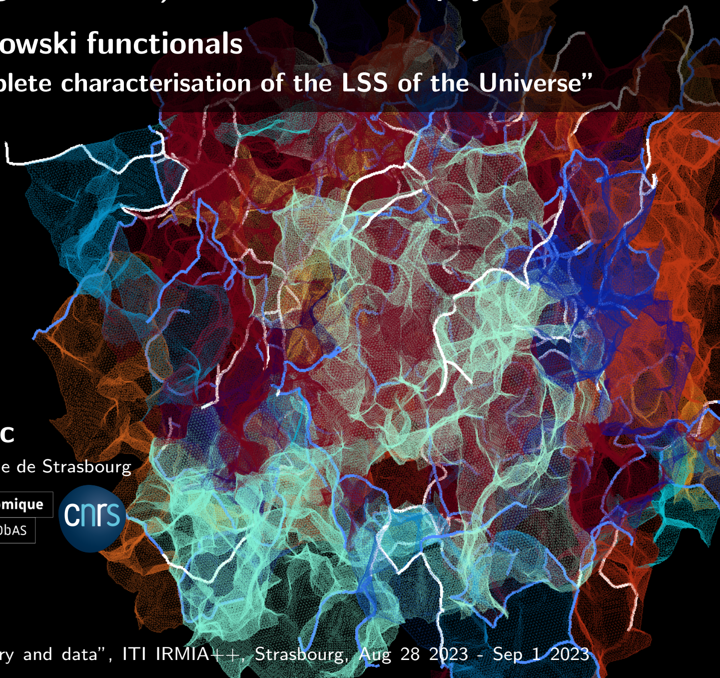
Topological (& geometrical) methods for astrophysical data

Lecture 1: Minkowski functionals

"Complete characterisation of the LSS of the Universe"



Katarina Kraljic

Observatoire astronomique de Strasbourg

Observatoire **astronomique** de Strasbourg | ObAS

cnrs

Summer school "Geometry and data", ITI IRMIA++, Strasbourg, Aug 28 2023 - Sep 1 2023

1 **Lecture 1: Minkowski functionals**

→ robust morphological characterisation of the large-scale structure of the Universe

1.1 Motivation

→ cosmic web

→ characterisation of the multi-scale Universe

1.2 Mathematical background

→ global Minkowski functionals

→ local Minkowski functionals

1.3 Applications

→ point processes (Poisson process, Galaxy distribution)

→ continuous fields (Hydrodynamic, N-body simulations)

Useful reading:

→ Mecke, Buchert & Wagner 1994 arXiv:astro-ph/9312028

→ Schmalzing & Buchert 1997 arXiv:astro-ph/9702130

→ Schmalzing, Kerscher & Buchert 1995 arXiv:astro-ph/9508154

2 **Lecture 2: Discrete persistent structures extractor (DisPerSE)**
→ coherent identification of persistent topological features within sampled distributions

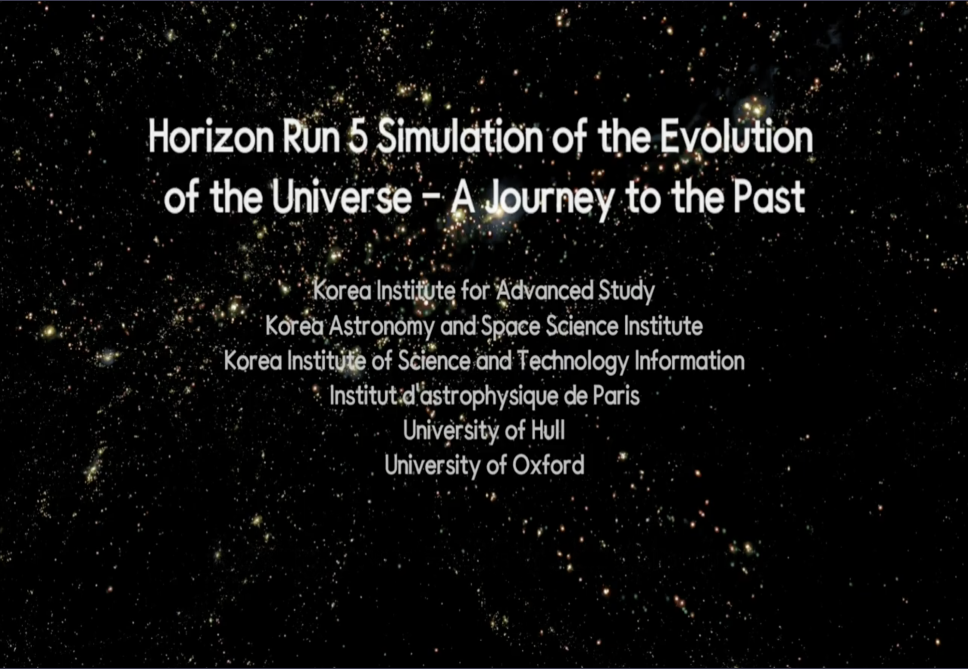
2.1 Motivation
→ characterisation of the web-like matter distribution in the Universe
→ cosmic web ↔ Morse theory

2.2 Morse theory
→ smooth Morse theory
→ discrete Morse theory

2.3 Theory of persistence

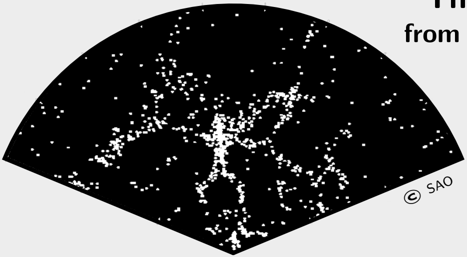
2.4 DisPerSE
→ applications

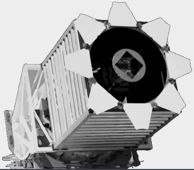
Useful reading:
→ Sousbie 2011 arXiv:astro-ph/1009.4015
→ Sousbie, Pichon, Kawahara 2011 arXiv:astro-ph/1009.4014
→ DisPerSE


Horizon Run 5 Simulation of the Evolution of the Universe – A Journey to the Past
Korea Institute for Advanced Study
Korea Astronomy and Space Science Institute
Korea Institute of Science and Technology Information
Institut d'astrophysique de Paris
University of Hull
University of Oxford

The Cosmic web
from observations . . .

de Lapparent et al. 1986





Sloan Foundation Telescope

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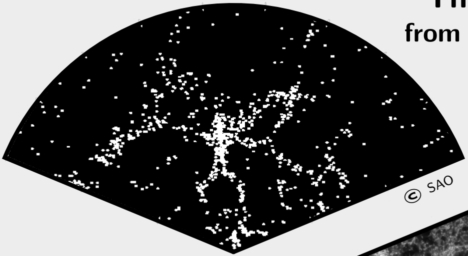

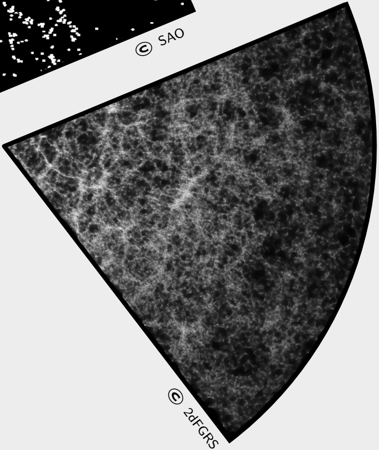

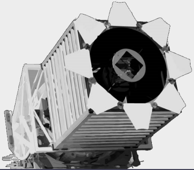
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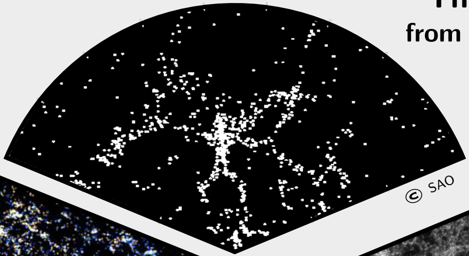

