}^d$ a translation vector.



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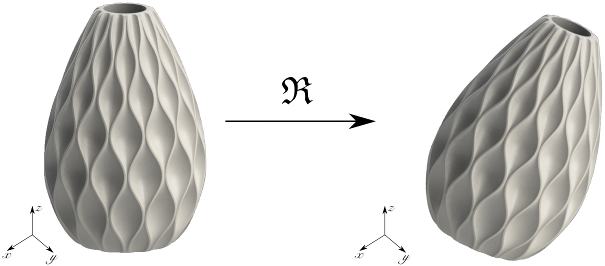

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A **digitized rigid motion** $\mathcal{R} : \mathbb{Z}^d \to \mathbb{Z}^d$ is defined as

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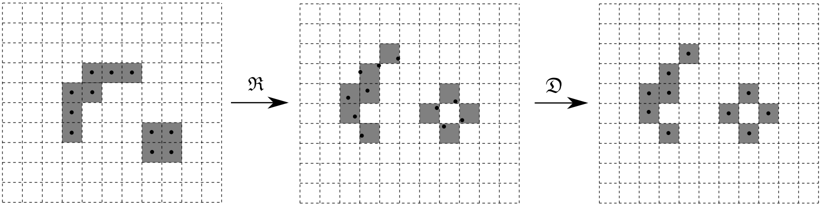

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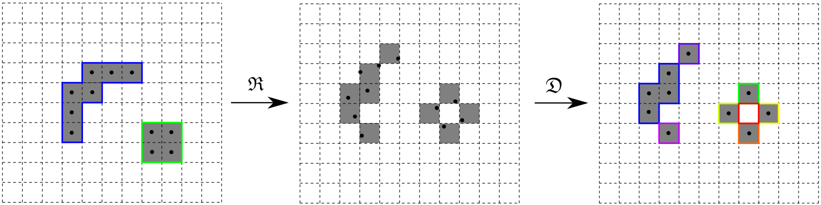

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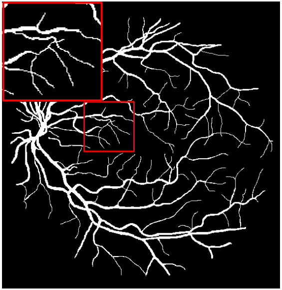
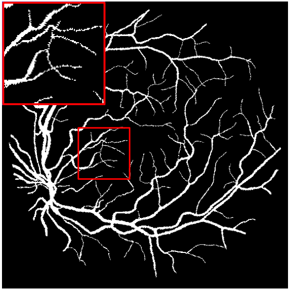


3D binary image



Transformed image by $\mathcal{R}$

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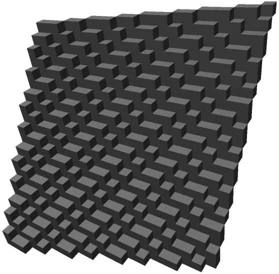
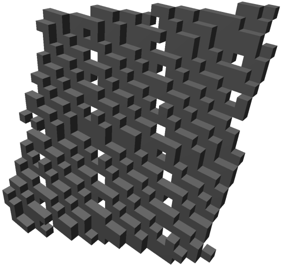
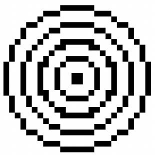


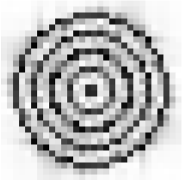
3D digital plane

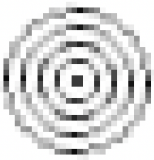


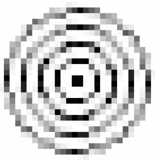
Transformed plane by $\mathcal{R}$

Rigid motion on $\mathbb{Z}^d$


Original image


Fourrier transf.

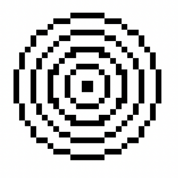

Linear interpolation


Cubical interpolation

Issues

- Interpolation techniques
  - ↪ Generating *new contents* in the transformed image
  - ↪ Visual artifacts: distortions, blurs, ...
- Continuous transformation methods (*e.g.* Fourier transform)
  - ↪ Precision/approximation, blurs, distortions, ...
- Digital transformation $\mathcal{R} = \mathfrak{D} \circ \mathfrak{R}$
  - ↪ Topology and geometry alteration, ...

Rigid motion on $\mathbb{Z}^d$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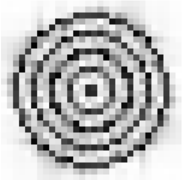
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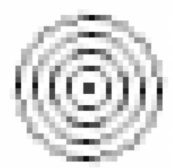
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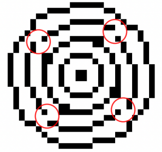
No interpolation: $\mathcal{R}$

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Rigid motion on $\mathbb{Z}^2$ and $\mathbb{Z}^3$

Topics covered in this course

1. Combinatorial structure of rigid motions on $\mathbb{Z}^2$

   $\hookrightarrow$ **Graph of discrete rigid motions** $\rightsquigarrow$ neighbouring relationships

2. Topological characterization of digital images under rigid motions

   $\hookrightarrow$ Notion of **regularity** and image regularization methods

3. Geometric characterization of continuous objects by Gauss discretization

   $\hookrightarrow$ Notion of **quasi-regularity** and verification of quasi-regular polygons

4. New models for geometric transformations on $\mathbb{Z}^2$ / $\mathbb{Z}^3$:

   $\hookrightarrow$ **Polygon/polyhedron-based models** for shape preservation of objects

   $\hookrightarrow$ Geometric transformations on $\mathbb{Z}^2$ as an **optimization scheme**

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Discrete rigid motion graph

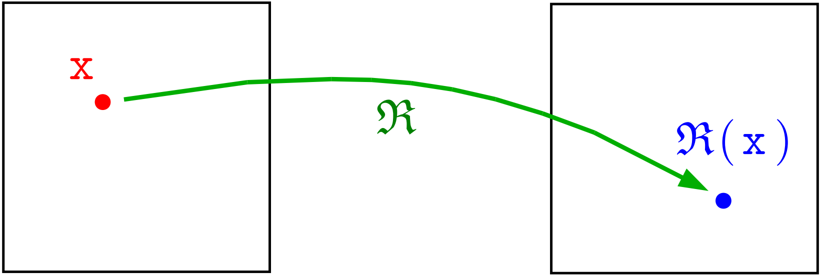
Rigid motion on $\mathbb{R}^2$

Definition

A rigid motion is a bijection defined for any $\mathrm{x} = (x, y) \in \mathbb{R}^2$ as

$$\mathfrak{R}_{ab\theta}(\mathrm{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$.



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Definition

A digitized rigid motion on $\mathbb{Z}^2$ is defined for any $\mathrm{p} = (p, q) \in \mathbb{Z}^2$ as

$$\mathcal{R}(\mathrm{p}) = \mathfrak{D} \circ \mathfrak{R}(\mathrm{p}) = \begin{pmatrix} [p\cos\theta - q\sin\theta + a] \\ [p\sin\theta + q\cos\theta + b] \end{pmatrix}$$

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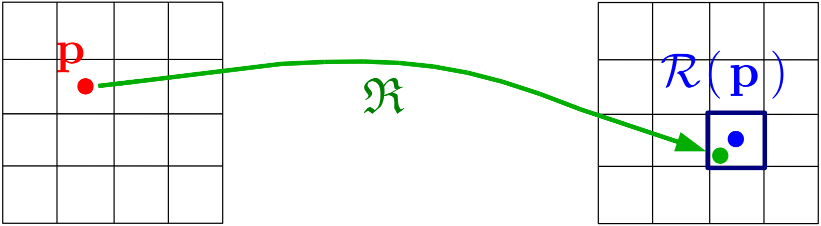

Lagrangian model – Forward transformation : $\mathcal{R} = \mathfrak{D} \circ \mathfrak{R}$

Rigid motion on $\mathbb{Z}^2$

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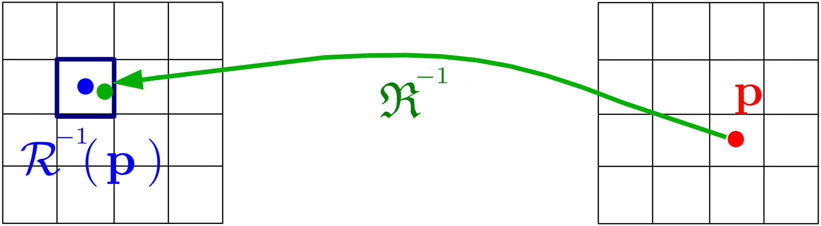
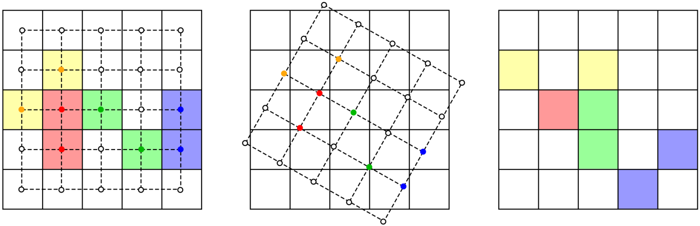


Eulerian model – Backward transformation: $\mathcal{R}^{-1} = \mathfrak{D} \circ \mathfrak{R}^{-1}$

Non-isometry of rigid motion on $\mathbb{Z}^2$



Distance alterations by digitized rigid motion

Before	After
1	$\sqrt{2}$
1	0
$\sqrt{2}$	1
$\sqrt{2}$	2

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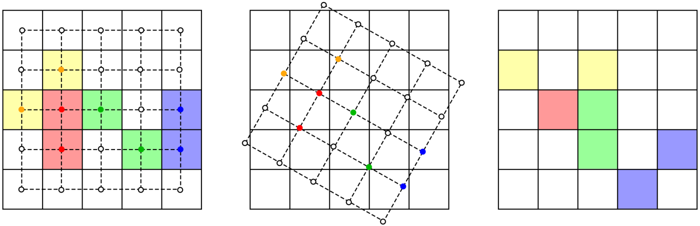

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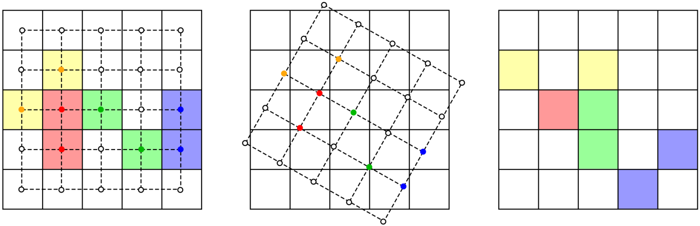

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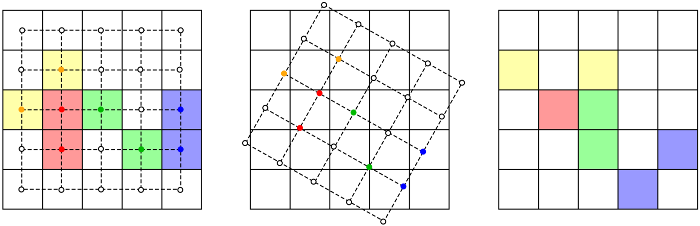

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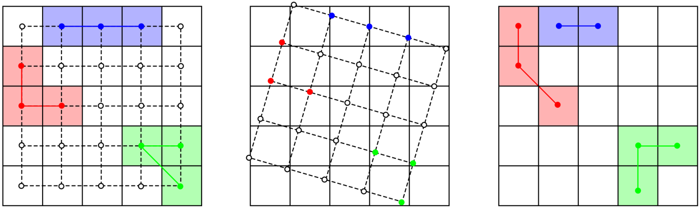

Angle alterations by digitized rigid motion

Before	After
90°	135°
180°	0°
45°	90°

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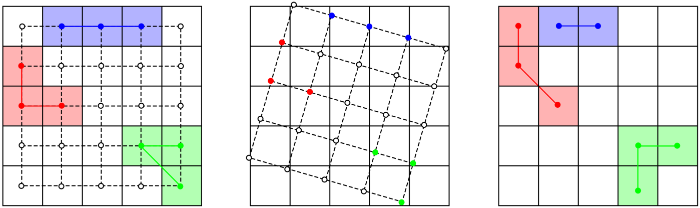

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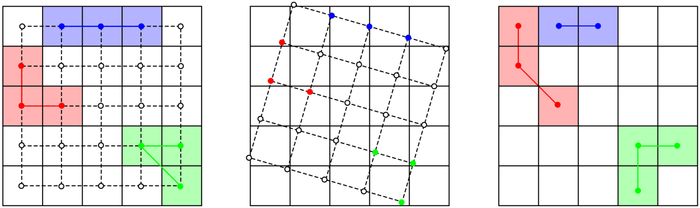

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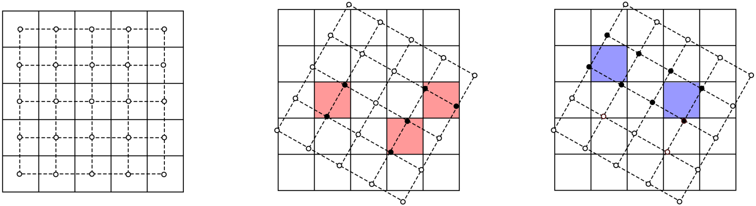
Non-bijectivity of rigid motion en $\mathbb{Z}^2$



Input grid

Double pixels

✗ Non surjective

Null pixels

✗ Non injective

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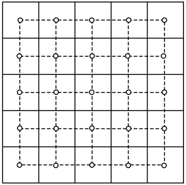

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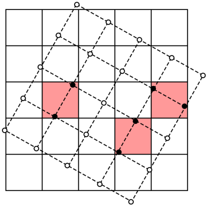

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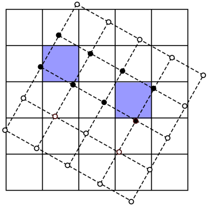

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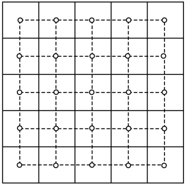

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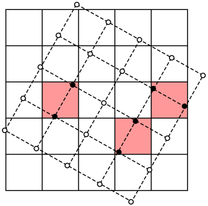

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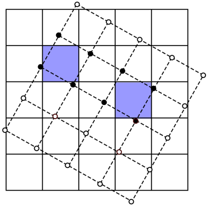

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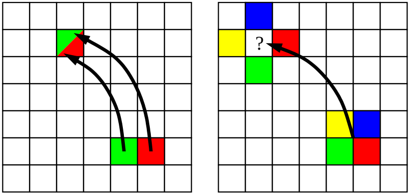
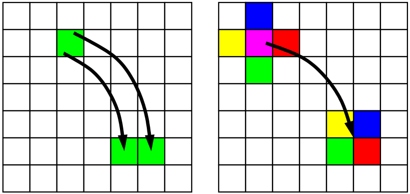


Lagrangian model

✗ Incomplete and ambiguous issues of color



Eulerian model

✓ No issue of color ⇝ use to generate images!

Discontinuities of rigid motion on $\mathbb{Z}^2$

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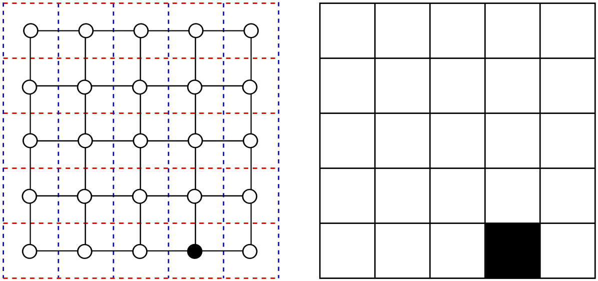

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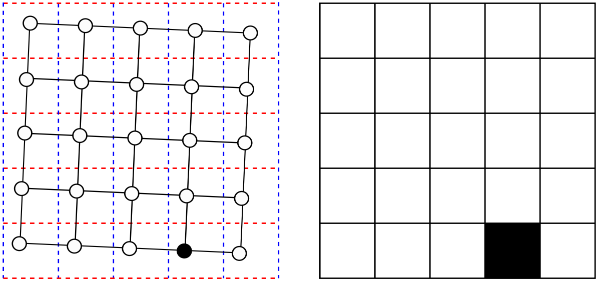

This model is also called **point-wise rigid motion** and noted by $\mathcal{R}_{Point}$.

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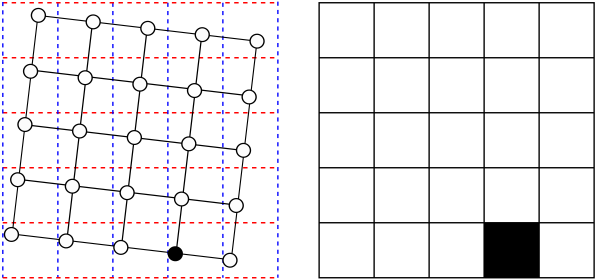

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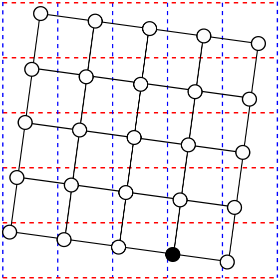

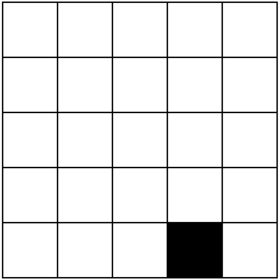

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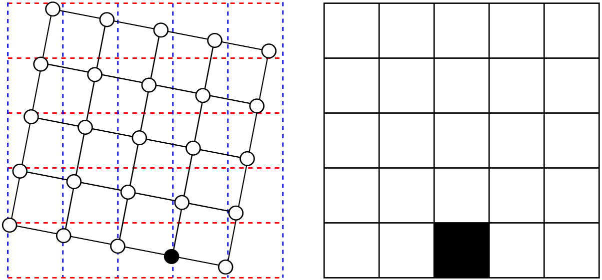

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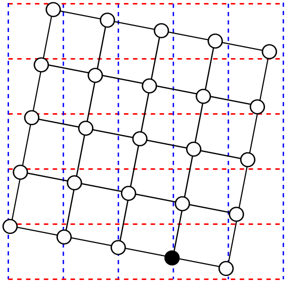
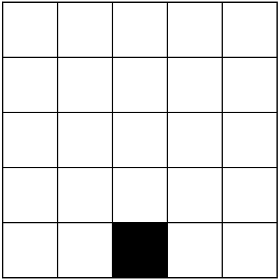
Discontinuities of rigid motion on $\mathbb{Z}^2$

Definition [Ngo et al., 2013]

A discrete rigid motion (DRM) is the set of all the rigid motions that generate a same image.



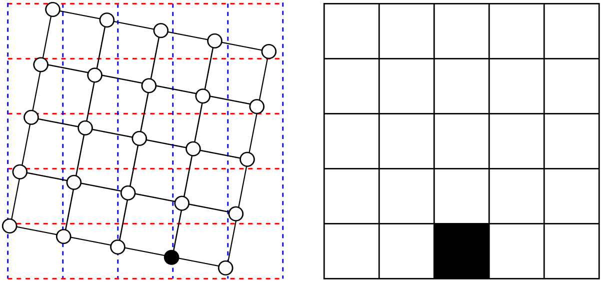


The parameter space (a, b, θ) is subdivided into disjoint sets of DRMs.

Critical rigid motions

Definition [Ngo et al., 2013]

A **critical rigid motion** moves at least one point of $\mathbb{Z}^2$ to a point on the vertical or horizontal half-grid.



The critical transformations correspond to the discontinuities of DRM.

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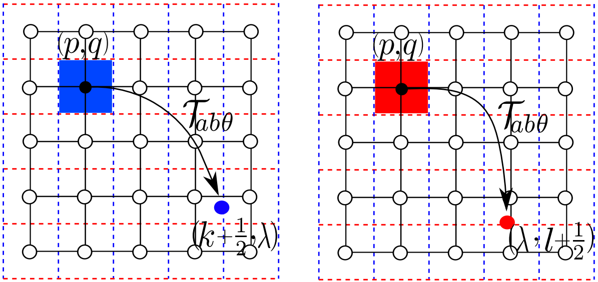

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Tipping surfaces

Definition [Ngo et al., 2013]

The **tipping surfaces** are the surfaces associated to critical transformations in the parameter space (a, b, θ).

$$\Phi_{pqk} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (b, \theta) & \longmapsto & a = k + \frac{1}{2} + q \sin\theta - p \cos\theta \end{array} \quad (\textit{vertical})$$

$$\Psi_{pql} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a, \theta) & \longmapsto & b = l + \frac{1}{2} - p \sin\theta - q \cos\theta \end{array} \quad (\textit{horizontal})$$

for $p, q, k, l \in \mathbb{Z}$.

Each **tipping surface**

- is indexed by a triplet of integers (p, q, k) (resp. (p, q, l)),
- indicates that the pixel (p, q) in a transformed image changes its value from the one at $(k, *)$ (resp. $(*, l)$) in the original image to the one at $(k + 1, *)$ (resp. $(*, l + 1)$).

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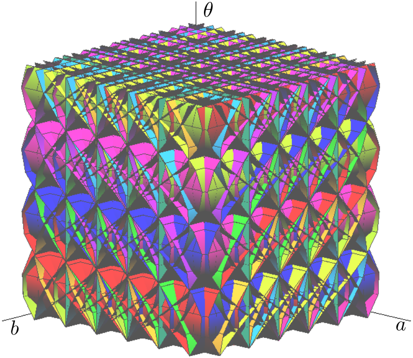
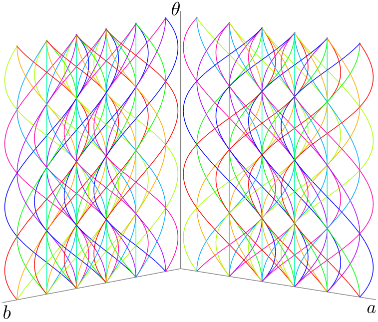
Example of tipping surfaces



Tipping surfaces



Tipping curves

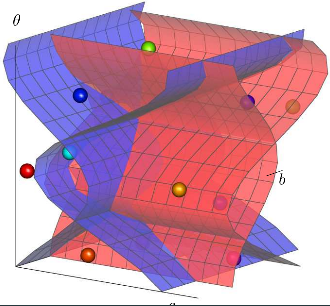
Vertical surfaces Φ_{pqk} and horizontal ones Ψ_{pql} for $p, q \in [0, 2]$ and $k, l \in [0, 3]$.

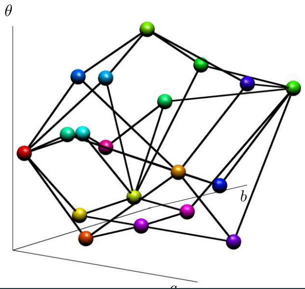
Graph of discrete rigid motions

Definition [Ngo et al., 2013]

A **graph of discrete rigid motions** (DRM graph) is a graph $G = (V, E)$ such that

- each vertex $v \in V$ corresponds to a DRM
- each edge $e \in E$ connects two DRMs sharing a tipping surface





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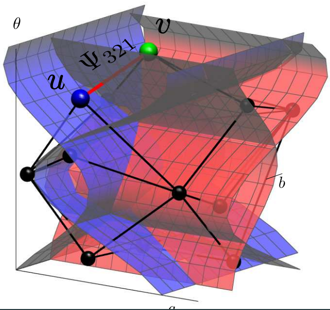

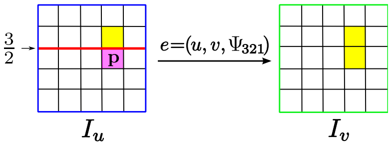

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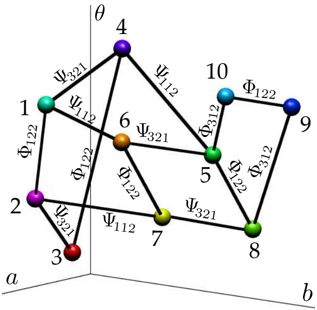

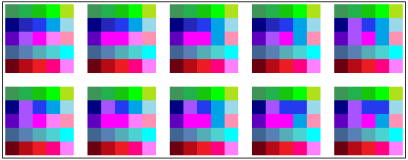

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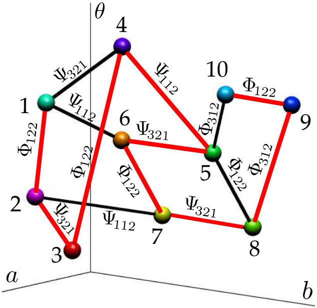

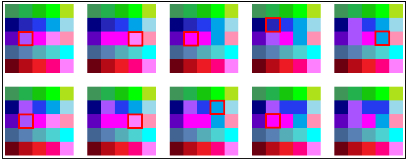

Properties of DRM graphs

Advantages

- DRMs are computed in **a discrete process** with exact calculation.
- Their combinatorial structure is represented by a **DRM graph** G whose complexity is $O(N^9)$ for images of size $N \times N$.
- G models **all the DRMs** with the topological information such that
  - ↪ a vertex corresponds to one transformed image
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- It enables to **generate exhaustively & incrementally all transformed images**.

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It has **high complexity** to generate the entire structure for large images.

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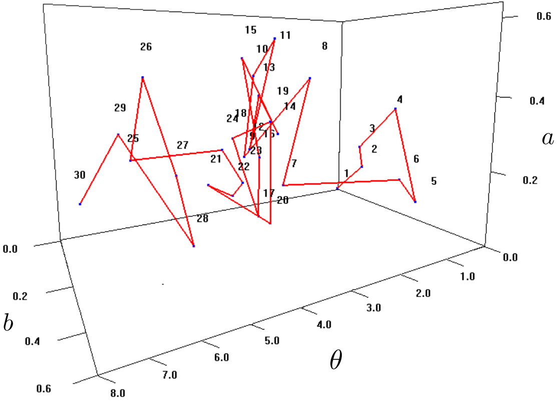
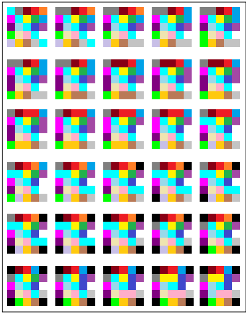
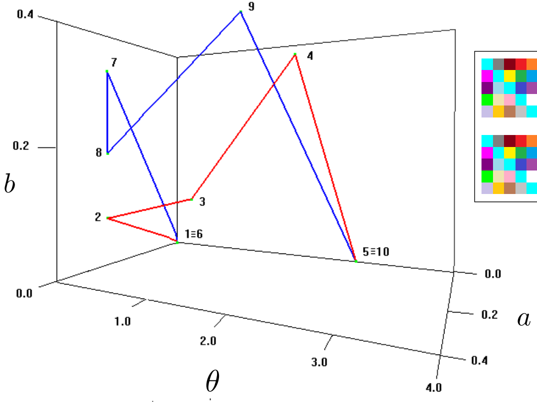
Application: Discrete transition path of transformed images





Application: Discrete rigid motion graph search



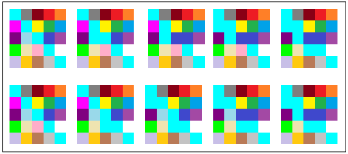


Image registration as a combinatorial optimisation problem

Problem formulation

Given two digital images A and B of size $N \times N$, image registration consists of finding a discrete rigid motion (DRM) such that

$$v^* = \arg\min_{v \in V} d(A, \mathcal{R}_v(B))$$

where $\mathcal{R}_v$ is the digitized rigid motion of a DRM v, and d is a given distance between two images.

Disadvantage

Exhaustive search on DRM graph costs $O(N^9)$ **in complexity**.

Advantage

A local search on DRM graph can determine a **local optimum**.

Image registration as a combinatorial optimisation problem

Problem formulation

Given two digital images A and B of size $N \times N$, image registration consists of finding a discrete rigid motion (DRM) such that

$$v^* = \arg\min_{v \in V} d(A, \mathcal{R}_v(B))$$

where $\mathcal{R}_v$ is the digitized rigid motion of a DRM v, and d is a given distance between two images.

Disadvantage

Exhaustive search on DRM graph costs $O(N^9)$ **in complexity**.

Advantage

A local search on DRM graph can determine a **local optimum**.

Image registration as a combinatorial optimisation problem

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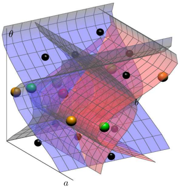
Local search on discrete rigid motion graph

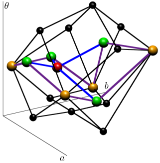
Local search

- **Input:** A reference image A, a target image B, an initial DRM $v_0 \in V$ and a distance metric d
- **Output:** A local optimum $\widehat{v} \in V$
- **Approach:** Gradient descent to find a better solution in neighbours.

DRM graph $G = (V, E)$ provides

- neighbourhood structure $N(v)$
  k-neighbourhood $N^k(v)$:
  $N^k(v) = N^{k-1}(v) \cup \bigcup_{u \in N^{k-1}(v)} N(u)$
- efficient computation of d
  We use **signed distance** with linear complexity w.r.t image size [Kimmel et al., 1996]





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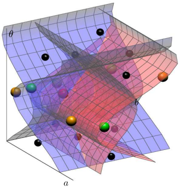

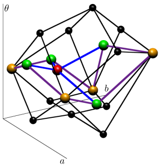

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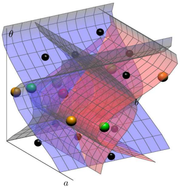

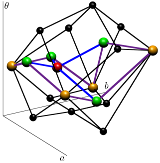

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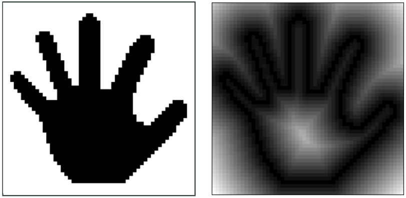
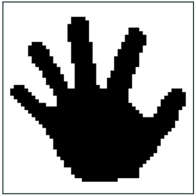
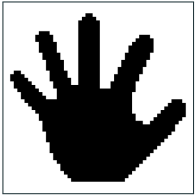


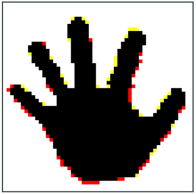
Experiment on binary images



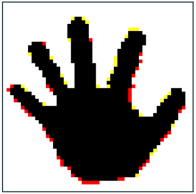
(a) reference image



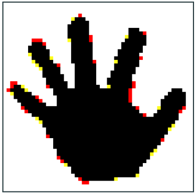
(b) target image



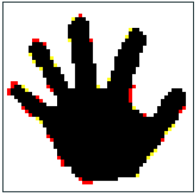
(c) initial solution



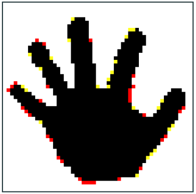
(d) solution: $k = 1$



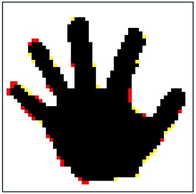
(e) $k = 3$



(f) $k = 5$

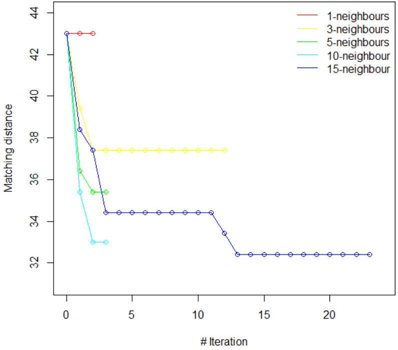


(g) $k = 10$

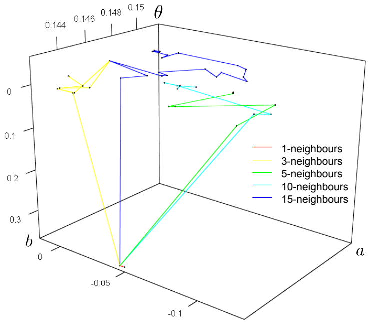


(h) $k = 15$

Experiment on binary images: distance evolutions

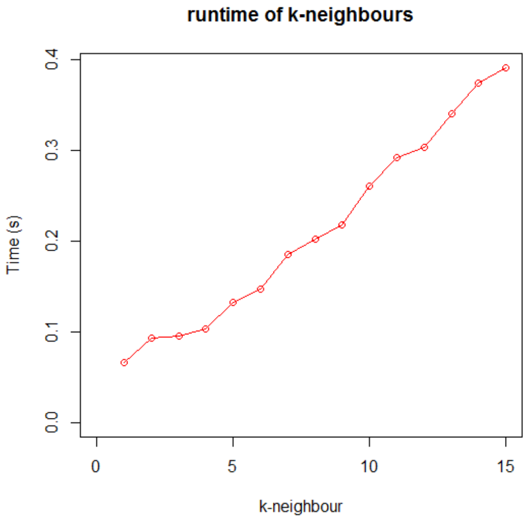


(a) Distance evolutions



(b) Transformation sequences

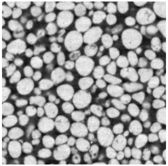
Experiment on binary images: runtime complexity

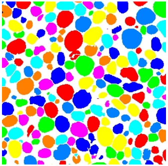


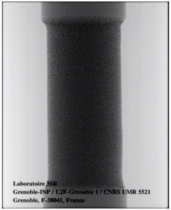
Experiment on gray images

Detect and follow the moving objects in a sequence of 3D grain images

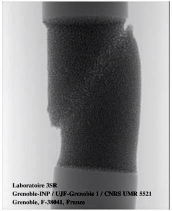
X-ray CT image: original and labelled cross-section images



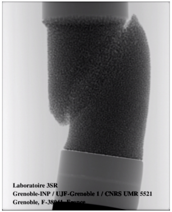




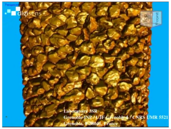
scanner 1



scanner 2



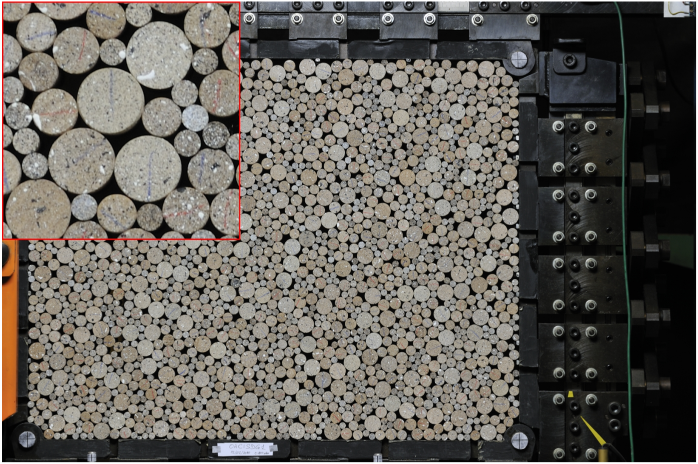
scanner 3



3D visualisation

Movements of Schneebeli rolls (Laboratoire 3SR, Grenoble)

Experiment on gray images



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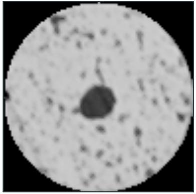

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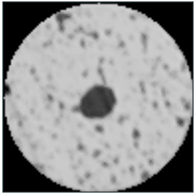

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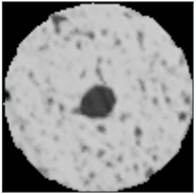

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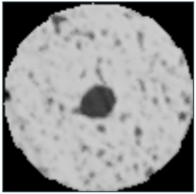

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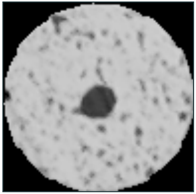

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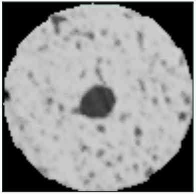

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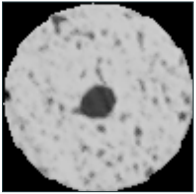
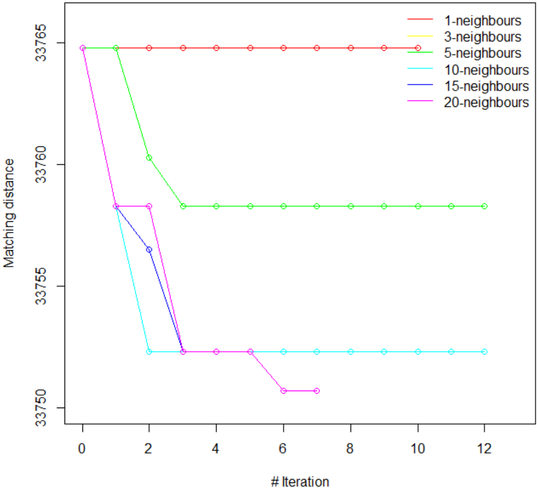


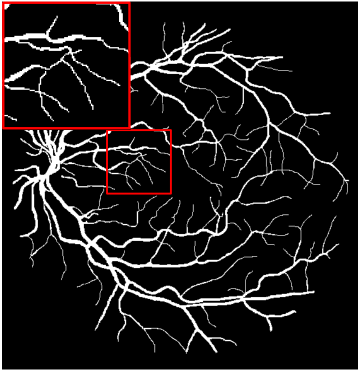
(g) $k = 20$

Experiment on gray images: distance evolutions

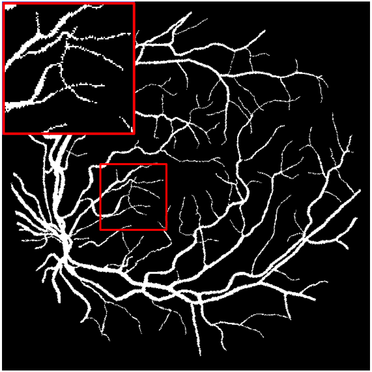


Topological aspect

Topological issue of rigid motion on digital images



Input retina image

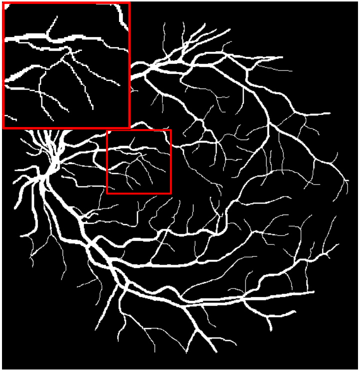


Transformed image

Questions

- Do binary images exist that preserve their topology under any rigid motions?
- What are conditions for images to preserve their topology?

Topological issue of rigid motion on digital images



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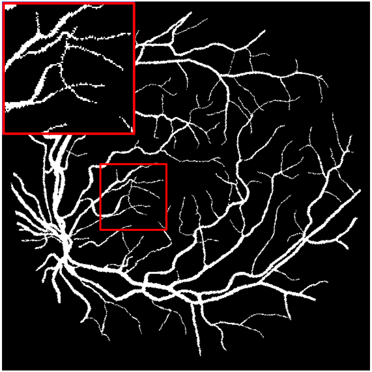

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Connectivity of digital set of points

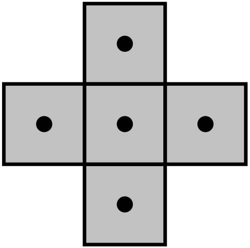
Definition [Latecki et al., 1995]

Two distinct grid points $p, q \in \mathbb{Z}^d$ are said **k-neighbours** if:

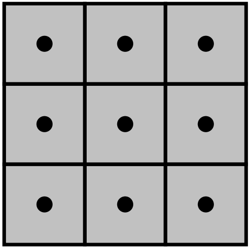
$$||p - q||_l < 1$$

with $k = 2d$ (resp. $3d - 1$) for $l = 1$ (resp. ∞).

- 2D: 4- and 8-neighbourhood $N_k(p) = \{q \in \mathbb{Z}^2 : ||p - q||_l < 1\}$
- 3D: 6- and 26-neighbourhood $N_k(p) = \{q \in \mathbb{Z}^3 : ||p - q||_l < 1\}$



4-neighbourhood



8-neighbourhood

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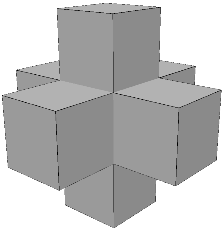
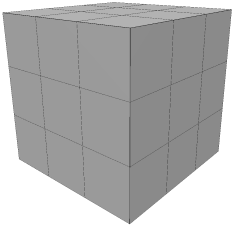


6-neighbourhood



26-neighbourhood

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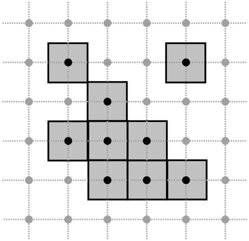
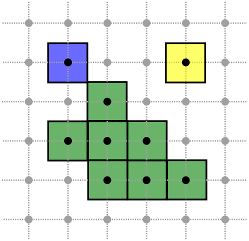


Grid points of $\mathbb{Z}^2$



4-connected components

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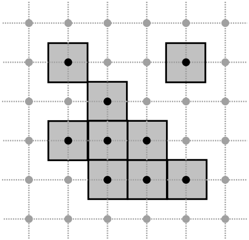

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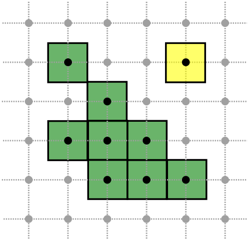
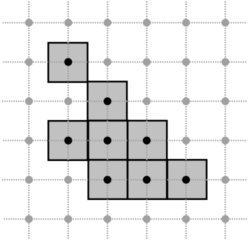


8-connected components

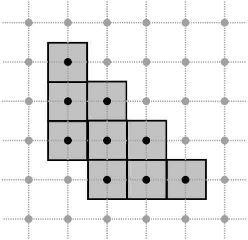
Well-composeness

Definition [Latecki et al., 1995]

A digital set $X \subset \mathbb{Z}^2$ is **well-composed** if each 8-connected component of X and of its complement $\overline{X}$ is also 4-connected.