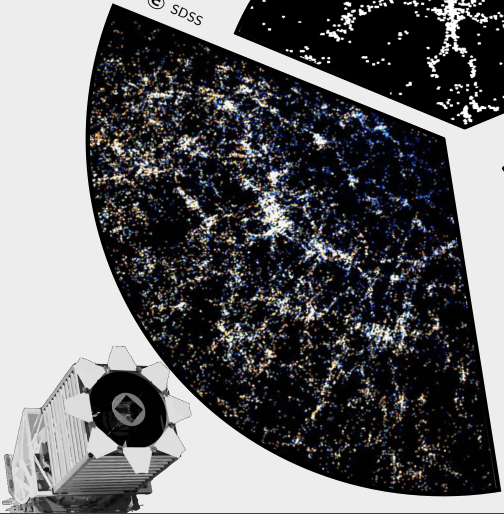

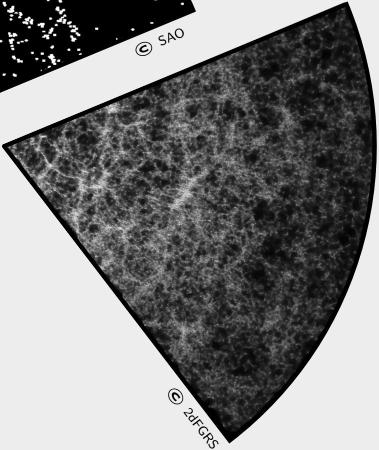

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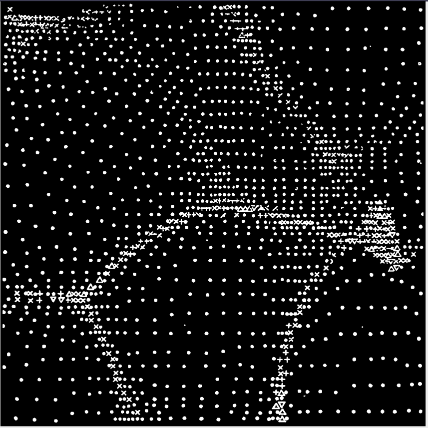

Sergei Shandarin
Zel'dovich 1970

The Cosmic web
. . . to theory

Klypin & Shandarin 1993
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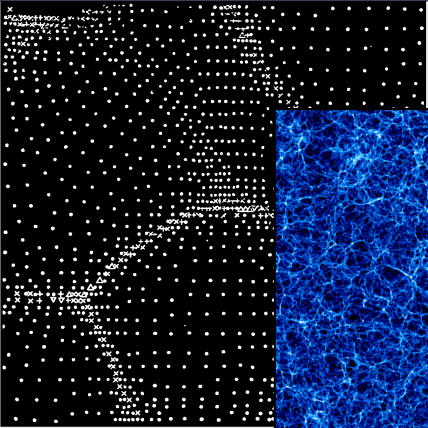

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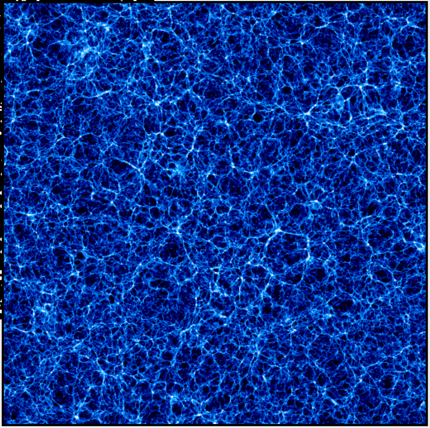

Marenostrum • Yepes et al. 2007

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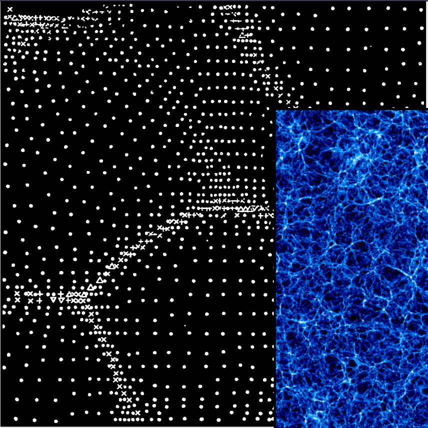

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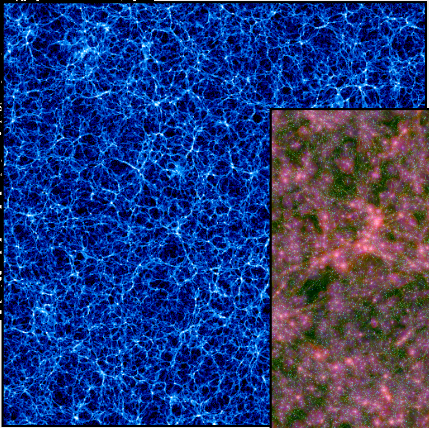

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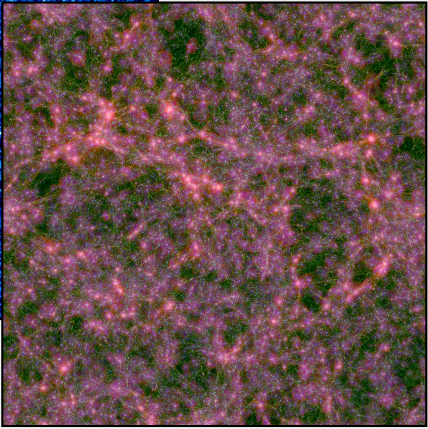

HORIZON-AGN • Dubois et al. 2014

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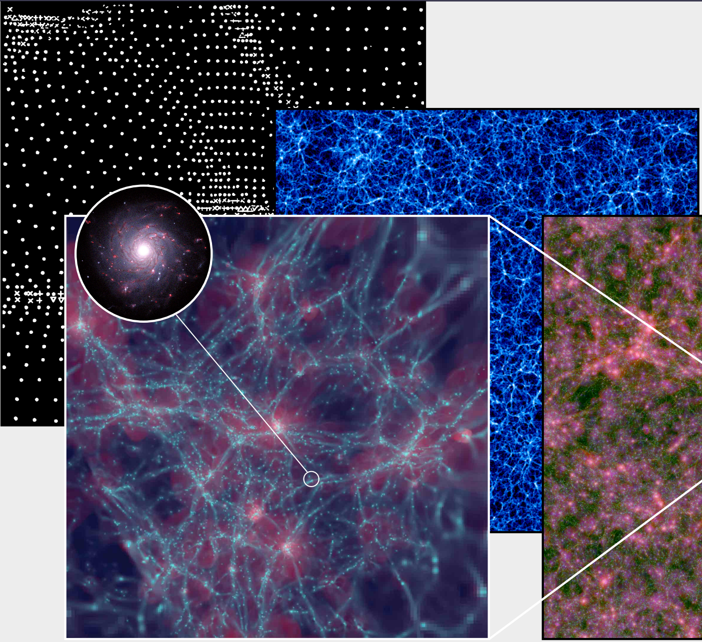
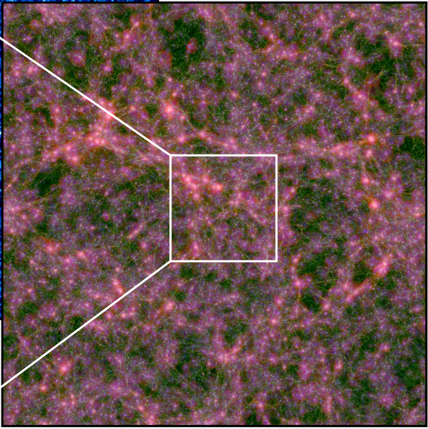


NewHorizon • Dubois et al. 2021



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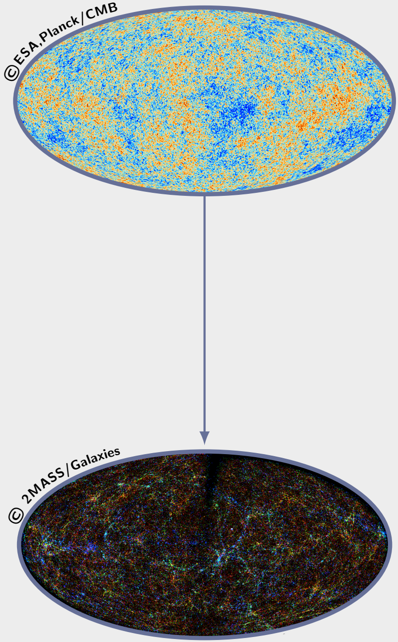
The Multi-scale Universe



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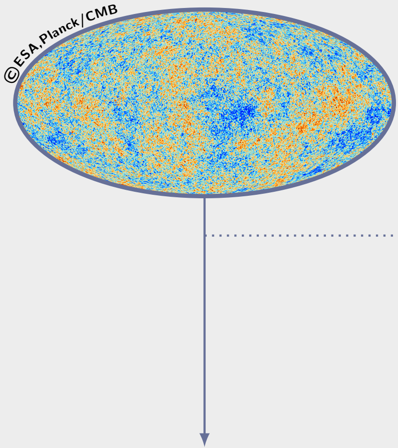
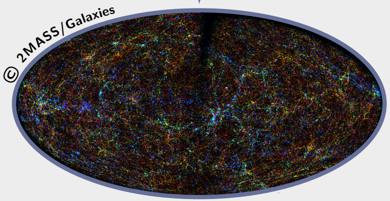
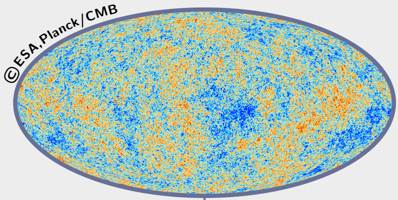
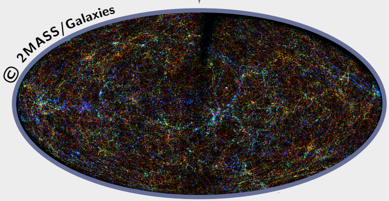




Theory

1. cosmological models
   - Gaussian random field (standard model) vs primordial non-Gaussianities
   - massive neutrinos
   - dark energy models ...
2. galaxy formation models
   - gravitational clustering
   - baryonic physics (scale-coupling) ...





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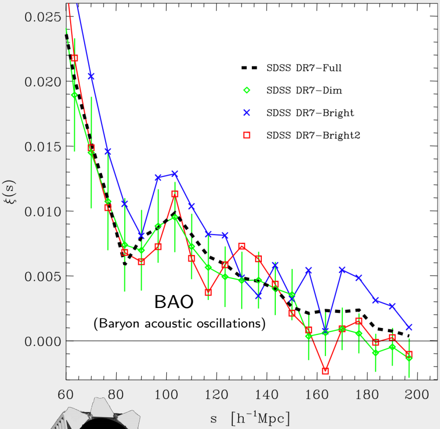
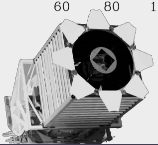
Method

1. statistical measures
   - two-point correlation function
   - higher-order correlations
   - topological/geometrical descriptors . . .
2. observables
   - galaxies
   - galaxy clusters
   - CMB . . .

Characterisation of Large-scale structure

2-point correlation function





SDSS

Kazin et al. 2010

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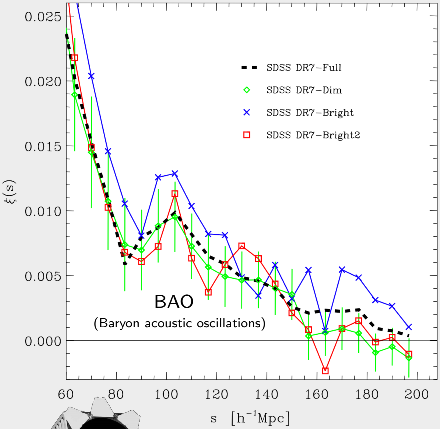
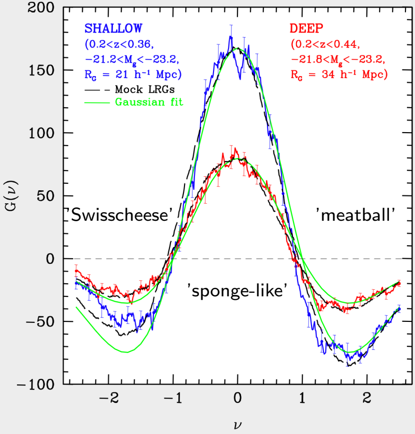


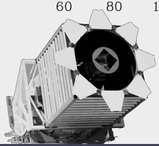
Kazin et al. 2010

Genus topology

(# of holes - # of isolated regions)



Gott et al. 2009



SDSS

- $\mathbb{E}^d$ - d-dimensional Euclidean space

  with

  $\mathcal{G}$ - group of motions operating on it:

$$\mathcal{G} = \mathcal{R} \otimes \mathcal{T}$$

  $\mathcal{R}$ - subgroup of rotations

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Hadwiger's theorem

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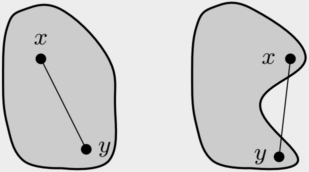

convex non-convex

- $\mathcal{K}$ - set of all compact and convex sets
  $\mathcal{R}$ - convex ring of all finite unions of convex bodies

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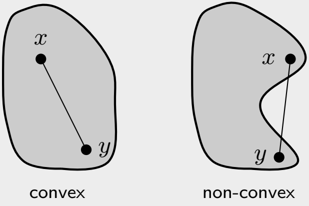

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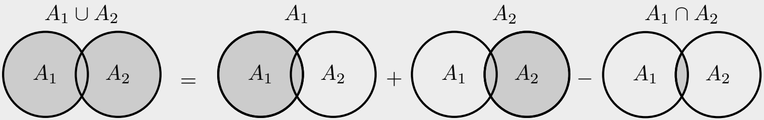
- To characterise the **topological** and **geometrical** properties of a body A from this convex ring, we wish to find functionals M satisfying:

  1. additivity
  2. motion-invariance
  3. conditional continuity

1 **Additivity:** $M(A_1 \cup A_2) = M(A_1) + M(A_2) - M(A_1 \cap A_2) \quad \forall A_1, A_2 \in \mathcal{R}$



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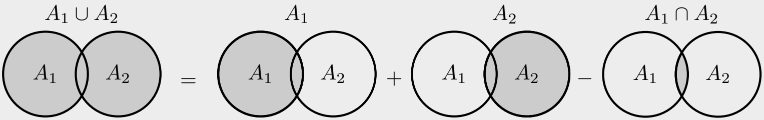
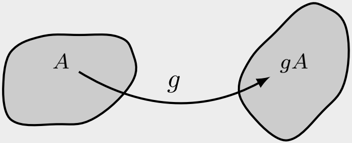


2. **Motion invariance:** $M(gA) = M(A) \quad \forall \quad g \in \mathcal{G}, A \in \mathcal{R}$