Non well-composed set



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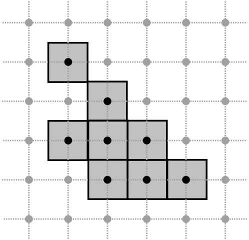

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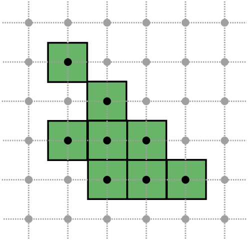

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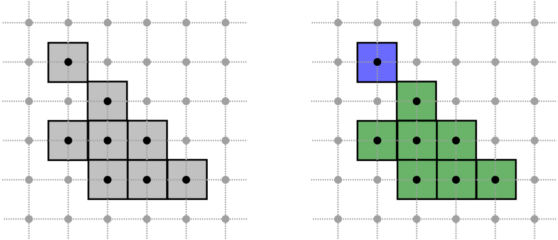

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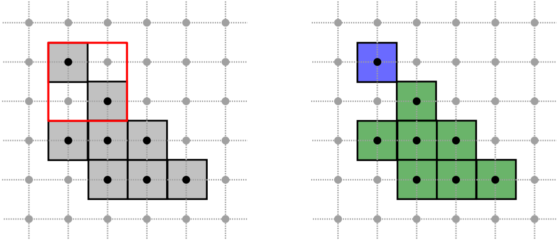
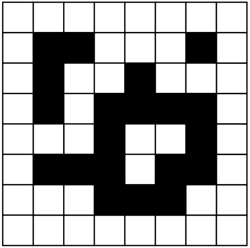
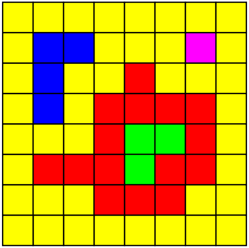


Critical configuration

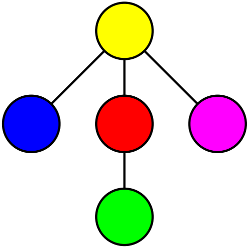
Topological preservation of digital image



Binary image I



Connected components of I

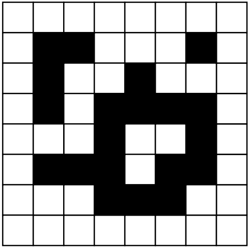


Adjacent tree $\mathfrak{T}(I)$

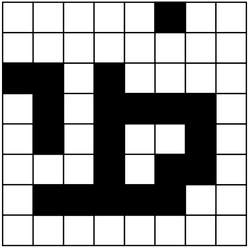
Définition [Ngo et al., 2014]

Let I be a binary image. We say that I is **topologically invariant** if, for *all rigid motions* $\mathfrak{R}$, $I_{\mathfrak{R}} = I \circ \mathfrak{R}$ induces a *isomorphism between adjacency trees* $\mathfrak{T}(I)$ and $\mathfrak{T}(I_{\mathcal{R}})$.

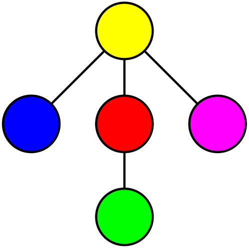
Topological preservation of digital image



Binary image I



Transformed image $I_{\mathfrak{R}}$



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Topological characterization: regularity

Définition [Ngo et al., 2014]

Let I be a binary image. We say that I is **regular** if it is :

- well-composed,
- non singular and
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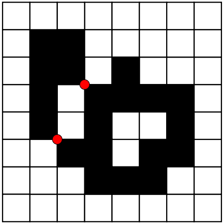
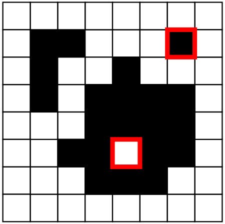
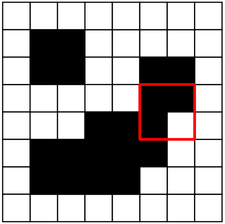

Non well-composed image


Singular image


Non squarely regular image

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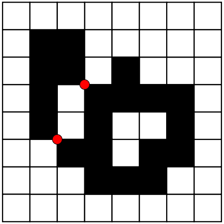
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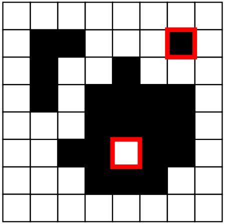
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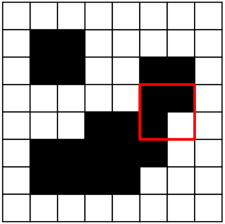
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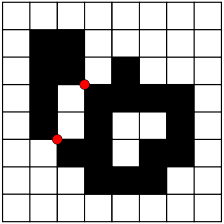
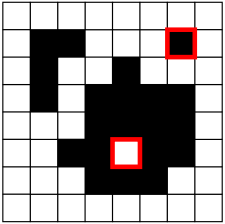
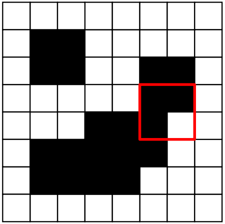


Non well-composed image



Singular image



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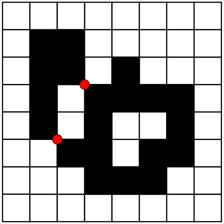
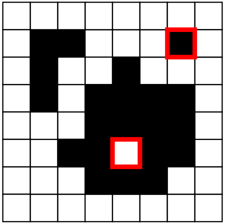
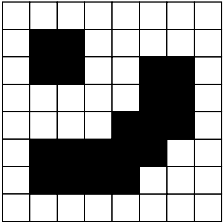


Non well-composed image



Singular image



Regular image

Topological characterization of binary images

Proposition [Ngo et al., 2014]

If a binary image I is regular then it is topologically invariant under any rigid motion.

Prohibited configurations

A binary image I is regular iff it does not contain the configurations:

$\mathfrak{C}_1$ $\mathfrak{C}_2$ $\mathfrak{C}_3$

The regularity of I can be verified locally and in linear time !

Extension

Regularity is extended to grayscale and Labelled images.

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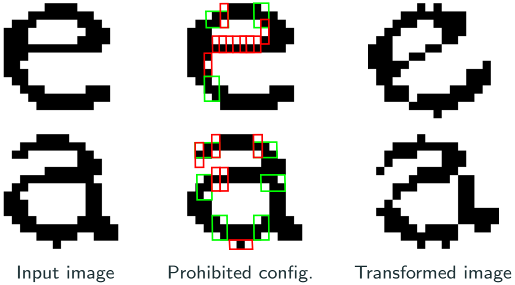

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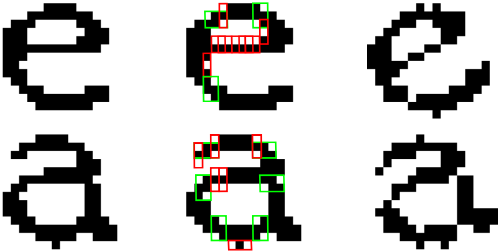
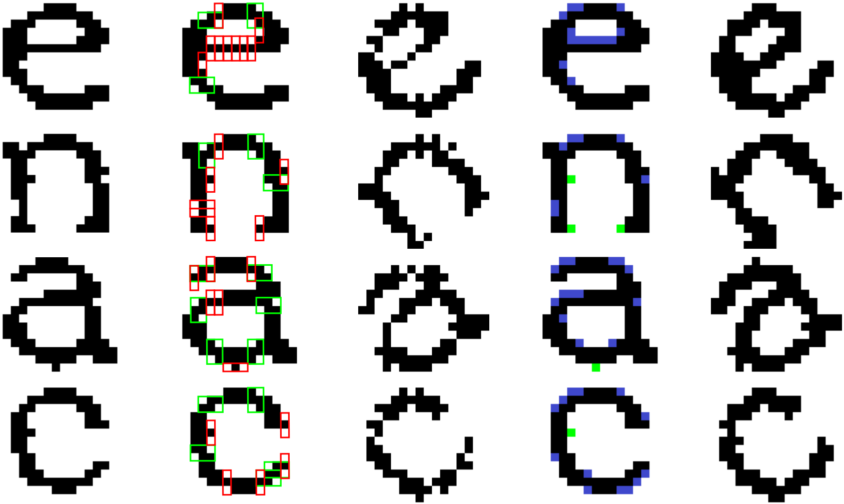


Input image   Prohibited config.   Transformed image

Extension

Regularity is extended to grayscale and Labelled images.

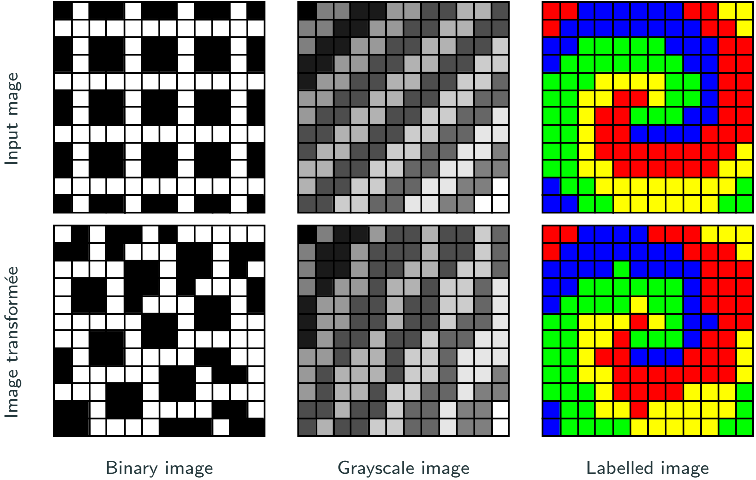
Regularization of images by homotopic transformation



Input image Prohibited config. Transformed image Regular Image Transformed image

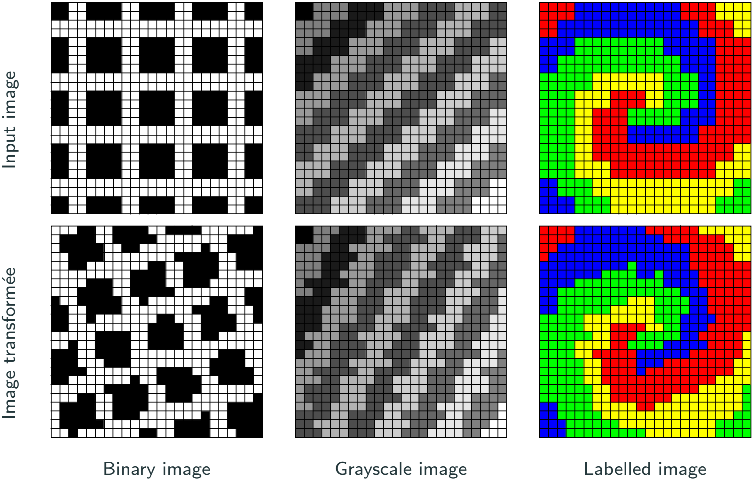
Regularization of images by homotopic transformation

No solution in the cases at the limit of the resolution:

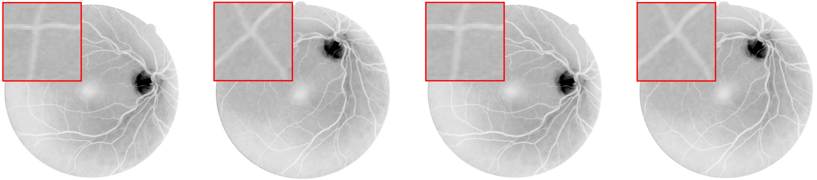


Regularization of images by oversampling

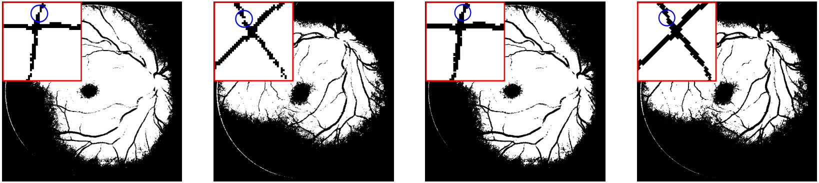
By doubling the resolution, well-composed images become regular:



Some experimental results



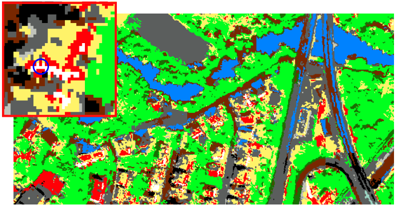
Input image Transformed image Regular image Transformed image



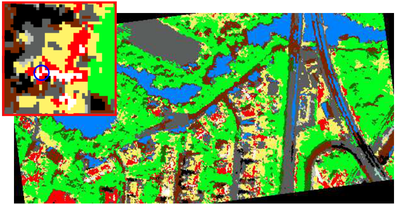
Thresholded images

Some experimental results

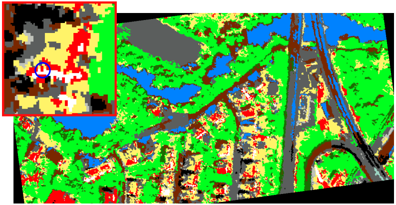
Input image



Regular image



Transformed image

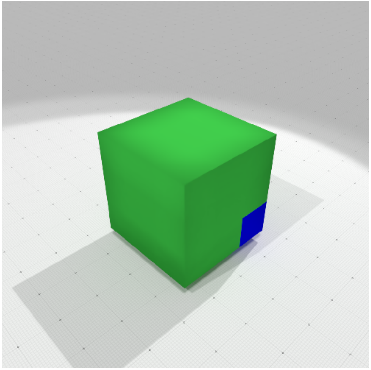


Transformed image

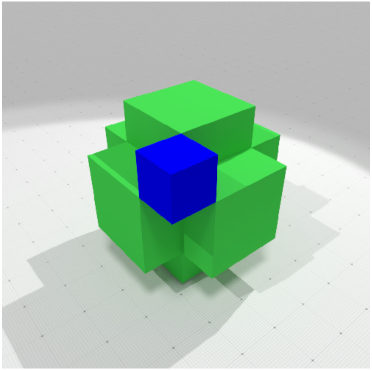
Extending the regularity in 3D

The 3D extension of the regularity would be to consider **a cover of cubes** $2 \times 2 \times 2$ that locally overlap everywhere.

Is such an object in $\mathbb{Z}^3$ topologically invariant? $\rightarrow$ No!



Regular object



Transformed object

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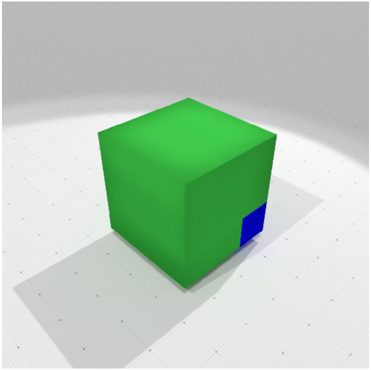
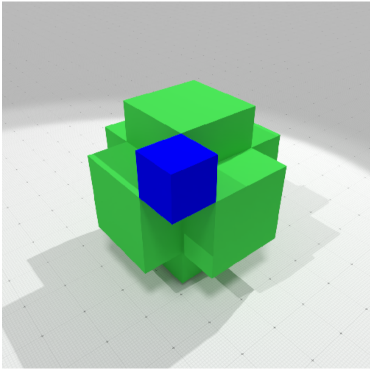


Regular object



Transformed object

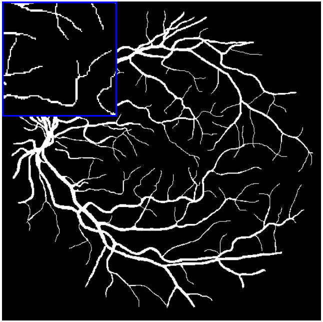
Topological characterizations of digital images

The point-to-point rigid motion model: $\mathcal{R}_{Point} = \mathfrak{D} \circ \mathfrak{R}_{|\mathbb{Z}^d}$

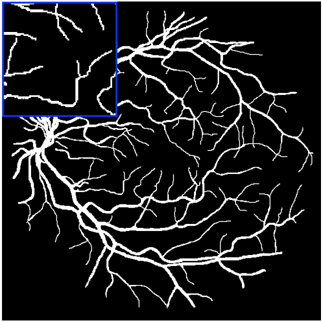
✓ Simple and easy to apply on digital images

✓ The notion of regularity allows a characterization of 2D images whose topological properties are preserved by $\mathcal{R}$

↪ Regularization: homotopic transformation or oversampling



Input image (non regular)

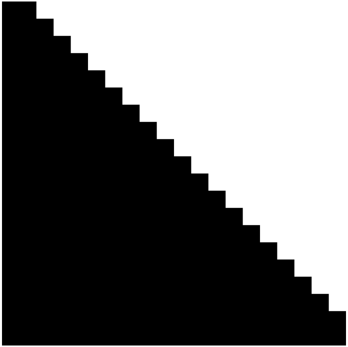


Regularized image

Topological characterizations of digital images

The point-to-point rigid motion model: $\mathcal{R}_{Point} = \mathfrak{D} \circ \mathfrak{R}_{|\mathbb{Z}^d}$

- ✗ The notion of regularity reaches its limit of validity by passing to $\mathbb{Z}^3$
- ✗ Geometric properties are not well preserved in general



Digitized rotations of a half-plane: linearity and convexity problems

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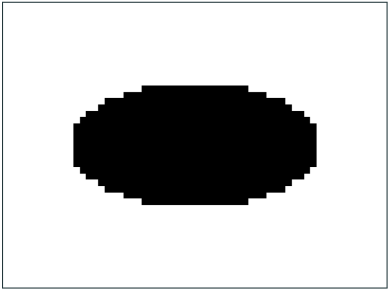
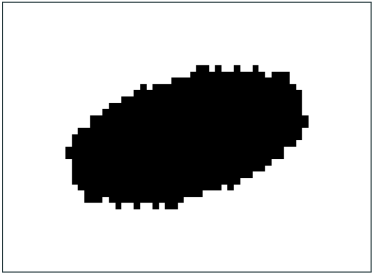


Ellipse



Transformed ellipse

Geometrical aspect

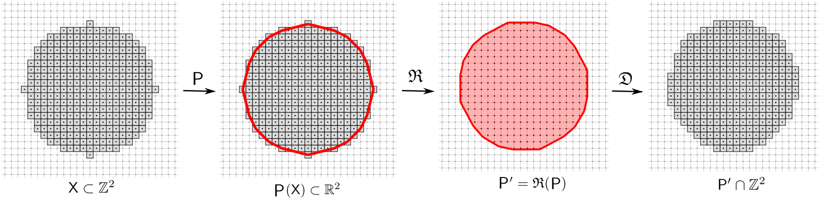
Geometrical preservation of rigid motion for discrete objects

New solutions for rigid transformations on $\mathbb{Z}^2$ and $\mathbb{Z}^3$:

$\hookrightarrow$ with **intermediate models** to transform a discrete object

$\hookrightarrow$ **better preserves the shape** of the object by the transformation

Polygons to represent the object's shape and used it for the transformation.



Digitalisation process

The transformed object model need to be digitized for a result in $\mathbb{Z}^d$.