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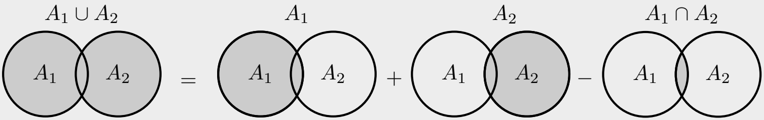

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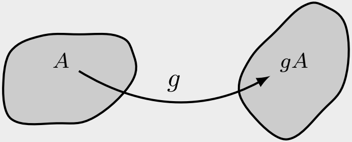
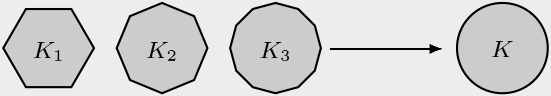


3. **Conditional continuity:** $M(K_i) \longrightarrow M(K)$ as $K_i \longrightarrow K$ for $K_i, K \in \mathcal{K}$



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Hadwiger's theorem

Let $\mathcal{R}$ be the convex ring in d-dimensional space. Then there $\exists$ $d+1$ functionals M_μ; $\mu = 0, 1, \ldots, d$ on $\mathcal{R}$ such that any *additive*, *motion invariant* and *conditionally continuous* functional M is a linear combination of them:

$$M = \sum_{\mu=0}^{d} c_\mu M_\mu \quad \text{with} \quad c_\mu \in \mathbb{R}$$

Hugo Hadwiger
(1908 - 1981)

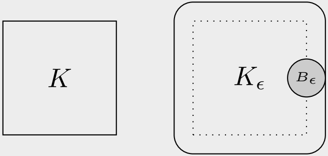
Steiner's theorem

- $K \in \mathcal{K}$ - convex body

  $K_\epsilon \in \mathcal{K}$ - parallel body of K

  $$K_\epsilon := \{\mathbf{x} \in \mathbb{R}^d \mid r_k(\mathbf{x}) \leq \epsilon\}$$

  with $r_k := \min_{y \in K} \|\mathbf{x} - \mathbf{y}\|$



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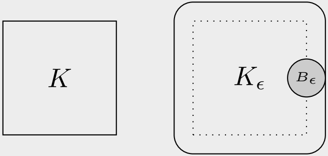

Steiner's theorem

Volume $V(K_\epsilon)$ of the parallel body K_ϵ of a convex body K is a polynomial in ϵ of degree d:

$$V(K_\epsilon) = \sum_{\nu=0}^{d} \binom{d}{\nu} W_\nu(K)\epsilon^\nu$$

Jacob Steiner
(1796 - 1863)

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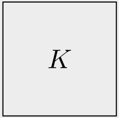

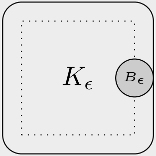

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$W_\nu(K)$ - depend only on the particular body K

- define a family of $d+1$ functionals on the convex ring

- termed Quermassintegrals or **Minkowski functionals**

(Minkowski 1903)

Hermann Minkowski
(1864 - 1909)

Inclusion-exclusion principle

- The continuation from the set $\mathcal{K}$ of convex bodies to the whole convex ring $\mathcal{R}$:

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  For a body $A \in \mathcal{R}$ given as the union of N convex bodies $K_i, i \in \mathcal{I}$ with $|\mathcal{I}| = N$

  $$A = \bigcup_{i \in \mathcal{I}} K_i$$

  any additive functional M is calculated as:

  $$M(A) = \sum_{\emptyset \neq \mathcal{J} \subseteq \mathcal{I}} (-1)^{|\mathcal{J}|-1} M(\bigcap_{j \in \mathcal{J}} K_j) \tag{1}$$

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- For a union set of 2 bodies, $\mathcal{I}=\{1,2\}$, we recover the additivity relation:

$$M(K_1 \cup K_2) = M(K_1) + M(K_2) - M(K_1 \cap K_2)$$

Application of Steiner's theorem

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  $\rightarrow$ topologically equivalent to a ball

  $\rightarrow$ Gaussian transformation - mapping between the two

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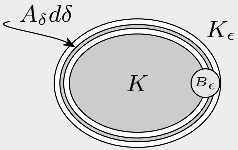

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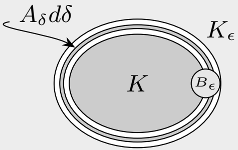

- A_δ - surface of the parallel body for radius δ:

$$A_\delta = \int_{S^2} (R_1 + \delta)(R_2 + \delta)dw = \int_{S^2} R_1 R_2 dw + \delta \int_{S^2} (R_1 + R_2)dw + \delta^2 \int_{S^2} dw =$$

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$$= A + 2H\delta + 4\pi\chi\delta^2$$

surface area — **integral mean curvature** — **Euler characteristic**

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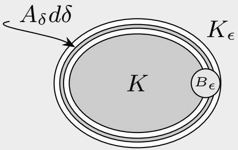

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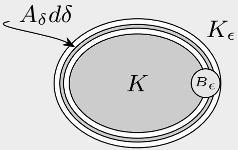

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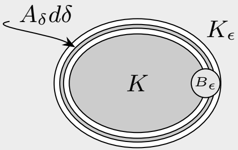

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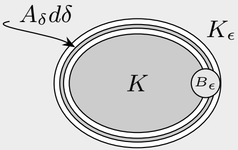

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➡ Minkowski functionals . . . geometric quantity:

$W_0 = V$	. . . volume V
$W_1 = \frac{1}{3}A$	. . . surface A
$W_2 = \frac{1}{3}H$	. . . mean curvature H
$W_3 = \frac{4\pi}{3}\chi$	. . . Euler characteristic χ

Minkowski functionals

- geometric interpretation in 3-dimensional space:

The most common notations for Minkowski functionals.

geometric quantity	μ	M_μ	V_μ	W_μ	$\omega_\mu^\star$
V volume	0	V	V	V	1
A surface	1	$A/8$	$A/6$	$A/3$	2
H mean curvature	2	$H/2\pi^2$	$H/3\pi$	$H/2$	π
χ Euler characteristic	3	$3\chi/4\pi$	χ	$4\pi\chi/3$	$4\pi/3$

$\star$ ω_μ - volume of μ-dimensional unit ball: $\omega_\mu := \frac{\pi^{\mu/2}}{\Gamma(1+\mu/2)}$

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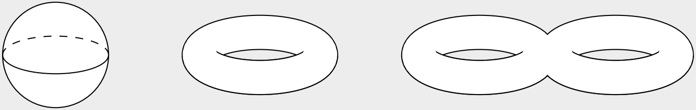
H - information about shape

χ - purely topological quantity

- related to *genus* g: $\chi = 1 - g$

$\chi = \#$ of components - $\#$ of tunnels + $\#$ of cavities



$\chi = 1$ $\qquad$ $\chi = 0$ $\qquad$ $\chi = -1$

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Principal kinematical formula

Consider $A, B \in \mathcal{R}$, and fix A, while allowing B to move through transformations $g \in \mathcal{G}$. Intersection $A \cap gB \in \mathcal{R}$ with Minkowski functionals $M_\mu(A \cap gB)$. With the Haar measure, integration of this quantity gives a factorisation into functionals of A and B:

$$\int\limits_{\mathcal{G}} dg\, M_\mu(A \cap gB) = \sum_{\nu=0}^{\mu} \begin{pmatrix} \mu \\ \nu \end{pmatrix} M_\nu(A)\, M_{\mu-\nu}(B)$$

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- **Solution:**
  1. generalised boundary
  2. Federer's local curvature measures
  3. suitable measure of integration

  $\Rightarrow$ **partition formula** analogous to (2)