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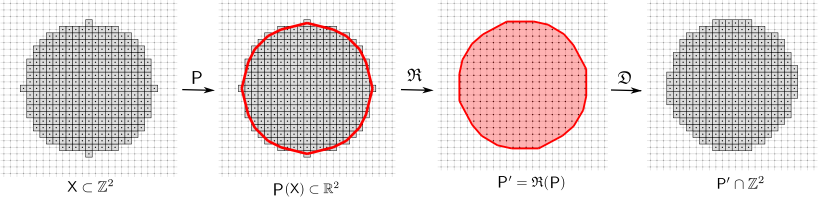

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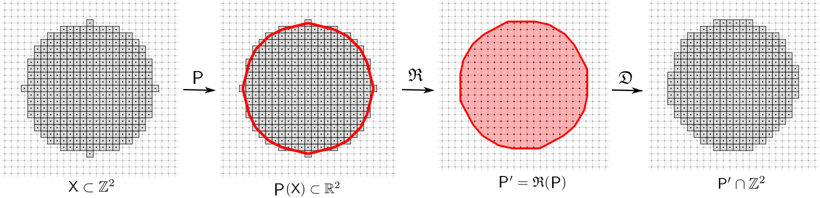

Digitalisation process

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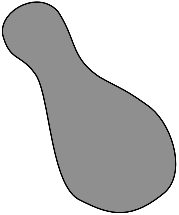
Digitization and topology preservation

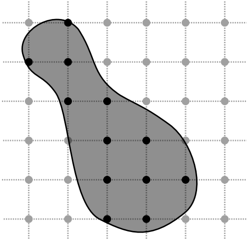
Definition [Klette and Rosenfeld, 2004]

Given a bounded and connected subset $\mathcal{X} \subset \mathbb{R}^d$, for $d \geq 2$, the **Gauss digitization** of $\mathcal{X}$ is a discrete object X defined as:

$$\mathsf{X} = \mathcal{X} \cap \mathbb{Z}^d$$



$\mathcal{X} \subset \mathbb{R}^2$



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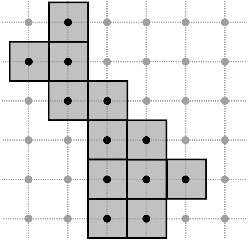

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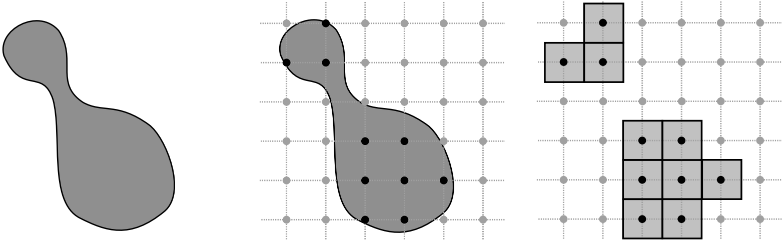

$\mathcal{X} \subset \mathbb{R}^2$ $\quad$ $\mathsf{X} = \mathcal{X} \cap \mathbb{Z}^2$ $\quad$ $\mathsf{X} \subset \mathbb{Z}^2$

Topology of the object can be altered under the digitization process.

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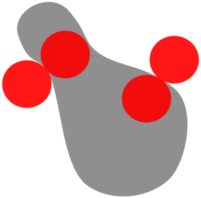
Questions

- What are conditions for continuous objects to preserve their topology under Gauss digitization?
- How to verify such conditions for a given continuous object?
- How to perform shape-preserving rigid motion of discrete objects?

r-regularity

Définition [Pavlidis, 1982]

A finite and connected subset $\mathcal{X} \subset \mathbb{R}^2$ is r-**regular** if for each boundary point of $\mathcal{X}$, there exist two tangent open balls of radius r, lying entirely in $\mathcal{X}$ and its complement $\overline{\mathcal{X}}$, respectively.



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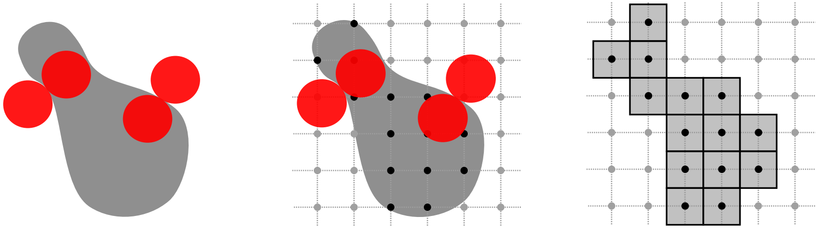

Proposition [Latecki et al., 1998]

If $\mathcal{X} \subset \mathbb{R}^2$ is r-regular, for $r \geq \frac{\sqrt{2}}{2}$, then $\mathrm{X} = \mathcal{X} \cap \mathbb{Z}^2$ is a well-composed.

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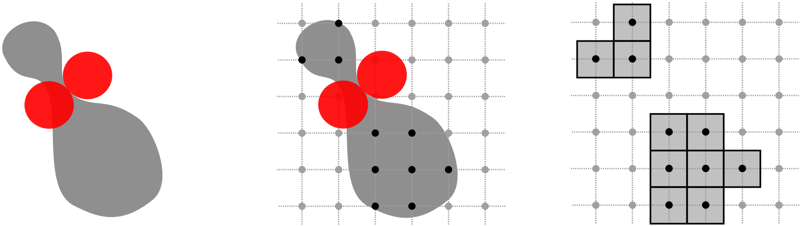

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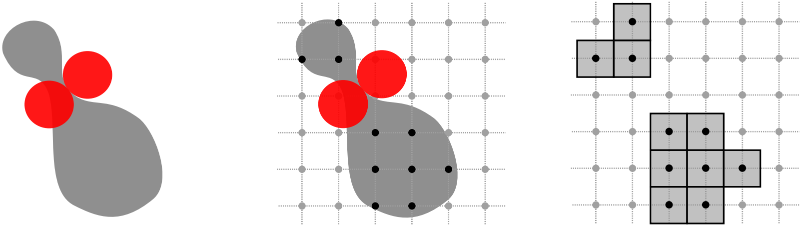


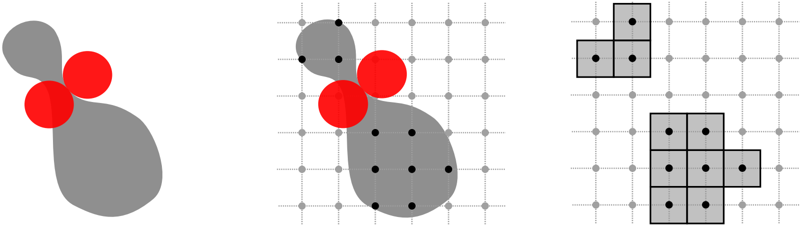
Object $\mathcal{X}$ must have a differentiable boundary.

What about objects with non-differentiable boundary (e.g. polygons)?

r-regularity

Définition [Pavlidis, 1982]

A finite and connected subset $\mathcal{X} \subset \mathbb{R}^2$ is ***r*-regular** if for each boundary point of $\mathcal{X}$, there exist two tangent open balls of radius r, lying entirely in $\mathcal{X}$ and its complement $\overline{\mathcal{X}}$, respectively.



Object $\mathcal{X}$ must have a differentiable boundary.

What about objects with non-differentiable boundary (e.g. polygons)?

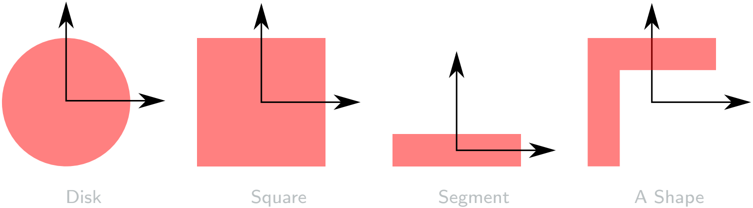
Morphological Operations: Erosion and dilation

Mathematical morphology [Serra, 1983]

The basic idea of mathematical morphology is to **compare** the set to be analyzed with a set with a known geometry called **structuring element**.

Structuring element B is a set with the following characteristics:

- has a known geometry,
- has a certain size $r > 0$,
- is located by its origin.



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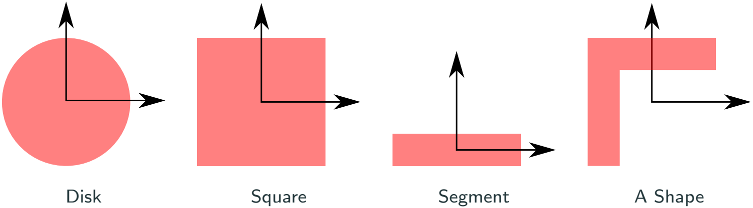

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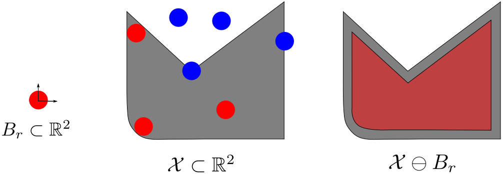
Definition [Serra, 1983]

Let $\mathcal{X} \subset \mathbb{R}^2$ be a set, and B be a structuring element located by its origin. The **erosion** of $\mathcal{X}$ by B in a space E is

$$\mathcal{E}_B(\mathcal{X}) = \mathcal{X} \ominus B = \{x \in E \mid B_x \subseteq \mathcal{X}\}$$

where B_x is the translation of B by x.

The erosion is a transformation relative to the inclusion.



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$$\delta_B(\mathcal{X}) = \mathcal{X} \oplus B = \{x \in E \mid B_x \cap \mathcal{X} \neq \emptyset\}$$

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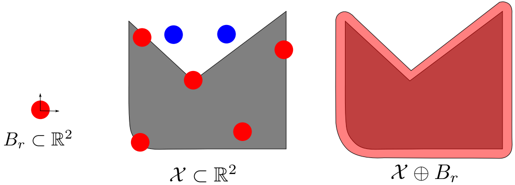


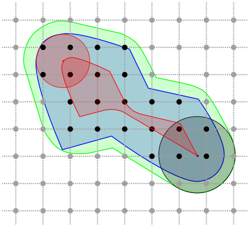
Quasi-r-regularity

Definition [Ngo et al., 2019]

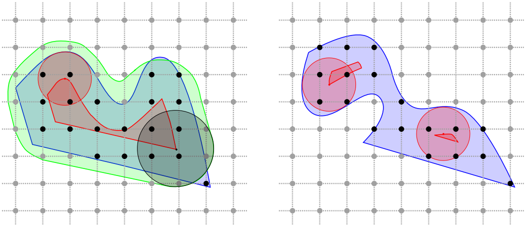
Let $\mathcal{X} \subset \mathbb{R}^2$ be a finite and simply connected set (*i.e.* connected and without hole). $\mathcal{X}$ is **quasi-r-regular** with *margin* $r' - r$, if

- $\mathcal{X} \ominus B_r$ (resp. $\overline{\mathcal{X}} \ominus B_r$) is non-empty and connected, and
- $\mathcal{X} \subseteq \mathcal{X} \ominus B_r \oplus B_{r'}$ (resp. $\overline{\mathcal{X}} \subseteq \overline{\mathcal{X}} \ominus B_r \oplus B_{r'}$)

où $\oplus, \ominus$ are the dilation and erosion operators and $B_r, B_{r'} \subset \mathbb{R}^d$ are respectively the balls of radius r and r', for $r' \geq r > 0$.



Quasi-r-regular object



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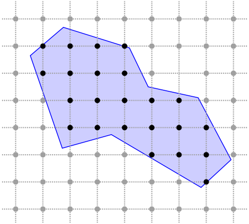
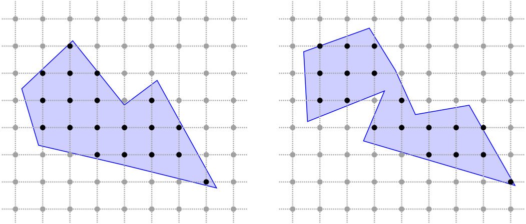


Quasi-r-regular polygon



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Proposition [Ngo et al., 2019]

If $\mathcal{X}$ is quasi-1-regular with margin $\sqrt{2} - 1$ (also called **quasi-regular**), then $\mathsf{X} = \mathcal{X} \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{\mathcal{X}} \cap \mathbb{Z}^2$ are both 4-connected. In particular, X is then well-composed.

Verify the quasi-regularity of polygonal objects? $\rightsquigarrow$ Medial axis

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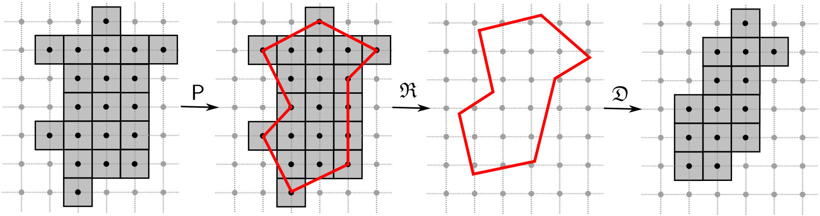
Topology and geometry preserving rigid motion on $\mathbb{Z}^2$



Approach via polygonization

- polygonal representation of discrete objects for rigid motion
- shape preservation of transformed object by the transformation
- quasi-regularity for topology preservation of object by the digitization

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If P is quasi-regular, then $\mathfrak{R}(\mathrm{P}) \cap \mathbb{Z}^2$ preserves connectivity.

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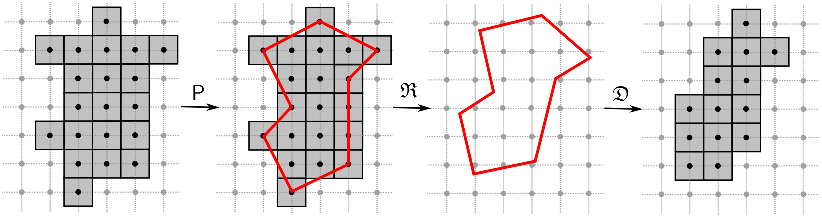

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Polygonalization method

Polygonal representation

The properties to satisfy for computing a polygonal representation $P(\mathrm{X})$ of a discrete object $\mathrm{X} \subset \mathbb{Z}^2$ are

- reversibility : $P(\mathrm{X}) \cap \mathbb{Z}^2 = \mathrm{X}$;
- vertices with rational coordinates (exact calculation).

For an object $\mathrm{X} \subset \mathbb{Z}^2$, different results can be obtained from different polygonalization techniques:

- Digital convex objects: convex hull + representation by half-planes
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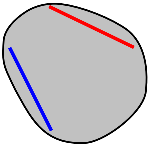
Digital convexity

Definition

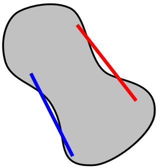
An object $\mathcal{X} \subset \mathbb{R}^2$ is said to be **convex** if, for any pair of points $\mathtt{x}, \mathtt{y} \in \mathcal{X}$, the line segment joining $\mathtt{x}$ and $\mathtt{y}$, defined by

$$[\mathtt{x}, \mathtt{y}] = \{\lambda \mathtt{x} + (1-\lambda)\mathtt{y} \in \mathbb{R}^2 \mid 0 \leq \lambda \leq 1\},$$

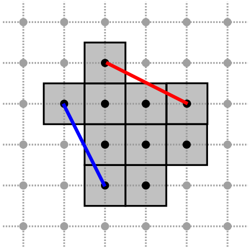
is included in $\mathcal{X}$.



Convex object in $\mathbb{R}^2$



Non-convex objet in $\mathbb{R}^2$



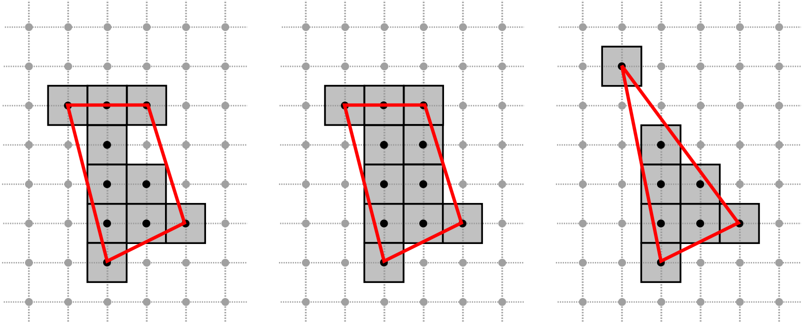
in $\mathbb{Z}^2$

Digital convexity

Definition [Kim, 1981]

A digital object $X \subset \mathbb{Z}^2$ is **H-convex**, for Conv(X) the convex hull of X

$$X = \mathrm{Conv}(X) \cap \mathbb{Z}^2$$



Non H-convex object H-convex objects

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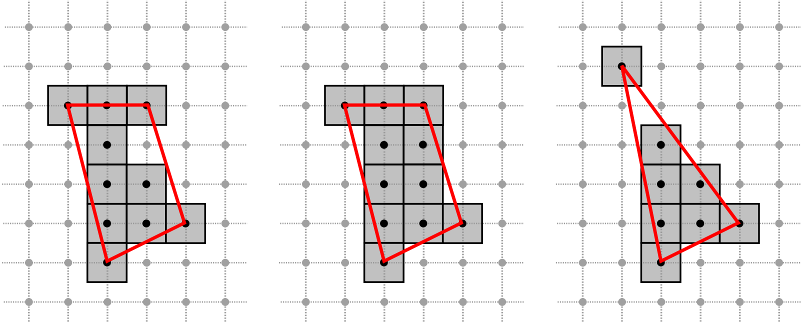

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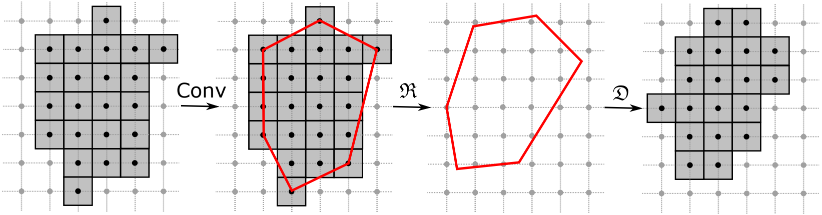
Convexity under rigid motion

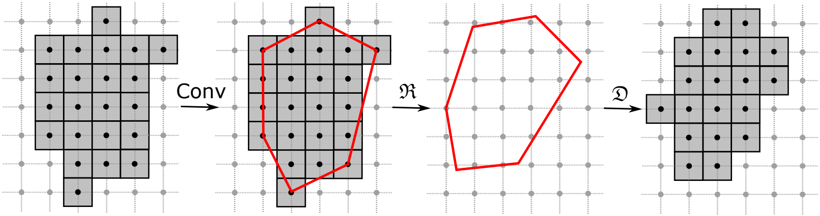


Proposition [Ngo et al., 2019]

Soit $X \subset \mathbb{Z}^2$ connexe et bien composé et Conv(X) son enveloppe convexe. Si X est *convexe* (*i.e.* $X = \text{Conv}(X) \cap \mathbb{Z}^2$) et Conv(X) est quasi-régulier, alors $\mathfrak{R}(\text{Conv}(X)) \cap \mathbb{Z}^2$ est convexe et bien composé.

The half-plane representation $\rightsquigarrow$ Gauss discretization in exact calculation!

Convexity under rigid motion



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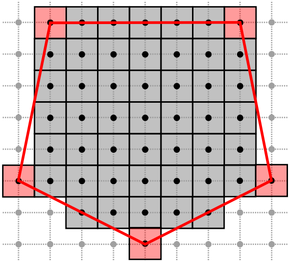
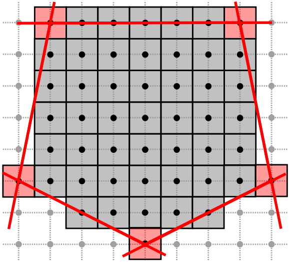
Half-plane representation of H-convex object

Let X be a H-convex object and Conv(X) be the convex hull of X. Then,

$$X = \mathrm{Conv}(X) \cap \mathbb{Z}^2 = \left(\bigcap_{\mathbb{H} \in R(X)} \mathbb{H} \right) \cap \mathbb{Z}^2 = \bigcap_{\mathbb{H} \in R(X)} \left(\mathbb{H} \cap \mathbb{Z}^2 \right)$$

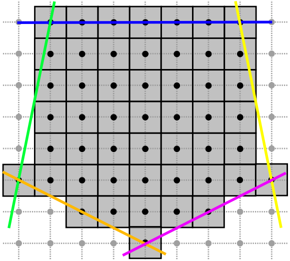
where R(X) is the minimal set of closed half-planes including X. Each half-plane $\mathbb{H}$ has coefficients defined by consecutive vertices of Conv(X).

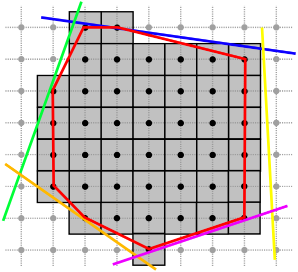




Rigid motion of H-convex objects via convex hull

$$\mathcal{R}_{\mathsf{Conv}}(\mathsf{X}) = \mathfrak{R}(\mathsf{Conv}(\mathsf{X})) \cap \mathbb{Z}^2 = \mathfrak{R}\left(\bigcap_{\mathrm{H} \in \mathsf{R}(\mathsf{X})} \mathrm{H}\right) \cap \mathbb{Z}^2$$





Property [Ngo et al., 2019]

$$\mathsf{Conv}(\mathcal{R}_{\mathsf{Conv}}(\mathsf{X})) \subseteq \mathfrak{R}(\mathsf{Conv}(\mathsf{X}))$$

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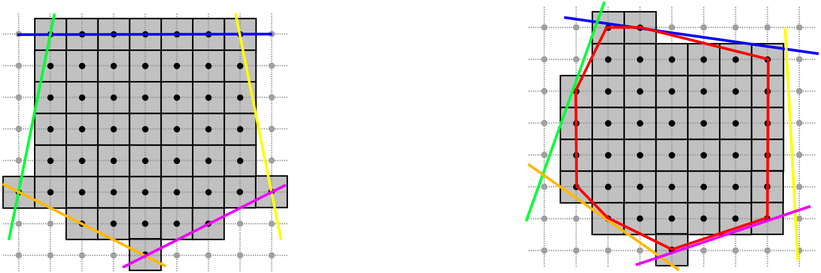

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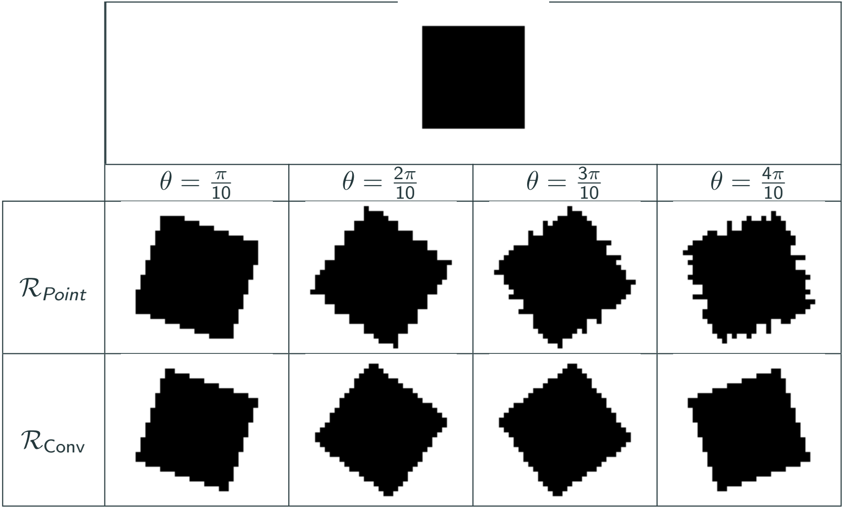
Experimental results

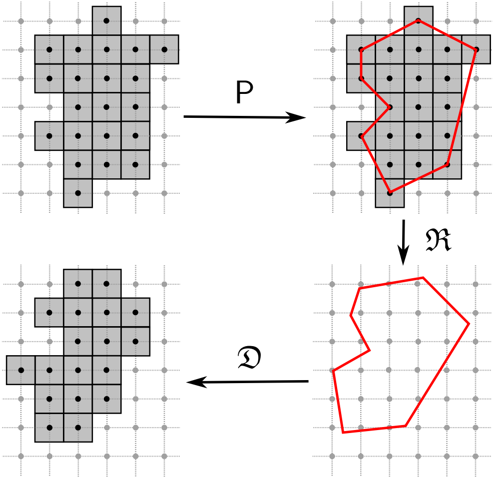


Experimental results

	$\theta = \frac{\pi}{10}$	$\theta = \frac{2\pi}{10}$	$\theta = \frac{3\pi}{10}$	$\theta = \frac{4\pi}{10}$
$\mathcal{R}_{Point}$				
$\mathcal{R}_{\text{Conv}}$				

Rigid motion of non-convex objects

$$\mathcal{R}_{\mathsf{Poly}}(X) = \mathfrak{R}(\mathsf{Poly}(X)) \cap \mathbb{Z}^2$$



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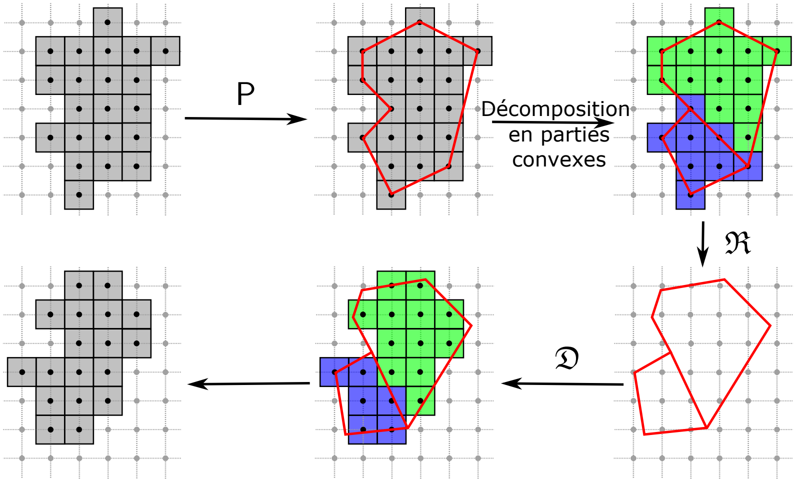
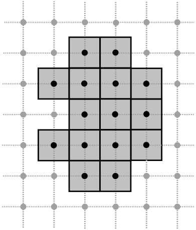


Polygonization of digital objects

Marching square method: Local configurations + lookup table (LUT)



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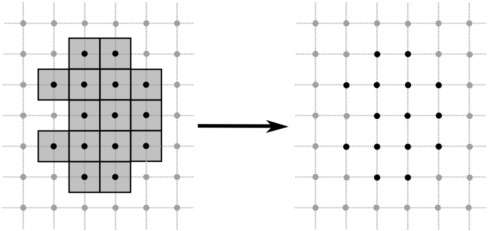

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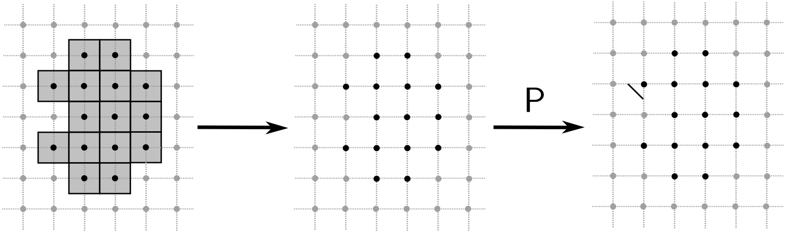

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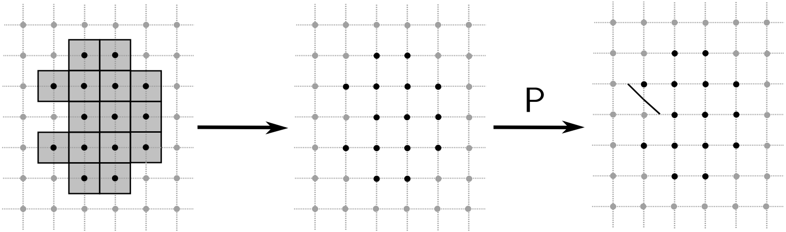

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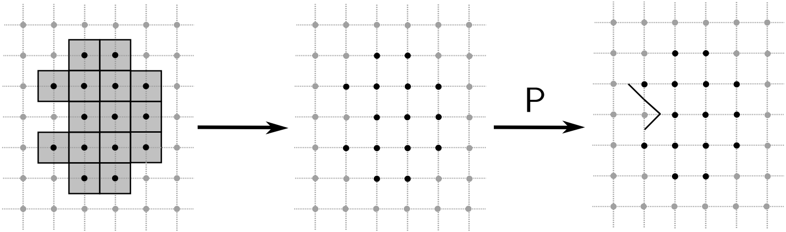

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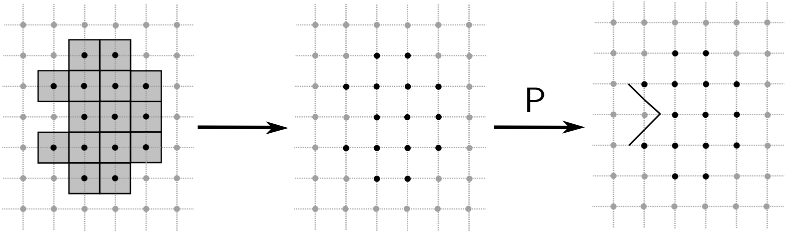

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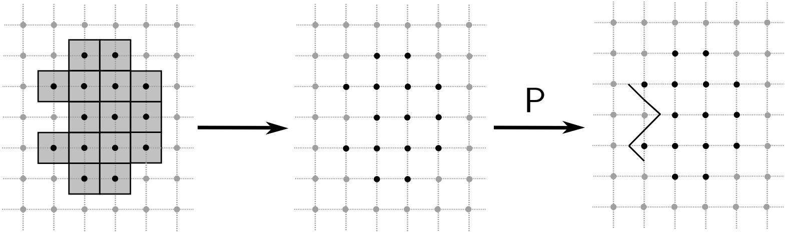

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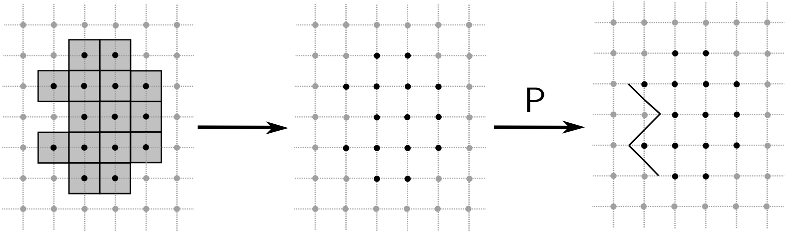

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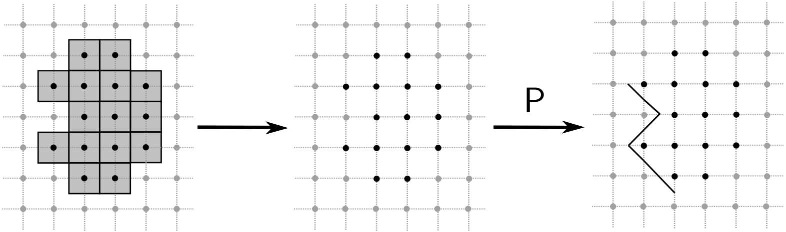

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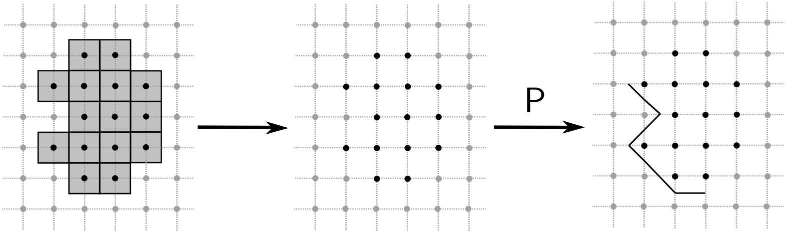

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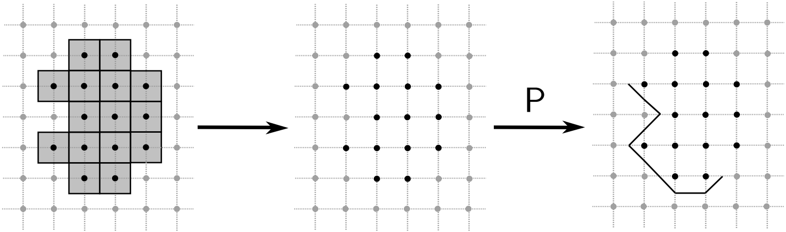

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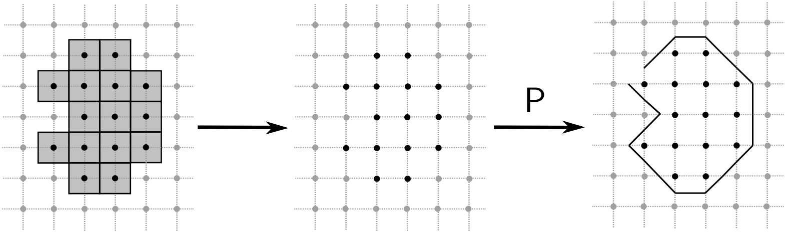

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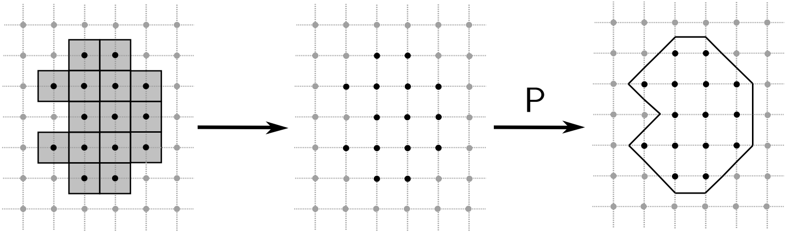

Polygonization of digital objects

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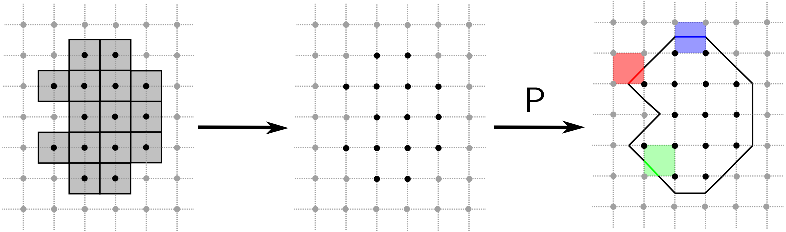

Polygonization of digital objects

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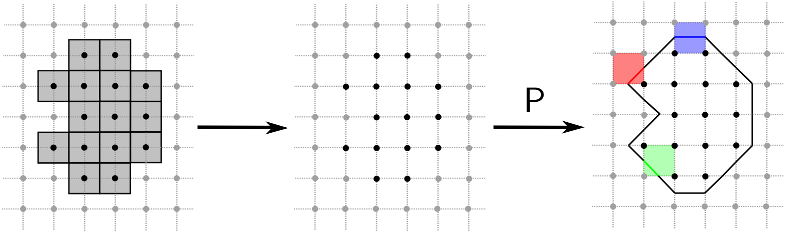
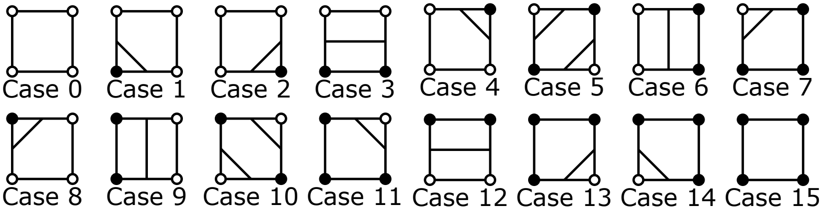


Look-up table contour lines



Polygonization of digital objects

Marching square method [Maple, 2003]

Generation of iso-contours for 2D scalar field (e.g., gray-scale images)

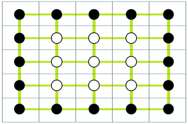
- Compute a binary image of the 2D field for an isovalue by a threshold
- Create contouring cells by 2x2 block of pixels in the binary image
  - ↪ Compute the binary code (=cell index) of each contouring cell
  - ↪ Access a pre-built LUT with the cell index for the contour lines
  - ↪ Apply interpolation between the original 2D field to find the exact contour lines

1	1	1	1	1
1	2	3	2	1
1	3	3	3	1
1	2	3	2	1
1	1	1	1	1

Threshold with iso-value →

0	0	0	0	0
0	1	1	1	0
0	1	1	1	0
0	1	1	1	0
0	0	0	0	0

Binary image to cells →



(Illustration from Wikipedia)

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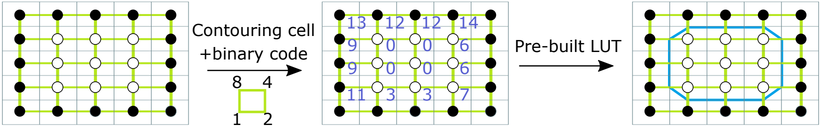

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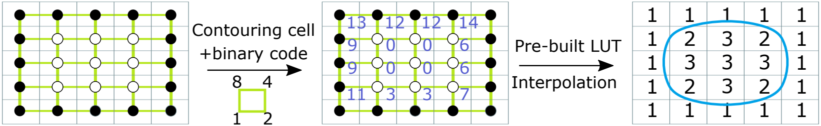
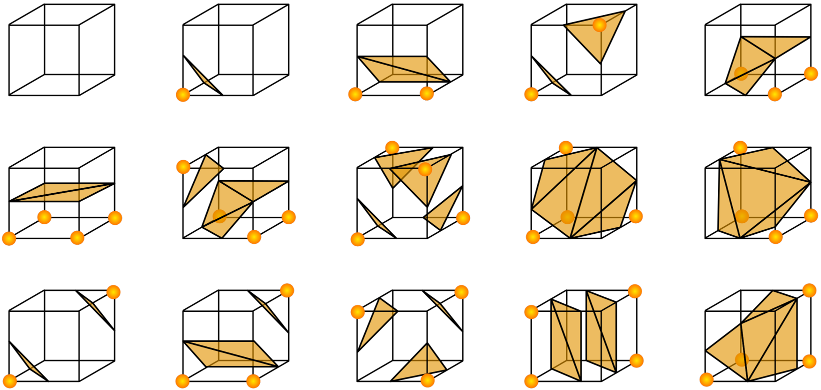


(Illustration from Wikipedia)

Polygonization of digital objects

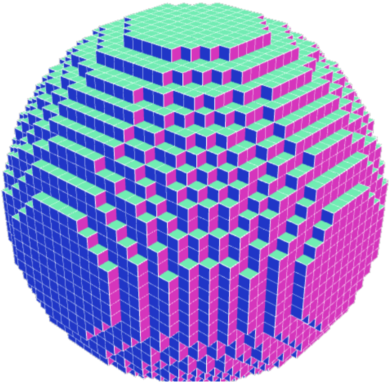
Extension to 3D: Marching cube method [Maple, 2003]

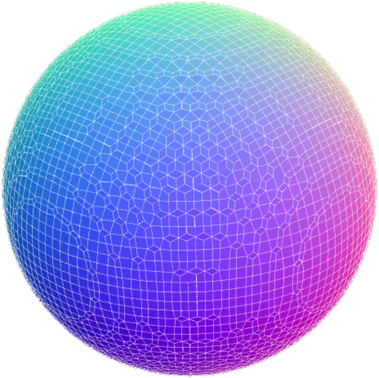


(Illustration from Wikipedia)

Polygonization of digital objects

Extension to 3D: Marching cube method [Maple, 2003]





(Illustration from Wikipedia)

Polygonization of digital objects

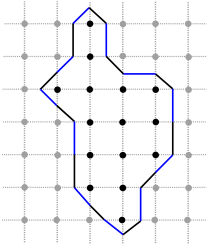
Marching square/cube method [Maple, 2003]

Advantages

- Simple and easy to implement
- Linear computation w.r.t image size
- Exact computation: polygon vertices with rational coordinates
- Extension to dimension 3

Disadvantages

- Polygon is composed of small segments
- It may not optimal/fit to the digital form



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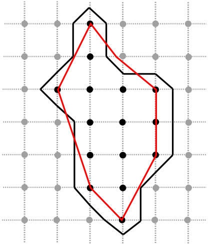
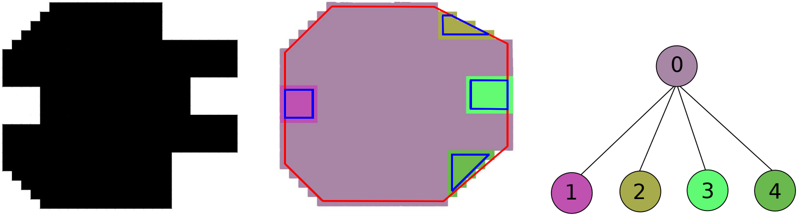


Polygonization of digital objects

Concavity tree by Sklansky [Sklansky, 1972]

- ↪ decompose an object into concavities
- ↪ encode description of a binary image
- ↪ possible to process each one separately
- ↪ measure/compare the concavities of digital objects



Input object — Concave parts — Concavity tree

Polygonization of digital objects

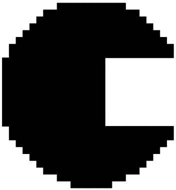
Concavity tree method [Sklansky, 1972]

Concavity tree structure for a digital object X:

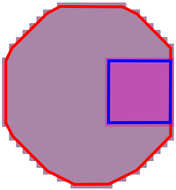
- The root corresponds to points in the convex hull: $\mathrm{Conv}(X) \cap \mathbb{Z}^2$
- Each node corresponds to points in the convex hull of a concave part (i.e., a connected component $\mathfrak{C}$) of its parents.