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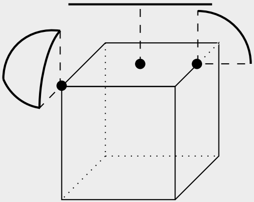
→ in 3D: $\partial^3 A$ - corners, $\partial^2 A$ - corners and edges,
$\partial^1 A$ - ordinary boundary

→ for $A \in \mathcal{R}$ given as the union of smooth bodies:

$$\partial^\nu A = \{\mathbf{x} \in \partial A | \exists \mathcal{J} \subseteq \mathcal{I} : |\mathcal{J}| = \nu, \mathbf{x} \in \cap_{j \in \mathcal{J}} \partial K_j\}$$

$\partial^\nu A$ - $(d-\nu)$-dimensional manifold composed of all intersection points of ν or more boundaries of smooth convex parts



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$C_{d-\mu}(K,\beta)$ - functionals on the set of Borel sets $\mathbb{B}^d$ for each convex body K
- body's curvature measures as defined by Federer (1959)
- depend only on $\beta \cap \partial K$
- if $\beta = \partial K \rightarrow$ local parallel set is equal to the complete parallel set, and the curvature measure is equal to the global MF:

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(1920 - 2010)

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- additivity on disjoint partition of ∂K into $\partial^\nu K$ & partition of each $\partial^\nu K$ into $N^{(\nu)}$ disjoint Borel sets $\beta_j^{(\nu)}$:

$$\partial K = \biguplus_{\nu=1}^{d} \partial^\nu K = \biguplus_{\nu=1}^{d} \biguplus_{j=1}^{N^{(\nu)}} \beta_j^{(\nu)}$$

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$$N^{(\nu)} \longrightarrow \infty, \qquad \max_{1 \le j \le N^{(\nu)}} V^{(d-\nu)}(\beta_j^{(\nu)}) \longrightarrow 0 \quad (4)$$

$$\sum_{j=1}^{N^{(\nu)}} V^{(d-\nu)}(\beta_j^{(\nu)}) \cdots \longrightarrow \int_{\partial^\nu K} d\lambda_{d-\nu}(\mathbf{x}) \ldots$$

2 Local Minkowski functionals (MFs)

- representation of MF:

$$W_\mu(K) = \sum_{\nu=1}^{\mu} \frac{1}{d} C_{d-\mu}(K, \partial^\nu K) = \sum_{\nu=1}^{\mu} \sum_{j=1}^{N^{(\nu)}} \frac{1}{d} C_{d-\mu}(K, \beta_j^{(\nu)}) =$$

$$= \sum_{\nu=1}^{\mu} \sum_{j=1}^{N^{(\nu)}} V^{(d-\nu)}(\beta_j^{(\nu)}) \frac{\frac{1}{d} C_{d-\mu}(K, \beta_j^{(\nu)})}{V^{(d-\nu)}(\beta_j^{(\nu)})} \tag{3}$$

$V^{(d-\nu)}(\cdot)$ - $(d-\nu)$-dimensional volume appropriate for the ν-th generalised boundary
- for a finer and finer partitions of the generalises boundaries

$$N^{(\nu)} \longrightarrow \infty, \quad \max_{1 \le j \le N^{(\nu)}} V^{(d-\nu)}(\beta_j^{(\nu)}) \longrightarrow 0 \tag{4}$$

$$\sum_{j=1}^{N^{(\nu)}} V^{(d-\nu)}(\beta_j^{(\nu)}) \cdots \longrightarrow \int_{\partial^\nu K} d\lambda_{d-\nu}(\mathbf{x}) \ldots$$

$\frac{\frac{1}{d} C_{d-\mu}(K, \beta_j^{(\nu)})}{V^{(d-\nu)}(\beta_j^{(\nu)})}$ - can be used to define local MFs with the limit (3)

2 Local Minkowski functionals (MFs)

- **Definition:** Let K be a convex body and $\mathbf{x} \in \partial K$ a point on its boundary belonging to the ν-th, but not the $(\nu+1)$-th generalised boundary, $\mathbf{x} \in \partial^\nu K \setminus \partial^{\nu+1} K$. Consider a sequence $(\beta_n)_{n\in\mathbb{N}}$ of Borel sets with

$$\beta_n \in \partial^\nu K, \qquad V^{(d-\nu)}(\beta_n) > 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = \{\mathbf{x}\}.$$

Then the *local Minkowski functional* $W_\mu(K, \mathbf{x})$ of K in the point $\mathbf{x}$ is well-defined by

$$W_\mu(K, \mathbf{x}) := \lim_{n\to\infty} \frac{\frac{1}{d} C_{d-\mu}(K, \beta_n)}{V^{(d-\nu)}(\beta_n)}$$

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- With these definitions & limit (4) in (3), replacing $\sum$ by $\int$:

$$W_\mu(K) = \sum_{\nu=1}^{\mu} \int_{\partial^\nu K} d\lambda_{d-\nu}(\mathbf{x}) W_\mu(K, \mathbf{x})$$

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- The local Minkowski functionals in a point $\mathbf{x} \in \partial A$, with $A \in \mathcal{R}$ (using the intersection of convex parts of A meeting in $\mathbf{x}$):

$$W_\mu(A,\mathbf{x}) = (-1)^{|\mathcal{J}|+1} W_\mu(K,\mathbf{x})$$

giving the global functional $W_\mu(A)$ when integrated over $\partial^\nu A$ with Lebesgue measures:

$$W_\mu(A) = \sum_{\nu=1}^{\mu} \int_{\partial^\nu A} d\lambda_{d-\nu}(\mathbf{x}) W_\mu(A,\mathbf{x})$$

3 Partition formula

- μ-th Minkowski functional can be re-written:

$$M_\mu(A) = \sum_{\nu=1}^{\mu} \int_{\mathbb{R}^d} d^d x M_\mu(A, \mathbf{x}) \chi_\nu(A, \mathbf{x})$$

partition formula

with *generalised characteristic function* of the ν-th generalised boundary defined as:

$$\chi_\nu(A, \mathbf{x}) = \int_{\partial^\nu A} d\lambda_{d-\nu}(\mathbf{y}) \delta(\mathbf{x} - \mathbf{y})$$

$\Longleftrightarrow$ $(d-\nu)$-dim. volume $V^{(d-\nu)}$ of the ν-th generalised boundary $\partial^\nu A$ is calculated by integrating with the $(d-\nu)$-dim. Lebesgue measure $d\lambda_{d-\nu}(\cdot)$:

$$V^{(d-\nu)} = \int_{\partial^\nu A} d\lambda_{d-\nu}(\mathbf{y}) = \int_{\partial^\nu A} d\lambda_{d-\nu}(\mathbf{y}) \int_{\mathbb{R}^d} d^d x \delta(\mathbf{x} - \mathbf{y}) =$$

$$= \int_{\mathbb{R}^d} d^d x \int_{\partial^\nu A} d\lambda_{d-\nu}(\mathbf{y}) \delta(\mathbf{x} - \mathbf{y})$$

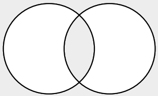
- simplification: for union of smooth convex bodies $A = \bigcup_{i \in \mathcal{I}} K_i \rightarrow$ the generalised boundary of ν-th order is the set of all points of ∂A where ν convex parts meet*:

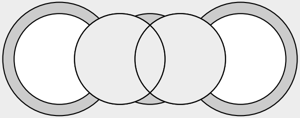
$$M_\mu(A) = \sum_{\nu=1}^{\mu} (-1)^{\nu-1} \sum_{\substack{\mathcal{J} \subset \mathcal{I} \\ |\mathcal{J}| = \nu}} \int_{\mathcal{V}} d^d x M_\mu(\bigcap_{j \in \mathcal{J}} K_j, \mathbf{x}) \chi_\nu(\bigcap_{j \in \mathcal{J}} K_j, \mathbf{x}) \chi(\partial A, \mathbf{x})$$