Then, X is represented as follows:

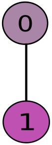
$$X = \left(\mathrm{Conv}(X) \cap \mathbb{Z}^2\right) \setminus \left(\bigcup_{X' \in \mathfrak{C}((\mathrm{Conv}(X) \cap \mathbb{Z}^2) \setminus X)} X' \right)$$



Input object



Concave parts



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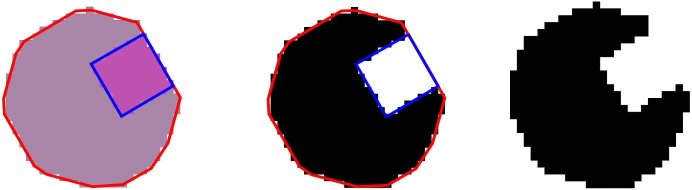


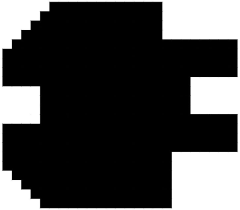
Transformed concave parts — Reconstructed object — Transformed object

Polygonization of digital objects

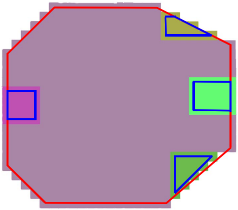
Concavity tree method [Sklansky, 1972]

Advantages

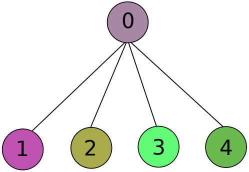
- Structural and hierarchical descriptions of 2D shape
- H-convex object = convex hull of the shape
- Exact computation: polygon vertices with integer coordinates
- Possible extension to dimension 3



Input object



Concave parts



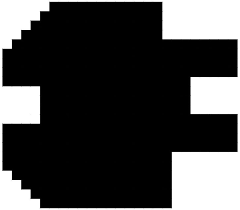
Concavity tree

Polygonization of digital objects

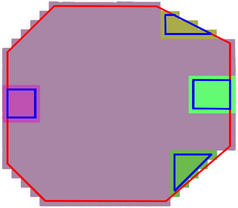
Concavity tree method [Sklansky, 1972]

Disadvantages

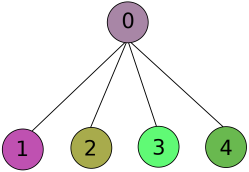
- Data structure for the concavity tree
- Operations performed to reconstruct the digital object
- Artifacts when applying geometric transformations on the structure



Input object



Concave parts



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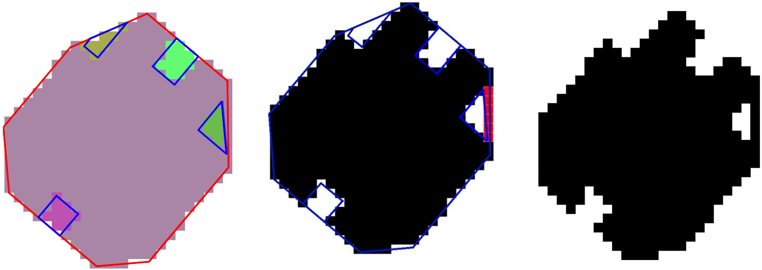


Transformed concave parts   Reconstructed object   Transformed object

Polygonization of digital objects

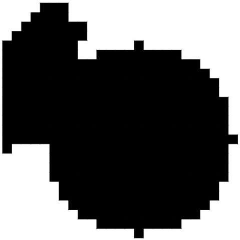
Contour-based polygonization:

↪ Extract 8-connected contour points $C(\mathrm{X})$ of X

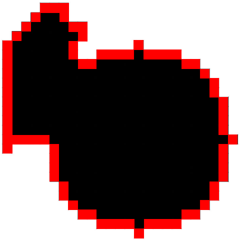
↪ Compute convex hull of $C(\mathrm{X})$ as part of $P(\mathrm{X})$

↪ Determine the polygon segments of $P(\mathrm{X})$ from the contour points that best fit the concave parts of X

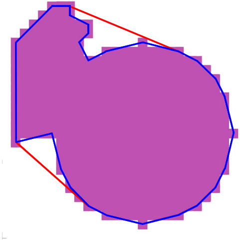
$$\mathrm{X} = P(\mathrm{X}) \cap \mathbb{Z}^2$$



Input object



8-connected contour

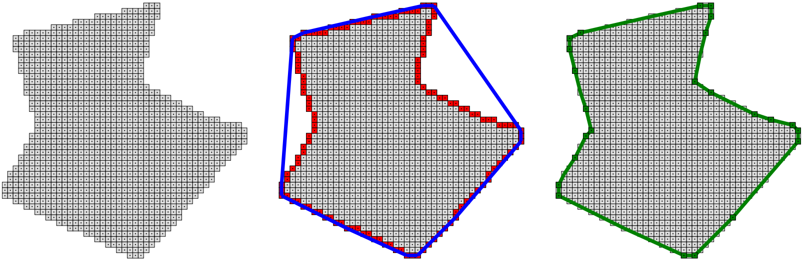


Polygon curve

Polygonization of digital objects

Contour-based polygonization

- Extract 8-connected contour $C(\mathrm{X})$ of X and compute $\mathrm{Conv}(C(\mathrm{X}))$
- Initialize $P(\mathrm{X})$ with $\mathrm{Conv}(C(\mathrm{X}))$ (in CW order), for each segment $[p_i, p_{i+1}] \in P(\mathrm{X})$, select $p \in C(p_i, p_{i+1})$, $C(\mathrm{X})$ between p_i, p_{i+1}, s.t.
$$p = \underset{q \in C(p_i, p_{i+1}) \setminus P}{\arg\max} \left\{ d(p_i, q) \mid \left(\Delta p_i q r \cap \mathbb{Z}^2\right) \cap \overline{\mathrm{X}} = \emptyset \wedge r \in C(p_i, q) \right\}$$
with $d(.,.)$ the Euclidean distance, $\Delta p_i q r$ the triangle whose vertices are p_i, q, r.



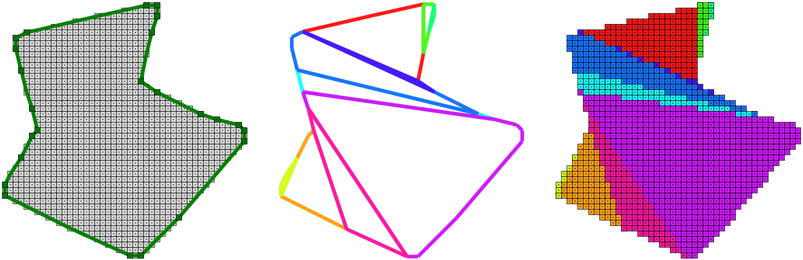
Convex decomposition of polygons

Convex decomposition [Lien and Amato, 2006]

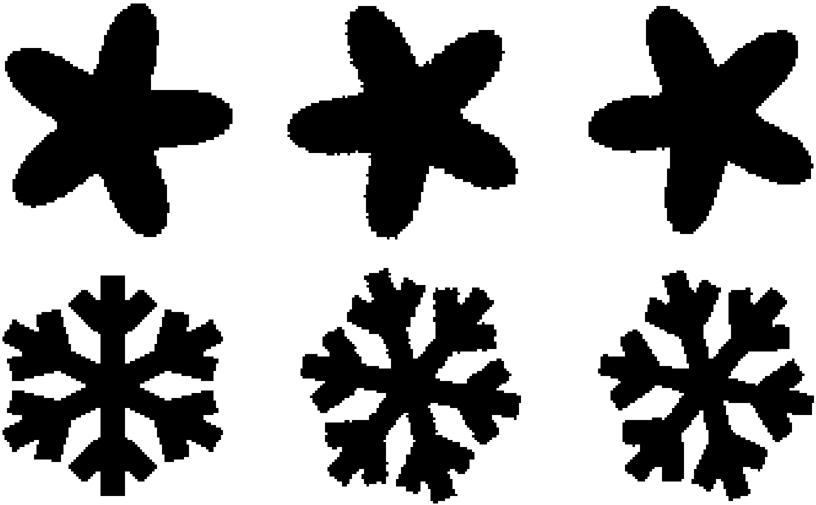
The method decomposes a simple polygon into convex pieces by iteratively removing the most significant non-convex features.

$$P = \bigcup P_i$$

$$\mathrm{X} = P(\mathrm{X}) \cap \mathbb{Z}^2 = \bigcup (P_i \cap \mathbb{Z}^2).$$



Experimental results of rigid motions on $\mathbb{Z}^2$



$X \subset \mathbb{Z}^2$ $\mathcal{R}_{Point}$ $\mathcal{R}_{Poly}$

Experimental results of rigid motions on $\mathbb{Z}^2$

$X \subset \mathbb{Z}^2$ $\qquad$ $\mathcal{R}_{Point}$ $\qquad$ $\mathcal{R}_{Poly}$

Extension to 3D

Definition [Ngo et al., 2019]

Let $\mathcal{X} \subset \mathbb{R}^3$ be a bounded, simply connected set. $\mathcal{X}$ is *quasi-r-regular* with *margin* $r' - r$, for $r' \geq r > 0$, if

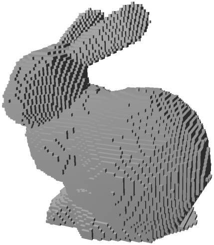
- $\mathcal{X} \ominus B_r$ (resp. $\overline{\mathcal{X}} \ominus B_r$) is non-empty and connected, and
- $\mathcal{X} \subseteq \mathcal{X} \ominus B_r \oplus B_{r'}$ (resp. $\overline{\mathcal{X}} \subseteq \overline{\mathcal{X}} \ominus B_r \oplus B_{r'}$)

où $\oplus, \ominus$ are the dilation and erosion operators and $B_r, B_{r'} \subset \mathbb{R}^d$ are respectively the balls of radius r and r'.

Proposition [Ngo et al., 2019]

Let $\mathsf{X} \subset \mathbb{Z}^3$ be a digital object. If X is quasi-1-regular with margin $\frac{2}{\sqrt{3}} - 1$, then $\mathsf{X} = \mathcal{X} \cap \mathbb{Z}^3$ and $\overline{\mathsf{X}} = \overline{\mathsf{X}} \cap \mathbb{Z}^3$ are both 6-connected.

Proposed method of rigid motions on $\mathbb{Z}^3$

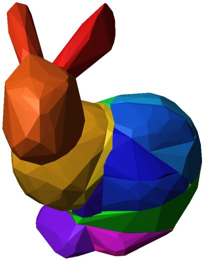


Polyhedrization of voxels

$\Longrightarrow$

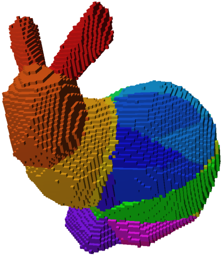
Convex decomposition

$\Downarrow$ **Rigid motion**



(Re)digitization

$\Longleftarrow$



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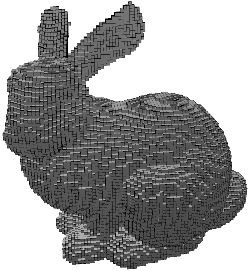
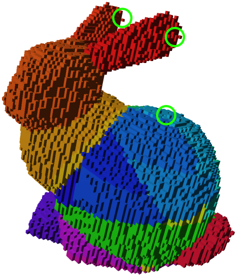
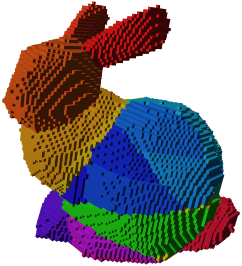


Input object



$\mathcal{R}_{Point}$



$\mathcal{R}_{Pomy}$: quasi-regular polyhedron

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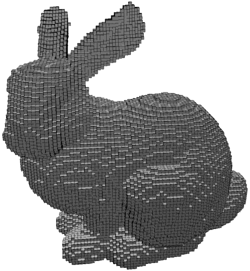

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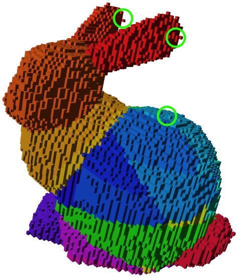

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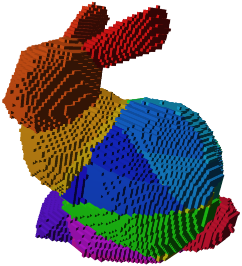

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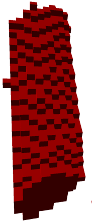

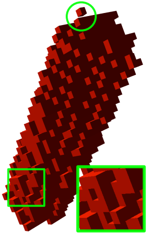

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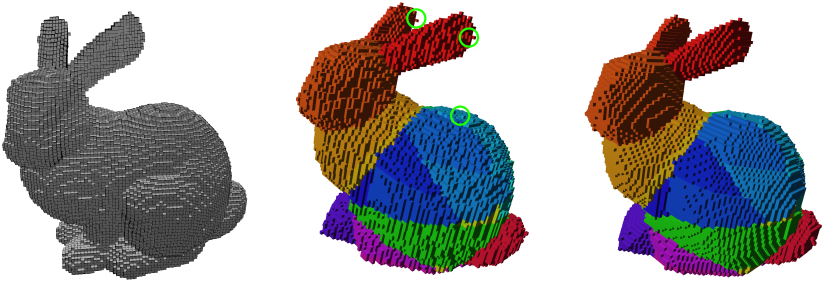
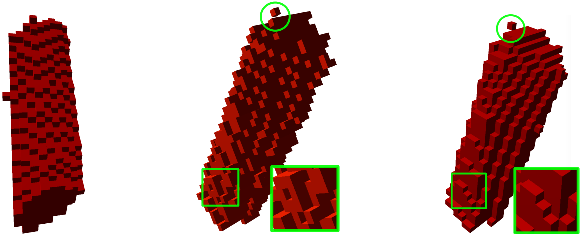


Input object — $\mathcal{R}_{Point}$ — $\mathcal{R}_{Pomy}$: quasi-regular polyhedron



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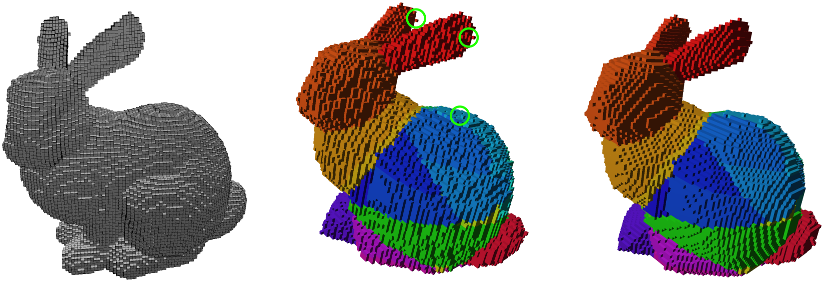

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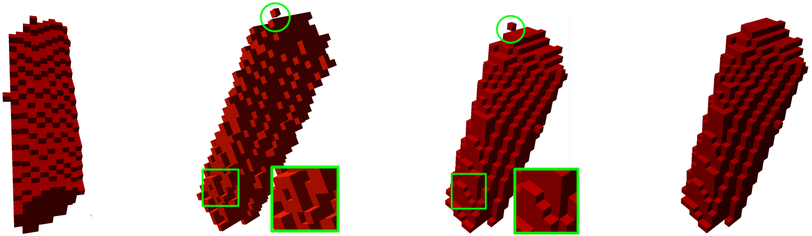

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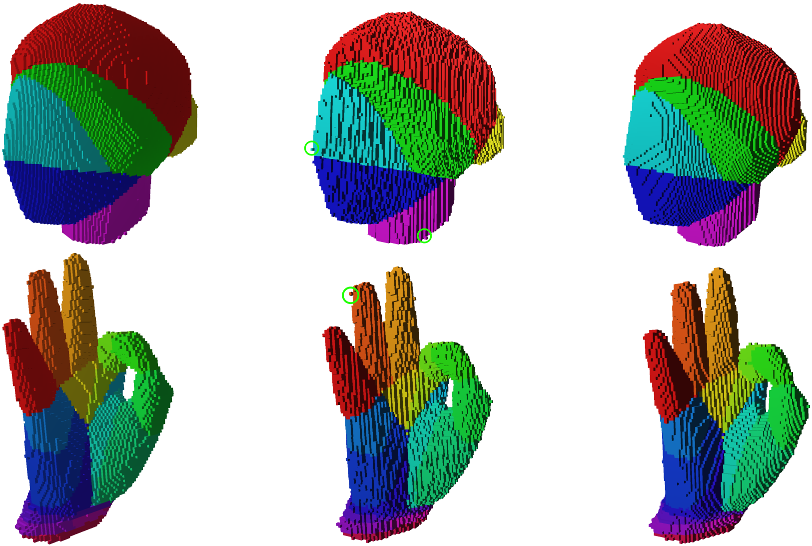


Affine transformation

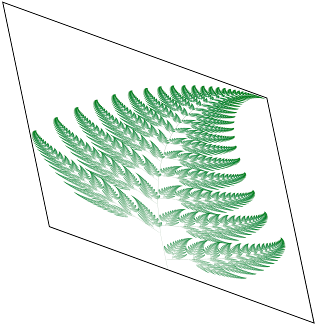
Affine transformation on $\mathbb{R}^2$

Definition

An affine transformation $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined, for any $\mathrm{p} \in \mathbb{R}^2$, by

$$\mathcal{A}(\mathrm{p}) = \mathrm{A} \cdot \mathrm{p} + t = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{p}_x \\ \mathrm{p}_y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

where $t = (t_x, t_y)^t \in \mathbb{R}^2$, $\mathrm{A} = [a_{i,j}]_{1 \leq i,j \leq 2}$, $\det(\mathrm{A}) \neq 0$, and $a_{i,j} \in \mathbb{R}$.



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The affine transformations include, in particular:

- translations ($\mathrm{A} = I_2$) ; et
- when $t = 0$:
  - ↪ rotations ($a_{11} = a_{22} = \cos\theta$, $-a_{12} = a_{21} = \sin\theta$ pour $\theta \in \mathbb{R}$) ;
  - ↪ symmetries ($a_{11} = \pm 1, a_{22} = \pm 1$, $a_{12} = a_{21} = 0$) ;
  - ↪ scalings ($a_{11} \neq 0$, $a_{22} \neq 0$ and $a_{12} = a_{21} = 0$) ;

and their compositions (*e.g.* rigid transformation: rotation + translation)

Affine transformation on $\mathbb{Z}^2$

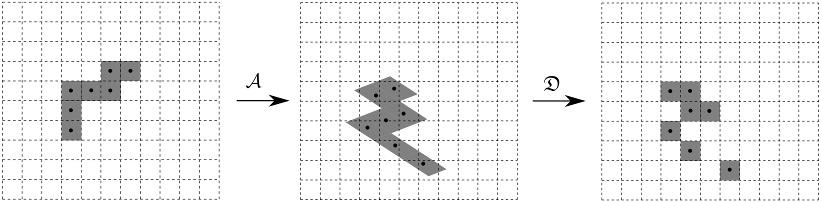
Definition

A digitized affine transformation $\mathsf{A} : \mathbb{Z}^2 \to \mathbb{Z}^2$ is defined as

$$\mathsf{A} = \mathfrak{D} \circ \mathcal{A}_{|\mathbb{Z}^2}$$

where $\mathfrak{D}$ is a digitization defined with the rounding operation:

$$\begin{array}{rclcl} \mathfrak{D} & : & \mathbb{R}^2 & \longrightarrow & \mathbb{Z}^2 \\ & & \mathrm{p} = (\mathrm{p}_x, \mathrm{p}_y) & \longmapsto & \mathrm{q} = (\mathrm{q}_x, \mathrm{q}_y) = ([\mathrm{p}_x], [\mathrm{p}_y]) \end{array}$$



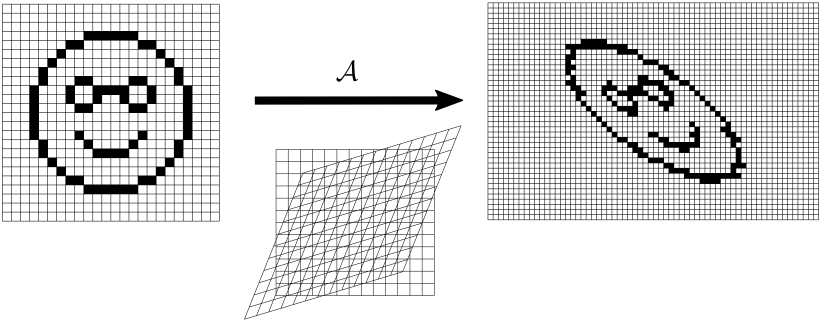
Digitized transformations can alter the topology of the transformed object.

Affine transformation on $\mathbb{Z}^2$

Goal

Given a binary object X and $\mathcal{A}$ an affine transformation, construct a transformed binary object $X_\mathcal{A}$ **preserving the homotopy type**.

The problem is formulated as an **optimization in the refined space** of the initial and transformed grids, called the *space of cellular complexes*.



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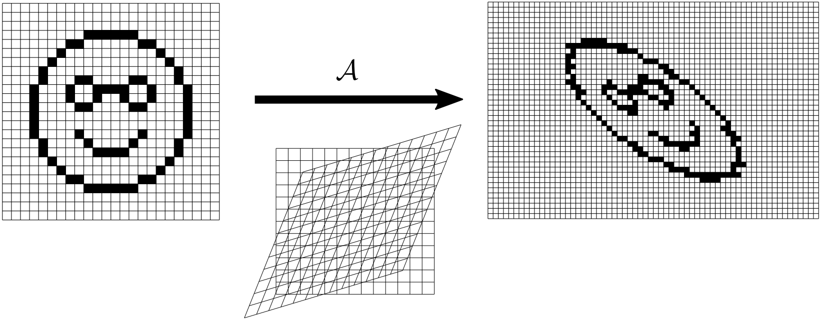

Problem formulation

Reaching the most similar $X_{\mathcal{A}} \subset \mathbb{Z}^2$ to $\mathcal{A}(X)$ can be formalized as:

$$X_{\mathcal{A}} = \arg_{Y \in 2^{\mathbb{Z}^2}} \min \mathcal{D}_{\mathcal{A},X}(Y)$$

where $\mathcal{D}_{\mathcal{A},X}(Y)$ is a dissimilarity measure between $\mathcal{A}(X)$ and Y.

Example of dissimilarity measure

Based on Gauss digitization:

$$\mathcal{D}^{\boxdot}_{\mathcal{A},X}(Y) = | \boxdot (\mathcal{A}(\square(X))) \setminus Y| + |Y \setminus \boxdot(\mathcal{A}(\square(X)))|$$

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- **Continuous analogue of $X \subset \mathbb{R}^2$:** $\square(X) = X \oplus \square = X$
  - $\hookrightarrow$ $\oplus$ is the dilation operator and
  - $\hookrightarrow$ $\square$ is the structuring element $[\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$.
- **Gauss digitization of $X \subset \mathbb{R}^2$:** $\boxdot(X) = X \cap \mathbb{Z}^2$.

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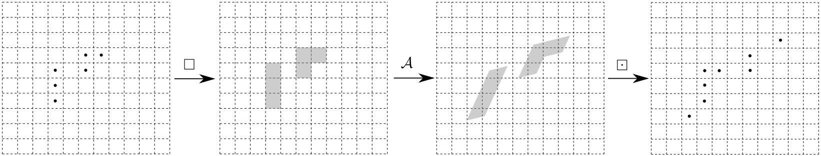

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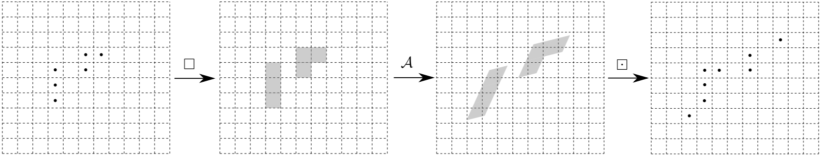

Topological constraint is missing!

Problem formulation

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Solution

Topological preservation via the **optimization in space of cellular complexes** with the notion of collapse on the complexes.

↪ **Simple cell**: Cells that can be removed/added without changing the topological structure

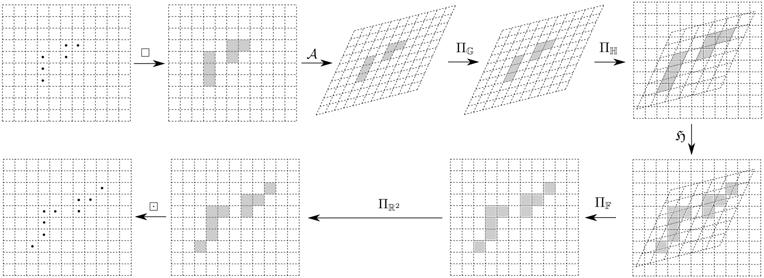
↪ **Collapse operation**: Detachment/Attachment of simple cells to the existing cellular complexes

Affine transformation on $\mathbb{Z}^2$ under topological constraint

Proposed method

The main steps to transform $X \subset \mathbb{Z}^2$ by $\mathcal{A}$:

1. Generate **refined cellular space** $\mathbb{H}$ from $\mathbb{F}$ and $\mathbb{G}$
2. Compute the complex H in $\mathbb{H}$ from G
3. Optimize by a **homotopic transformation** $\mathfrak{H}$ from H to $\widehat{H}$
4. Embed the **digitized complex** $\widehat{H}$ in $\mathbb{F}$, *i.e.* $\widehat{F} = \Pi_{\mathbb{F}}(\widehat{H}) \subset \mathbb{Z}^2$.



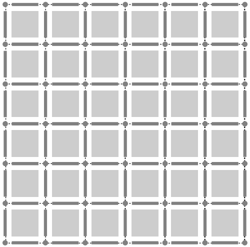
Cellular space $\mathbb{F}$ induced by $\mathbb{Z}^2$

Definition

Let $\Delta = \mathbb{Z} + \frac{1}{2}$.The induced **cellular complex space** $\mathbb{F}$ is composed of:

- set of 0-faces $\mathbb{F}_0 = \{\{d\} \mid d \in \Delta^2\}$
- set of 1-faces $\mathbb{F}_1 = \bigcup_{i=1,2}\{]d, d + e_i[\mid d \in \Delta^2\}$
- set of 2-faces $\mathbb{F}_2 = \{]d, d + e_1[\times]d, d + e_2[\mid d \in \Delta^2\}$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.



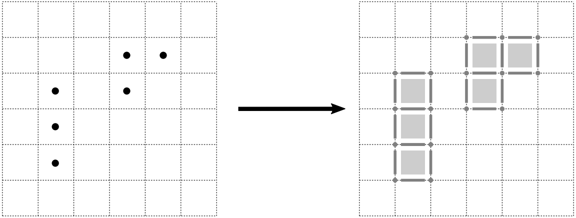
Cellular space $\mathbb{F}$ induced by $\mathbb{Z}^2$

Definition

Given a digital object $\mathrm{X} \subset \mathbb{Z}^2$, the **associated complex** $F = \Pi_{\mathbb{F}}(\square(\mathrm{X}))$ is defined as:

$$F = \bigcup_{\mathrm{x} \in \mathrm{X}} C(\blacksquare(\mathrm{x}))$$

where $\blacksquare(\mathrm{p}) = \mathrm{p} \oplus \,]-\frac{1}{2}, \frac{1}{2}[^2$ for $\mathrm{p} \in \mathbb{Z}^2$.

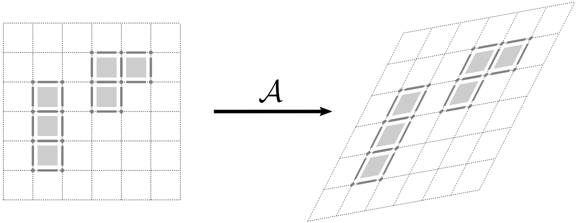


Transformed cellular space $\mathbb{G}$ induced by $\mathcal{A}(\mathbb{Z}^2)$

Definition

The **cellular space** $\mathbb{G}$ induced by an affine transformation $\mathcal{A}$ and $\mathbb{Z}^2$ is composed of the three sets of d-faces ($0 \leq d \leq 2$):

$$\mathbb{G}_d = \mathcal{A}(\mathbb{F}_d) = \{\mathcal{A}(\mathfrak{f}) \mid \mathfrak{f} \in \mathbb{F}_d\}$$

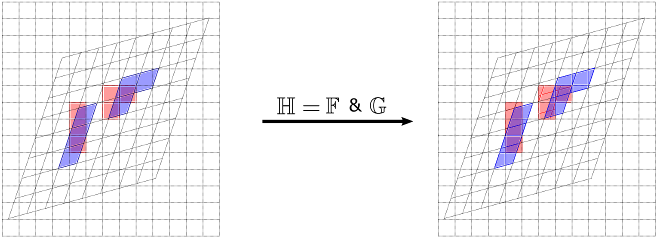


The continuous object $X_\mathcal{A}$ is modeled by the complex $G = \Pi_\mathbb{G}(X_\mathcal{A})$, which is defined by

$$G = \mathcal{A}(F) = \mathcal{A}(\Pi_\mathbb{F}(X)) = \{\mathcal{A}(\mathfrak{f}) \mid \mathfrak{f} \in \Pi_\mathbb{F}(X)\}$$

Cellular space $\mathbb{H}$ refining $\mathbb{F}$ and $\mathbb{G}$

A new cellular space $\mathbb{H}$ that refines both $\mathbb{F}$ and $\mathbb{G}$ is built.



For each 2-face $\mathfrak{h}_2$ of $\mathbb{H}$, there exists exactly one 2-face $\mathfrak{f}_2$ of $\mathbb{F}$ and one 2-face $\mathfrak{g}_2$ of $\mathbb{G}$ such that $\mathfrak{h}_2 = \mathfrak{f}_2 \cap \mathfrak{g}_2$. We can define

- $\phi : \mathbb{H}_2 \to \mathbb{F}_2$ such that $\phi(\mathfrak{h}_2) = \mathfrak{f}_2$
- $\gamma : \mathbb{H}_2 \to \mathbb{G}_2$ such that $\gamma(\mathfrak{h}_2) = \mathfrak{g}_2$.

and reversely,

- $\Phi : \mathbb{F}_2 \to 2^{\mathbb{H}_2}$ such that $\Phi(\mathfrak{f}_2) = \{\mathfrak{h}_2 \in \mathbb{H}_2 \mid \phi(\mathfrak{h}_2) = \mathfrak{f}_2\}$
- $\Gamma : \mathbb{G}_2 \to 2^{\mathbb{H}_2}$ such that $\Gamma(\mathfrak{g}_2) = \{\mathfrak{h}_2 \in \mathbb{H}_2 \mid \gamma(\mathfrak{h}_2) = \mathfrak{g}_2\}$.

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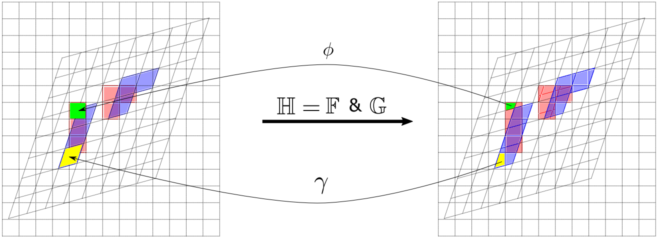

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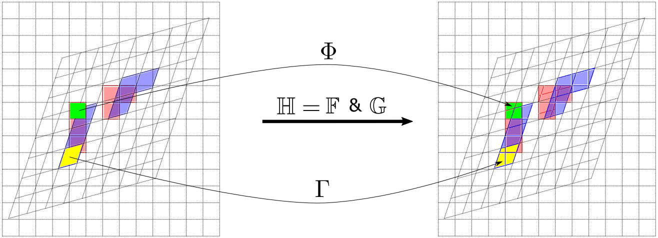

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Transformation affine sur $\mathbb{Z}^2$ sous contrainte topologique

Proposed method

The main steps to transform $X \subset \mathbb{Z}^2$ by $\mathcal{A}$:

1. Generate refined cellular space $\mathbb{H}$ from $\mathbb{F}$ and $\mathbb{G}$
2. Compute the complex H in $\mathbb{H}$ from G
3. **Optimize by a homotopic transformation $\mathfrak{H}$ from H to $\widehat{H}$**
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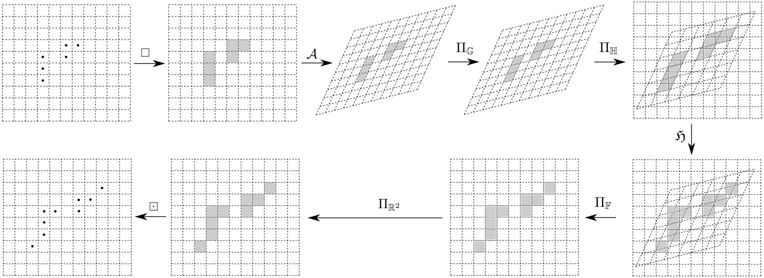
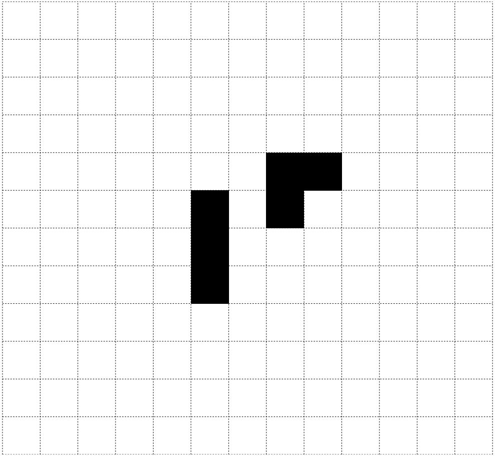


Homotopic transformation $\mathfrak{H}$ on $\mathbb{H}$



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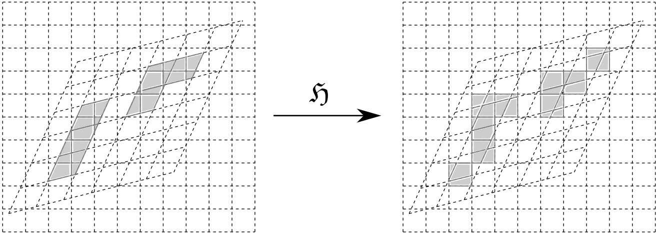

A discrete optimization process with topological constraint on $\mathbb{H}$

- **Topology**: $\mathfrak{H}$ is a homotopic transformation of H to $\widehat{H}$
  $\hookrightarrow$ a sequence of additions/removals of simple 2-cells
- **Digitization**: $\widehat{H}$ can be embedded into $\mathbb{F}$, *i.e.* $\widehat{F} = \Pi_{\mathbb{F}}(\widehat{H})$
- **Geometry**: the digital analogue $\mathsf{X}_{\mathcal{A}} = \boxdot(\Pi_{\mathbb{R}^2}(\widehat{H})) \subset \mathbb{Z}^2$ of $\widehat{H}$ is as close as possible to the solution of the optimization problem

$$\mathsf{X}_{\mathcal{A}} = \arg_{\mathsf{Y} \in 2^{\mathbb{Z}^2}} \min \mathcal{D}_{\mathcal{A},\mathsf{X}}(\mathsf{Y})$$

Optimization-based affine transformation with constraints

The cost function:

$$C = \underbrace{\mathcal{E}_{\text{topo}}}_{\mathcal{E}_{\text{topo}}(H,\widetilde{H})=0} + \underbrace{\mathcal{E}_{\text{digi}}}_{\mathcal{E}_{\text{digi}}(\widetilde{H})\geq 0} + \underbrace{\mathcal{E}_{\text{geom}}}_{\mathcal{E}_{\text{geom}}(H,\widetilde{H})\geq 0}$$

With

- **Topological energy**: $\mathcal{E}_{\text{topo}} : \mathsf{C}_{\mathbb{H}} \times \mathsf{C}_{\mathbb{H}} \to \mathbb{R}_+$
  $\hookrightarrow$ $\mathcal{E}_{\text{topo}}(H, \widetilde{H}) = 0$, *i.e.* H and $\widetilde{H}$ have the same topology
- **Digitization energy**: $\mathcal{E}_{\text{digi}} : \mathsf{C}_{\mathbb{H}} \to \mathbb{R}_+$
  $\hookrightarrow$ $\mathcal{E}_{\text{digi}}(\widetilde{H}) = 0$ if there exists $\widetilde{F}$ in $\mathsf{C}_{\mathbb{F}}$ s.t $\widetilde{F} \equiv \Pi_{\mathbb{F}}(\widetilde{H})$
- **Geometrical energy**: $\mathcal{E}_{\text{geom}} : \mathsf{C}_{\mathbb{H}} \times \mathsf{C}_{\mathbb{H}} \to \mathbb{R}_+$
  $\hookrightarrow$ $\mathcal{E}_{\text{geom}}(H, \widetilde{H})$ measures the dissimilarity between H and $\widetilde{H}$
  $\hookrightarrow$ $\mathcal{E}_{\text{geom}}(H, \widetilde{H}) = 0$, *i.e.* H and $\widetilde{H}$ are the same.

Optimization-based affine transformation with constraints

The cost function:

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Conditions and objectives of the optimization process:

- $\mathcal{E}_{\text{topo}}(H, \widetilde{H}) = 0$ throughout the optimization process
- $\mathcal{E}_{\text{digi}}(\widetilde{H}) = 0$ at the end of the process to have $\widetilde{H}$ embeddable in $\widetilde{F}$
- $\mathcal{E}_{\text{geom}}(H, \widetilde{H})$ is as small as possible at the end of the process.

Optimization-based affine transformation with constraints

The cost function:

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Let $\widetilde{H}$ be the current solution of the optimization process. At each step:

- we add/remove a simple 2-face $\mathfrak{h}_2 \in \mathbb{H}$ (*i.e.* $\mathcal{E}_{\text{topo}}(H, \widetilde{H}) = 0$) that minimizes $\mathcal{E}_{\text{digi}}(\widetilde{H})$ and $\mathcal{E}_{\text{geom}}(H, \widetilde{H})$
- we are interested in the 2-faces $\mathfrak{h}_2$ belonging *to the boundary* of $\widetilde{H}$. This set is defined by

$$\mathbb{B}_{0,1}(\widetilde{H}) = \{\mathfrak{h}_{0,1} \in \mathbb{H}_0(\widetilde{H}) \cup \mathbb{H}_1(\widetilde{H}) \mid S(\mathfrak{h}_{0,1}) \subsetneq \widetilde{H}\}$$
$$\mathbb{B}_2(\widetilde{H}) = \{\mathfrak{h}_2 \in \mathbb{H}_2 \mid C(\mathfrak{h}_2) \cap \mathbb{B}_{0,1}(\widetilde{H}) \neq \emptyset\}$$

(They are the 2-faces $\mathfrak{h}_2$ whose 0- and 1-faces *belong* to the background)

Dissimilarity measures

We search $X_{\mathcal{A}} \subset \mathbb{Z}^2$ resulting from $X \subset \mathbb{Z}^2$ by the affine transformation $\mathcal{A}$ *as close as possible* to the solution of the optimization problem:

$$X_{\mathcal{A}} = \arg_{Y \in 2^{\mathbb{Z}^2}} \min \mathcal{D}_{\mathcal{A},X}(Y)$$

Examples of $\mathcal{E}_{\mathrm{geom}}$

- based on majority vote digitization:
$$\mathcal{D}^{\square}_{\mathcal{A},X}(Y) = |\mathcal{A}(\square(X)) \setminus \square(Y)| + |\square(Y) \setminus \mathcal{A}(\square(X))|$$
- based on Gauss digitization:
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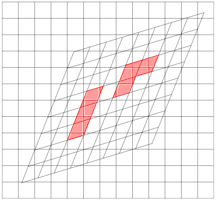
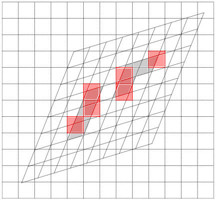
$\mathcal{D}^{\square}_{\mathcal{A},X}$



$\mathcal{D}^{\boxdot}_{\mathcal{A},X}$



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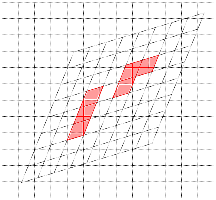
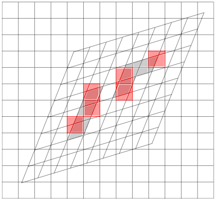
$\mathcal{D}^{\square}_{\mathcal{A},X}$  $\mathcal{D}^{\boxdot}_{\mathcal{A},X}$ 

General algorithm of homotopic affine transformation on $\mathbb{Z}^2$

Algorithm 1: Construction of $\widehat{H}$ from H by $\mathfrak{H}$.