* w inclusion-exclusion principle

3 Partition formula

- The partition formula in 2D for 2 intersecting bodies:





→ the MFs of the union set: 3 boundary contributions - $M_{\mu,1}$, $M_{\mu,12}$, $M_{\mu,2}$
(2 for single surfaces: $M_{\mu,1}$, $M_{\mu,2}$, 1 for intersection points $M_{\mu,12}$)

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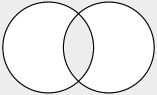

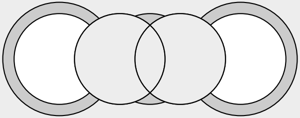

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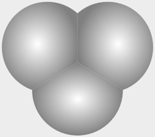
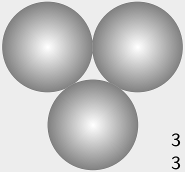
- The partition formula in 3D for 3 intersecting bodies:

$$M_\mu(A) = \underbrace{\sum_{i\in\mathcal{I}} M_{\mu,i}(A)}_{\text{surfaces of single partial bodies}} + \underbrace{\frac{1}{2}\sum_{i,j\in\mathcal{I}} M_{\mu,i,j}(A)}_{\text{intersection lines}} + \underbrace{\frac{1}{6}\sum_{i,j,k\in\mathcal{I}} M_{\mu,i,j,k}(A)}_{\text{intersection points where 3 partial bodies meet}}$$



3 faces
2 triple points
3 intersection lines



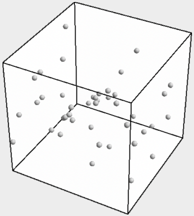
3 faces
3 intersection lines

Germ grain model

- $\{\mathbf{x}_i\}_{i=1\ldots N}$ - single realisation of a point process in a region $\mathcal{V}$ of space
  $\rightarrow$ decorate each point with a ball $K_i(R)$ of radius R
  $\rightarrow$ content, shape and connectivity of the union set

$$A(R) = \bigcup_{i=1}^{N} K_i(R)$$

  reflect the structure of the point process on a scale of the order of R
  $\rightarrow$ $A(R) \in \mathcal{R}$ (convex ring) $\Rightarrow M_\mu(A(R))$ - quantitative measures of its morphology



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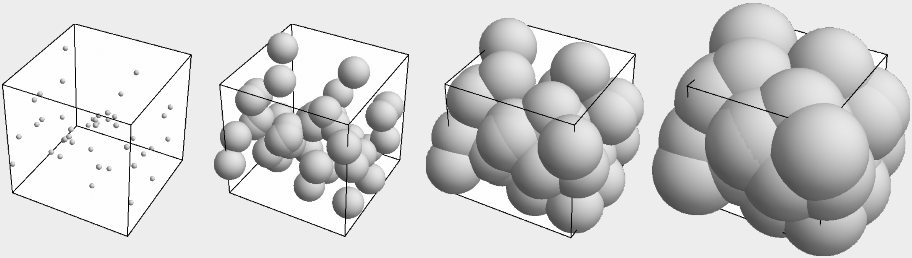
  $\rightarrow$ union's set morphology changes with variation of $R \rightarrow$ diagnostic parameter



The Poisson process

- principal kinematical formula $\oplus$ the Haar measure:
  $\rightarrow$ independent distribution of N bodies K_i corresponds to applying N motions g_i to single body and weighting with a product measure of the individual Haar measures:

$$A_N := \bigcup_{i=1}^{N} g_i K_i, \qquad d\mu_N := \prod_{i=1}^{N} \frac{dg_i}{|\mathcal{V}|} \quad \text{with} \int_{\mathcal{G}^N} d\mu_N = 1$$

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- N identical bodies in 3D with MFs $M_0 \dots M_3$ $\rightarrow$ volume densities $m_0 \dots m_3$ of MFs:

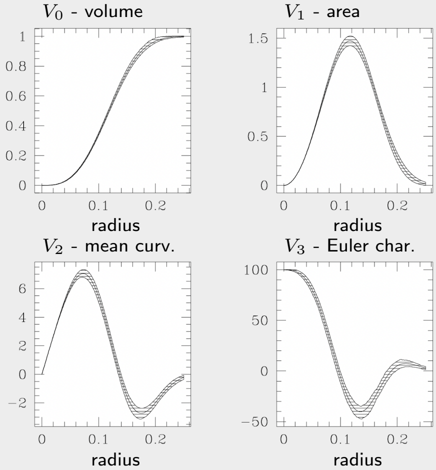
$$\begin{aligned} m_0 &= 1 - e^{-nM_0}, \\ m_1 &= e^{-nM_0} nM_1, \\ m_2 &= e^{-nM_0}(nM_2 - n^2 M_1^2), \\ m_3 &= e^{-nM_0}(nM_3 - 3n^2 M_1 M_2 + n^3 M_1^3), \end{aligned}$$

(Mecke & Wagner 1991)

with $n = N/|\mathcal{V}|$ and Minkowski functionals M_ν of balls of radius R:

$$M_0 = \tfrac{4\pi}{3} R^3, \quad M_1 = \tfrac{\pi}{2} R^2, \quad M_2 = \tfrac{2}{\pi} R, \quad M_3 = \tfrac{3}{4\pi}$$

The Poisson process

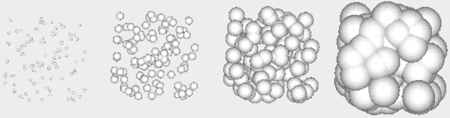


V_0 - grows monotonically with increasing radius r until complete filling

V_1 - increases with increasing r
- increase slows down and finally turns around ↔ intersecting neighbours

V_2 - positive maximum
- negative minimum ↔ formation of tunnels and networks

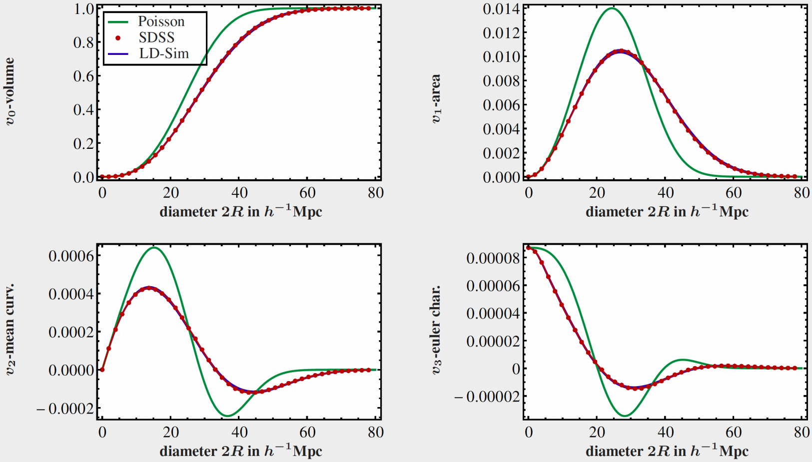
V_3 - small r = nb. of isolated points
- tunnels at intermediate r → negative
- cavities at large r → 2nd maximum