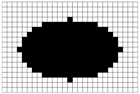
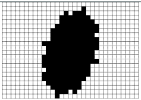
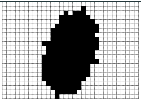
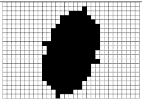
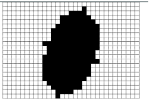
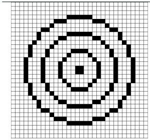
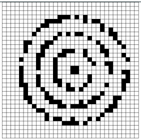
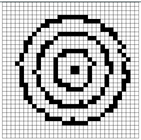
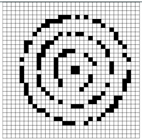
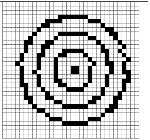
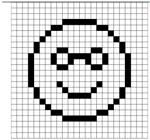
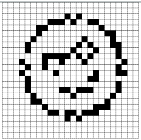
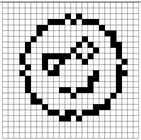
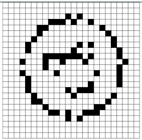
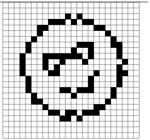
Input : $H \in \mathsf{C}_{\mathbb{H}}$
Input : $\mathcal{E}_{\text{geom}} : \mathsf{C}_{\mathbb{H}} \times \mathsf{C}_{\mathbb{H}} \to \mathbb{R}_+$
Output : $\widehat{H} \in \mathsf{C}_{\mathbb{H}}^{H} \cap \mathsf{C}_{\mathbb{H}}^{\mathbb{F}}$

$\widetilde{H} \leftarrow H$
Build $\mathbb{B}_2(\widetilde{H})$
while $\mathcal{E}_{\text{digi}}(\widetilde{H}) > 0$ **do**
    Choose $\mathfrak{h}_2 \in \mathbb{B}_2(\widetilde{H})$ s.t $\underbrace{\widetilde{H} \circledcirc C(\mathfrak{h}_2) \frown_h \widetilde{H}}_{\mathfrak{h}_2 \text{ is a simple 2-face}}$ that minimizes $\mathcal{E}_{\text{digi}}$ and $\mathcal{E}_{\text{geom}}(H, \cdot)$

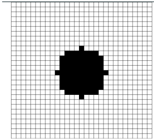
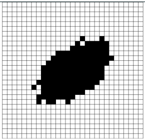
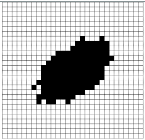
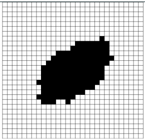
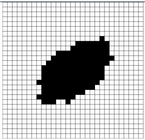
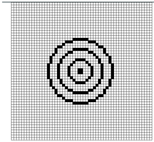
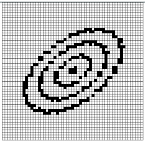
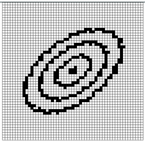
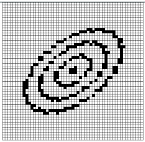
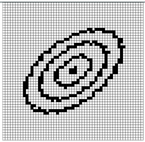
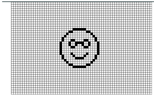
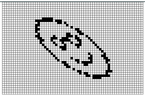
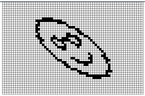
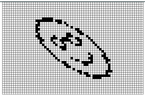
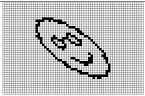
$$\widetilde{H} \leftarrow \widetilde{H} \circledcirc C(\mathfrak{h}_2) = \begin{cases} \widetilde{H} \oslash C(\mathfrak{h}_2) & \text{if } \mathfrak{h}_2 \in \widetilde{H} \\ \widetilde{H} \cup C(\mathfrak{h}_2) & \text{if } \mathfrak{h}_2 \notin \widetilde{H} \end{cases}$$

    Update $\mathbb{B}_2(\widetilde{H})$
$\widehat{H} \leftarrow \widetilde{H}$

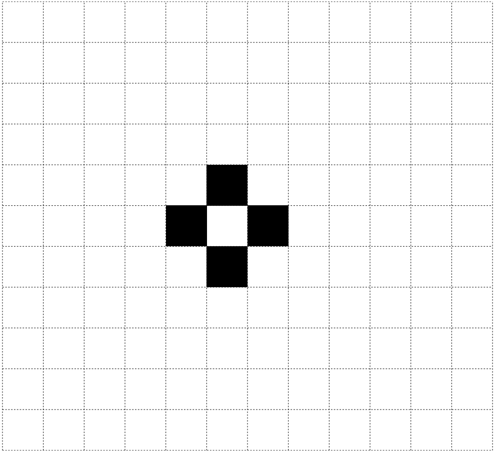
Results of rotation on $\mathbb{Z}^2$ with/without topological constraint

Original image	Gauss digitization		Majority vote	
	w.o cont. topo	with cont. topo	w.o cont. topo	with cont. topo
				
				
				

Results of affine transformation on $\mathbb{Z}^2$

Original image	Gauss digitization		Majority vote	
	w.o cont. topo	with cont. topo	w.o cont. topo	with cont. topo
				
				
				

Non-existence of solutions



Conclusion

Overview

Rigid motion
on digital images

Graph of
discret
rigid motion

Image
registration

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Rigid motion on digital images

Graph of discret rigid motion

Regular images

Image registration

Topological characterization of digital images

Overview

Rigid motion on digital images

Graph of discret rigid motion

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Quasi-regular shapes

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Topological characterization of digital images

Geometrical & topological characterization of 2D/3D shapes

Overview

Rigid motion on digital images

Affine transform on $\mathbb{Z}^2$

Graph of discret rigid motion

Regular images

Quasi-regular shapes

Complex space

Geometrical & topological characterization of 2D/3D shapes

Image registration

Topological characterization of digital images

Take home messages

- **Topological issues** when applying geometric transformations on digital images/digital shapes
- Several **solutions exist** for topology-preserving transformations
  - ↪ Regularity, quasi-regularity, ...
  - ↪ Transformation model: complex cellular, intermediate model of digital object, ...
  - ↪ Multi-grid strategies, continuous techniques,...
- Still many open questions, especially in **higher dimensions**
- **geometric properties** of transformed objects ...
- and **other families of transformations** (projective transformations, free deformation, diffeomorphism,...)

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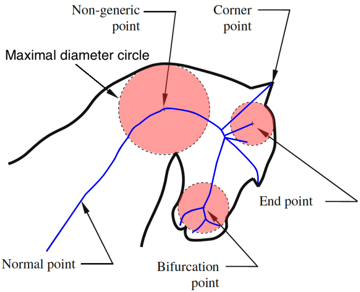
Medial axis

Définition [Blum, 1967]

Let $\mathcal{X} \subset \mathbb{R}^2$ be a closed, bounded set such that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is a 1-manifold. The **medial axis of** $\mathcal{X}$ is defined as the locus of the centers of the maximal balls included in $\mathcal{X}$:

$$\mathcal{M}(\mathcal{X}) = \{x \in \mathcal{X} \mid \nexists y \in \mathcal{X}, B(x, r(x)) \subset B(y, r(y))\}$$

where $B(y, r) \subseteq \mathcal{X}$ is the ball of center y and radius $r \in \mathbb{R}_+$.



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By definition, we have $\mathcal{M}(\mathcal{X}) \subseteq \mathcal{X}$ and

$$\mathcal{X} = \bigcup_{x \in \mathcal{M}(\mathcal{X})} B(x, r(x))$$

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where $B(y, r) \subseteq \mathcal{X}$ is the ball of center y and radius $r \in \mathbb{R}_+$.

We define the λ-level medial axis, noted $\mathcal{M}_\lambda(\mathcal{X})$, by

$$\mathcal{M}_\lambda(\mathcal{X}) = \{x \in \mathcal{M}(\mathcal{X}) \mid r(x) \geq \lambda\}$$

In particular, $\lambda_1 \leq \lambda_2 \Rightarrow \mathcal{M}_{\lambda_2}(\mathcal{X}) \subseteq \mathcal{M}_{\lambda_1}(\mathcal{X})$, and $\mathcal{M}_0(\mathcal{X}) = \mathcal{M}(\mathcal{X})$.

We also define

$$\mathcal{M}_{\lambda_1}^{\lambda_2}(\mathcal{X}) = \{x \in \mathcal{M}(\mathcal{X}) \mid \lambda_1 \leq r(x) \leq \lambda_2\}$$

Properties of medial axis

Proposition [Lieutier, 2004]

$\mathcal{X}$ and $\mathcal{M}(\mathcal{X})$ have the same homotopy type, and noted $\mathcal{X} \frown \mathcal{M}(\mathcal{X})$.

Proposition [Serra, 1983]

Let B_λ be the ball of center $0_{\mathbb{R}^2}$ and of radius $\lambda \geq 0$. We have

$$\mathcal{X} \ominus B_\lambda = \bigcup_{x \in \mathcal{M}_\lambda(\mathcal{X})} B(x, r(x) - \lambda)$$

$$\mathcal{X} \oplus B_\lambda = \bigcup_{x \in \mathcal{M}(\mathcal{X})} B(x, r(x) + \lambda)$$

$$\mathcal{M}(\mathcal{X} \ominus B_\lambda) = \mathcal{M}_\lambda(\mathcal{X})$$

We now verify the quasi-regularity of polygon via its medial axis.

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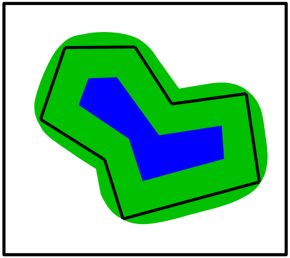
Verification of quasi-regularity

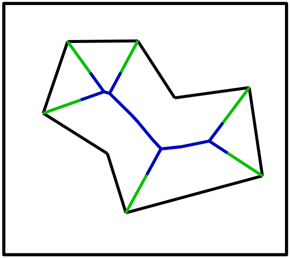
Property [Ngo et al., 2021]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected polygon. If $\mathcal{M}(\mathcal{X}) \frown \mathcal{M}_1(\mathcal{X})$ and $\mathcal{M}(\overline{\mathcal{X}}) \frown \mathcal{M}_1(\overline{\mathcal{X}})$ then

(i) $\mathcal{X} \ominus B_1$ is non-empty and connected

(ii) $\overline{\mathcal{X}} \ominus B_1$ is connected





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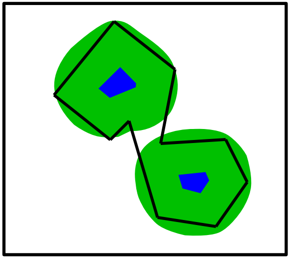
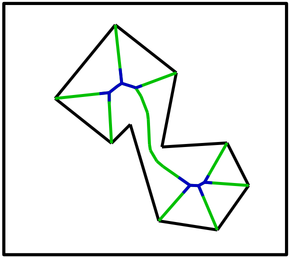


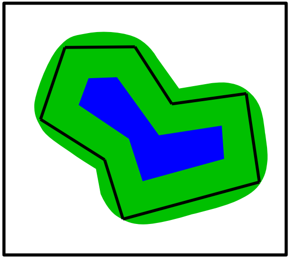


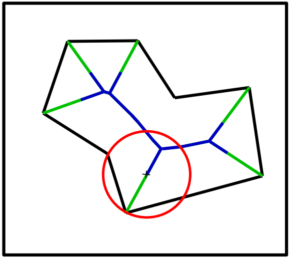
Verification of quasi-regularity

Let $Y \in \{\mathcal{X}, \overline{\mathcal{X}}\}$ and $M \subseteq \mathcal{M}_0^1(Y)$ a connected component of $\mathcal{M}_0^1(Y)$. M contains a set of k points, noted z_i $(1 \leq i \leq k)$, with $r(z_i) = 0$ (they are convex vertices of the polygon Y), and a point y with $r(y) = 1$.

Let $(\mathcal{P})$: $\forall 1 \leq i \leq k, \|y - z_i\|_2 \leq \sqrt{2}$. We have

$$(\mathcal{P}) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \ominus B_1 \oplus B_{\sqrt{2}}$$





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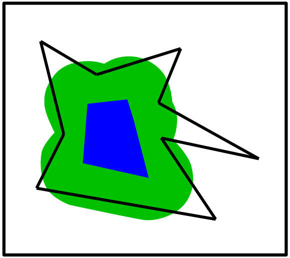
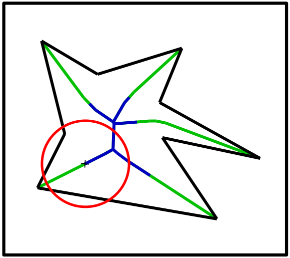
$$(\mathcal{P}) \Rightarrow \bigcup_{x \in M} B(x, r(x)) \subseteq Y \ominus B_1 \oplus B_{\sqrt{2}}$$





Verification of quasi-regularity

Let $Y \in \{\mathcal{X}, \overline{\mathcal{X}}\}$ and $M \subseteq \mathcal{M}_0^1(Y)$ a connected component of $\mathcal{M}_0^1(Y)$. M contains a set of k points, noted z_i $(1 \leq i \leq k)$, with $r(z_i) = 0$ (they are convex vertices of the polygon Y), and a point y with $r(y) = 1$.

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Proposition [Ngo et al., 2021]

Let $X \subset \mathbb{R}^2$ be a simply connected polygon. If $\mathcal{M}(\mathcal{X}) \frown \mathcal{M}_1(\mathcal{X})$, $\mathcal{M}(\overline{\mathcal{X}}) \frown \mathcal{M}_1(\overline{\mathcal{X}})$ and, for each connected component of $\mathcal{M}_0^1(X)$ and $\mathcal{M}_0^1(\overline{X})$, the property $(\mathcal{P})$ holds. Then, X is quasi-regular.

Quasi-regularity verification method

The method consists in verifying the following two conditions:

(i) $\mathcal{M}(\mathcal{X}) \frown \mathcal{M}_1(\mathcal{X})$ and $\mathcal{M}(\overline{\mathcal{X}}) \frown \mathcal{M}_1(\overline{\mathcal{X}})$

(ii) $(\mathcal{P})$ holds for each connected component of $\mathcal{M}_0^1(\mathcal{X})$ and $\mathcal{M}_0^1(\overline{\mathcal{X}})$.

Algorithm 2: Quasi-regularity verification.

Input : A simply connected polygonal object $\mathcal{X} \subset \mathbb{R}^2$

Output : A boolean indicating whether $\mathcal{X}$ is quasi-regular

for $\mathcal{Y} \in \{\mathcal{X}, \overline{\mathcal{X}}\}$ **do**
- **if** *not* $\mathcal{M}(\mathcal{Y}) \frown \mathcal{M}_1(\mathcal{Y})$ **then** **return** false
- **foreach** *connected component* $M \in \mathcal{M}_0^1(\mathcal{Y})$ **do**
  - Let $y \in M$ such that $r(y) = 1$
  - **foreach** $z_i \in M$ *such that* $r(z_i) = 0$ **do**
    - **if** $||y - z_i||_2^2 > 2$ **then** **return** false

return true

Quasi-regularity verification method

The method consists in verifying the following two conditions:

(i) $\mathcal{M}(\mathcal{X}) \frown \mathcal{M}_1(\mathcal{X})$ and $\mathcal{M}(\overline{\mathcal{X}}) \frown \mathcal{M}_1(\overline{\mathcal{X}})$

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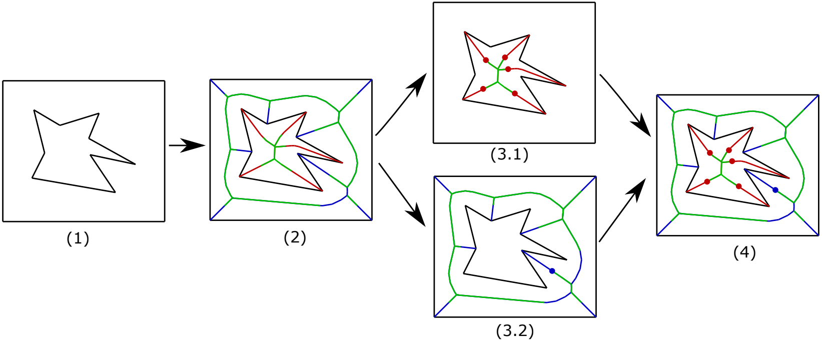
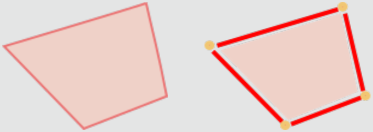


Definition of cellular space

A closed convex polygon P and its partition $\mathcal{F}(P)$

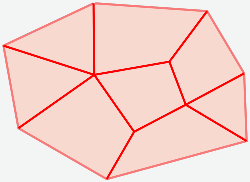


$\mathcal{F}(P)$ contains:

- 2-face (interior of P, $\mathring{P}$),
- 1-faces (edges of P), and
- 0-faces (vertices of P).

A union of closed convex polygons Ω and its partition $\mathbb{K}(\Omega)$

Let $\Omega = \bigcup \mathcal{K}$ where $\mathcal{K}$ is a set of closed, convex polygons such that for any pair $P_1, P_2 \in \mathcal{K}$, $\mathring{P}_1 \cap \mathring{P}_2 = \emptyset$. Then, $\mathbb{K}(\Omega) = \bigcup_{P \in \mathcal{K}} \mathcal{F}(P)$.



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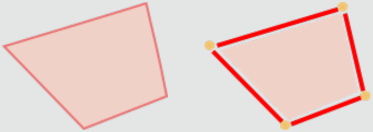

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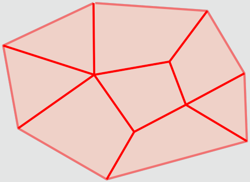

Definition of cellular complexes

Let $\mathbb{K}$ be a cellular space and $\mathfrak{f} \in \mathbb{K}$ be a face.

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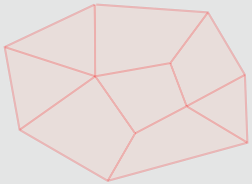

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The **embedding of K into $\mathbb{R}^2$** is defined by $\Pi_{\mathbb{R}^2}(K) = \bigcup K$.

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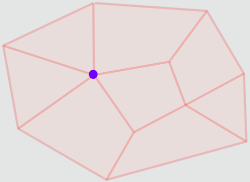

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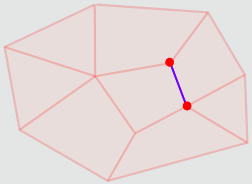

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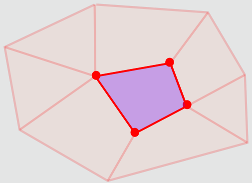

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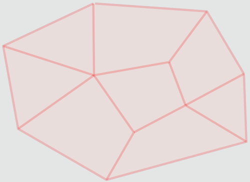

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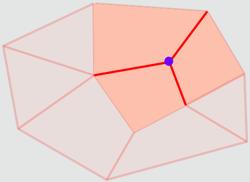

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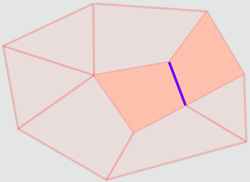

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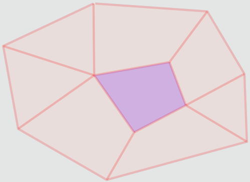

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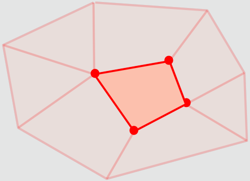

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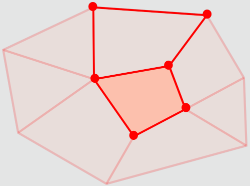

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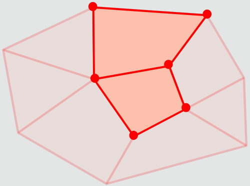

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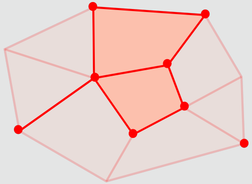

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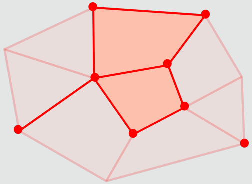

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Collapse on complexes

Let K be a complex defined in a cellular space $\mathbb{K}$.

Elementary collapse

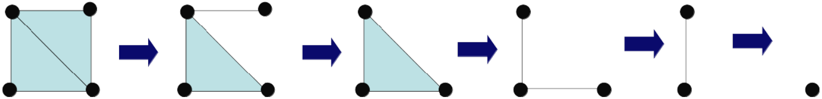
Suppose that τ and σ are two faces of K such that

$\tau \subset \sigma$ with $dim(\tau) = dim(\sigma) - 1$ and

σ is a maximal face of K and no other maximal face of K contains τ,

then τ is called a **free face** and the removal of the faces, $K \setminus \{\tau, \sigma\}$, is called an **elementary collapse**.

If there is a sequence of elementary collapses from K to a complex K', we say that K **collapses** to K'.



Simple cells

Let K be a complex defined in a cellular space $\mathbb{K}$ on $\mathbb{R}^2$.

Let $\mathfrak{f}_2$ be a 2-face of K.

Let $D_d(\mathfrak{f}_2)$, $d = 0, 1$, be the subset of $C(\mathfrak{f}_2)$ composed by the d-faces $\mathfrak{f}$ such that $S(\mathfrak{f}) \cap K = S(\mathfrak{f}) \cap C(\mathfrak{f}_2)$.

Simple cells

If $|D_1(\mathfrak{f}_2)| = |D_0(\mathfrak{f}_2)| + 1$, $C(\mathfrak{f}_2)$ is called a **simple** 2-**cell** for K.

Detachment of a simple 2-cell $C(\mathfrak{f}_2)$ from K: collapse operation from K to $K \oslash C(\mathfrak{f}_2) = K \setminus (\{\mathfrak{f}_2\} \cup D_1(\mathfrak{f}_2) \cup D_0(\mathfrak{f}_2))$

Attachment of a simple 2-cell $C(\mathfrak{f}_2)$ for $K \cup C(\mathfrak{f}_2)$ where $\mathfrak{f} \in \mathbb{K} \setminus K$: the inverse collapse operation from K into $K \cup C(\mathfrak{f}_2)$