Schmalzing et al. 1996

SDSS: galaxy distribution

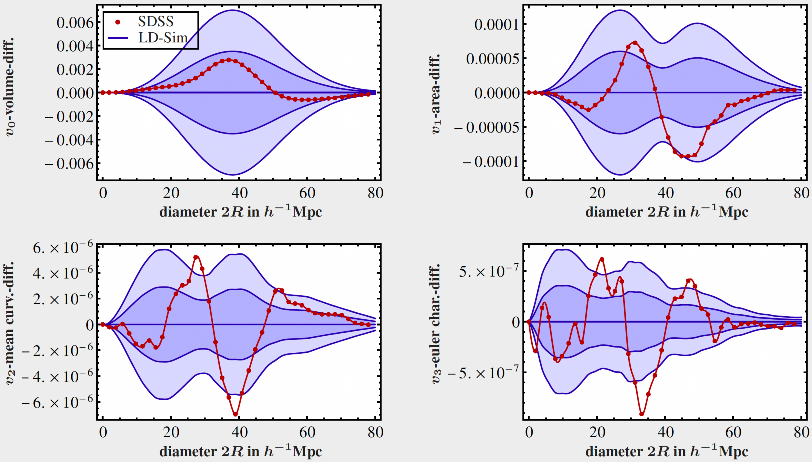
Minkowski functionals



Wiegand et al. 2014

SDSS: galaxy distribution

Differences



Wiegand et al. 2014

Koendrink invariants

- random field $\nu(\mathbf{x})$ on a d-dimensional support $\mathcal{D} \subseteq \mathbb{R}^d$

  $\rightarrow F_\nu$ - excursion set for a given threshold ν

$$F_\nu = \{\mathbf{x} \in \mathcal{D} | \nu(\mathbf{x}) \geq \nu\}$$

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  $\rightarrow$ Minkowski functionals (per unit volume) for $d = 3$:

$$v_0(\nu) = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \Theta \left[\nu - \nu(\mathbf{x})\right]$$

$$v_k(\nu) = \frac{1}{|\mathcal{D}|} \int_{\partial F_\nu} d^2A(\mathbf{x}) \, v_k^{(\mathrm{loc})}(\nu, \mathbf{x})$$

with

$$v_1^{(\mathrm{loc})}(\nu, \mathbf{x}) = \frac{1}{6}$$

$$v_2^{(\mathrm{loc})}(\nu, \mathbf{x}) = \frac{1}{6\pi} \left(\frac{1}{R_1(\mathbf{x})} + \frac{1}{R_2(\mathbf{x})}\right)$$

$$v_3^{(\mathrm{loc})}(\nu, \mathbf{x}) = \frac{1}{4\pi} \frac{1}{R_1(\mathbf{x}) R_2(\mathbf{x})}$$

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- transform the surface integrals into volume integrals:

$$v_k(\nu) = \frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \delta\left[\nu - \nu(\mathbf{x})\right] |\nabla \nu(\mathbf{x})| v_k^{(\mathrm{loc})}(\nu, \mathbf{x})$$

  with $v_k^{(\mathrm{loc})}(\nu, \mathbf{x})$ - in terms of Koendrink invariants (Konedrink 1984) formed from the 1st and 2nd derivatives of the field (e.g. Schmalzing & Buchert 1997)

Koendrink invariants

- density field $u(\mathbf{x})$ sampled at the grid points of a cubic lattice

  $\rightarrow F_\nu$ - excursion set for a given threshold ν

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  $\rightarrow$ Minkowski functionals (per unit volume):

$$\hat{v}_0(\nu) = \tfrac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \Theta\left[u - \nu\right]$$

$$\hat{v}_1(\nu) = \tfrac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \frac{1}{6} \delta(u - \nu)(u_{,i}u_{,i})^{1/2}$$

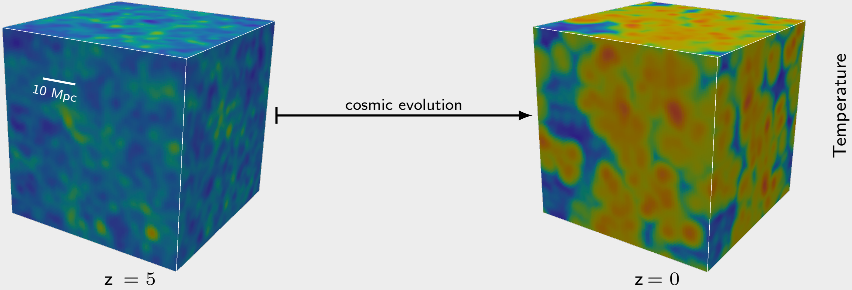
$$\hat{v}_2(\nu) = \tfrac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \frac{1}{3\pi} \delta(u - \nu) \frac{\epsilon_{ijm}\epsilon_{klm}u_{,i}u_{,jk}u_{,l}}{2u_{,n}u_{,n}}$$

$$\hat{v}_3(\nu) = \tfrac{1}{|\mathcal{D}|} \int_{\mathcal{D}} d^3x \, \frac{1}{4\pi} \delta(u - \nu) \frac{\epsilon_{ijk}\epsilon_{lmn}u_{,i}u_{,l}u_{,jm}u_{,kn}}{2(u_{,p}u_{,p})^{3/2}}$$

with $u_{,i}$, $u_{,ij}$ 1st and 2nd field's derivatives

Hydrodynamic simulation: baryonic physics

Gas distribution



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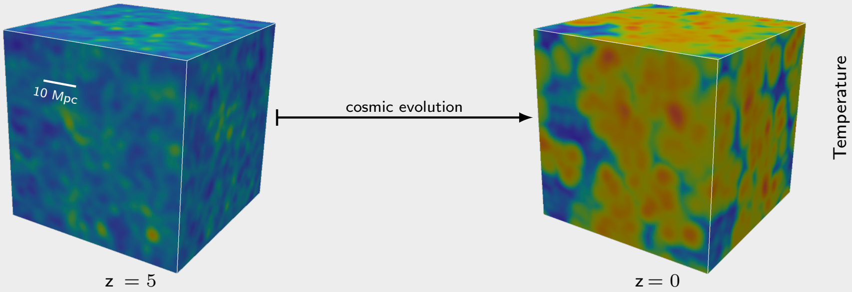


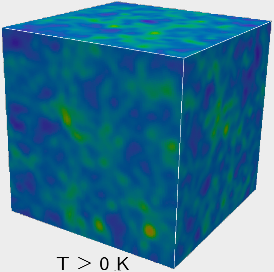
	Fiducial	NoX	NoJet	NoAGN	NoFeedback
Stellar feedback	✓	✓	✓	✓	
AGN winds	✓	✓	✓		
Jets	✓	✓			
X-ray heating	✓				

Summary of main ingredients of different SIMBA runs.

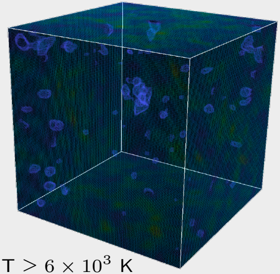
SIMBA (Davé et al. 2019)

Hydrodynamic simulation: baryonic physics

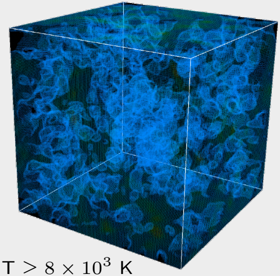
Excursion sets



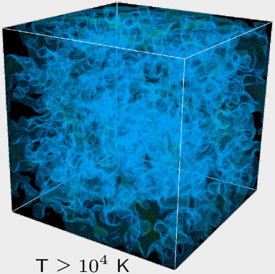
T ≥ 0 K



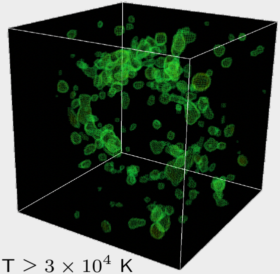
T $\geq 6 \times 10^3$ K



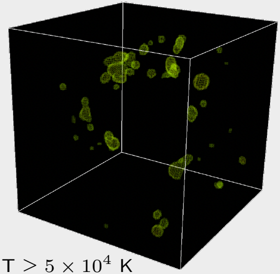
T $\geq 8 \times 10^3$ K



T $\geq 10^4$ K



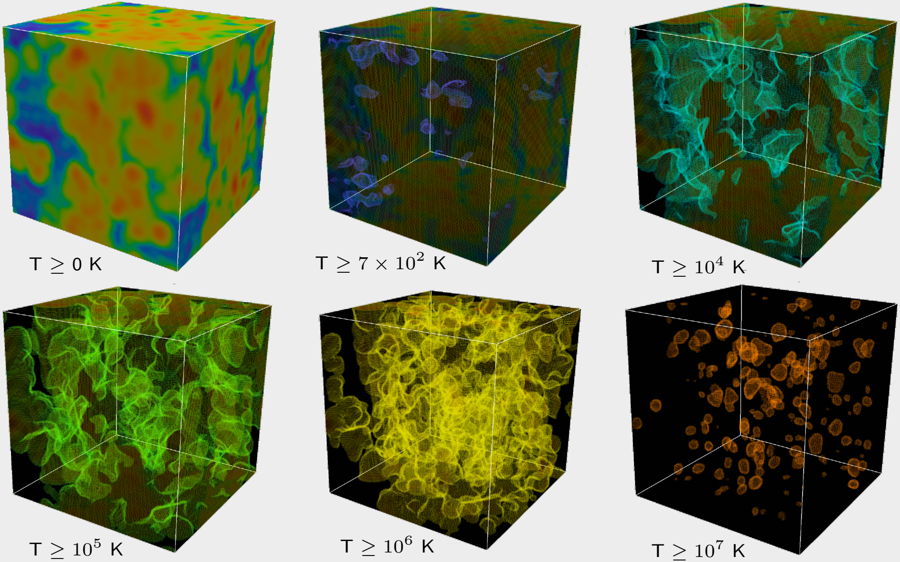
T $\geq 3 \times 10^4$ K



T $\geq 5 \times 10^4$ K

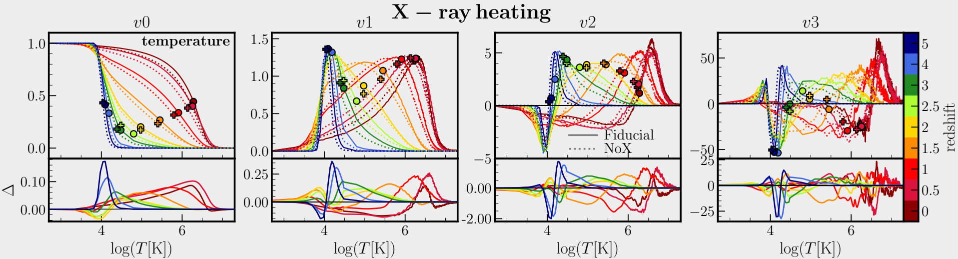
Hydrodynamic simulation: baryonic physics

Excursion sets



Hydrodynamic simulation: baryonic physics

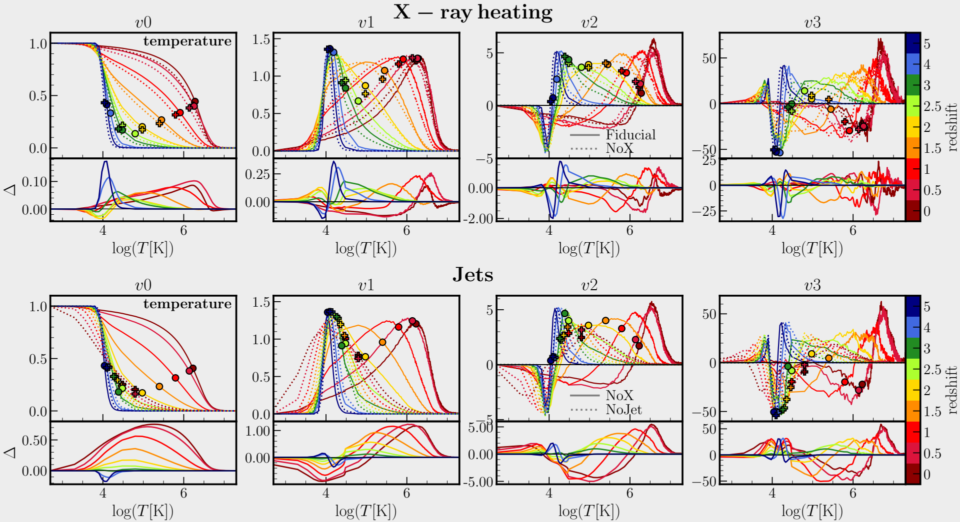
Minkowski functionals



Other properties: density, HI, H_2, pressure, metallicity, . . .

Hydrodynamic simulation: baryonic physics

Minkowski functionals



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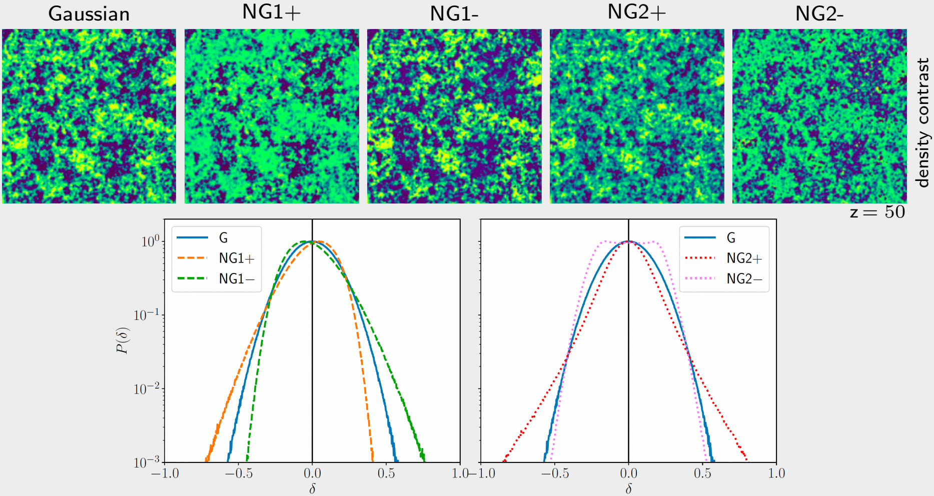
N-body simulations: primordial non-Gaussianity

Matter distribution



Stahl et al. 2023

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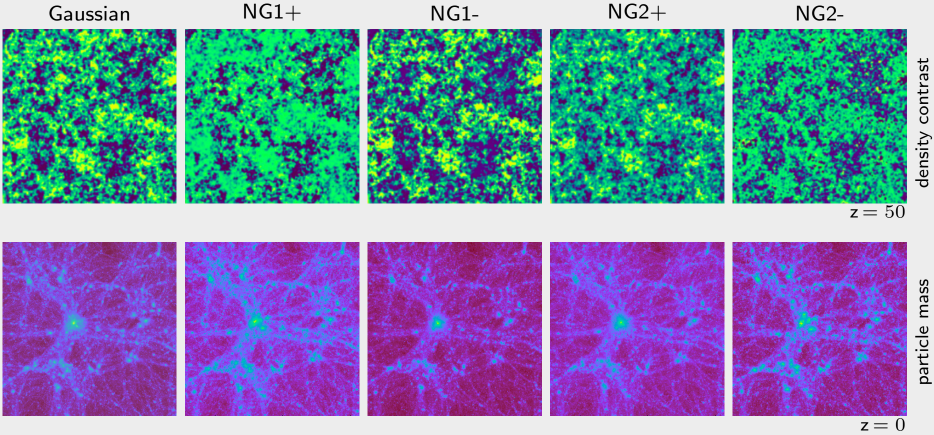

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