Souslin Space Theory

unnoticed connection between topologies and measures

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Who is Souslin?

Luzin's student (Moscow, 1913–1917), Souslin,
died during the Russian Civil War (1919).

Made major contributions to the fields of
general topology and descriptive set theory.

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Luzin's theorem

Measurable functions on $\mathbb{R}$ are nearly continuous.

Who is Galois?

Luzin's student (Moscow, 1913–1917), Souslin, died during the Russian Civil War (age 24).

Comparing to Galois (died at 20)

Who is Abel?

Luzin's student (Moscow, 1913–1917), Souslin,
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Comparing to Galois (died at 20), Abel (died at 26).

Who is Lebesgue?

Luzin's student (Moscow, 1913–1917), Souslin,
died during the Russian Civil War (age 24).

He found an error in Lebesgue's argument (1905):

the projection of a Borel set of $\mathbb{R}^2$ onto the real axis
was also a Borel set.

Who is Borel?

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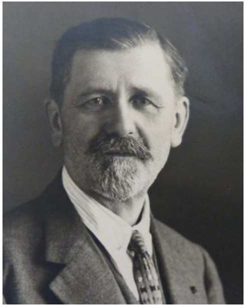

Borel sets

Given a topology space (X, τ), we construct the Borel σ-algebra

$$\mathcal{B}(X) \text{ set of Borel sets} \quad \Longleftarrow \quad \mathcal{T} \text{ set of open sets}$$

through countable union, countable intersection, and relative complement.

Remark (Lebesgue's error)

Projection mappings preserve open sets, but not Borel sets.

Remark

$(X, \mathcal{B}(X))$ is a measurable space, Borel sets are thus called measurable sets.

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Borel sets

WAIT! That is right, but the projection $(x, y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$ is a 1-Lipschitz map, so its image of a Lebesgue measurable set is Lebesgue measurable.

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What are Souslin spaces?

Definition

A set in a Hausdorff space is called Souslin if it is the image of a complete separable metric space under a continuous mapping.
A Souslin space is a Hausdorff space that is a Souslin set.

Remark

Souslin sets are also called analytic sets.

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A topological space homeomorphic to a complete separable metric space is called Polish Space.

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Examples

1. Polish spaces are Souslin spaces: manifolds, L^p spaces with separable base space.
2. Borel subsets of Souslin spaces are Souslin sets.

General structure of Souslin spaces: $\displaystyle\bigcup_{(n_i)\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=0}^{\infty} A_{n_0,\dots,n_k}$ ($A_{n_0,\dots,n_k}$ are closed sets)

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Unleash the power of Souslin spaces — step one

Theorem

Every nonempty Polish space is the image of $\mathbb{N}^{\mathbb{N}}$ *under a continuous mapping.*

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Proof ideas

1. Equip $\mathbb{N}^{\mathbb{N}}$ with a metric so that it has the product topology.
2. Represent a complete and separable metric space X, by induction, as

$$X = \bigcup_{j=0}^{\infty} E(j), \quad \ldots \quad , E(n_0, \ldots, n_k) = \bigcup_{j=0}^{\infty} E(n_0, \ldots, n_k, j),$$

   where $E(n_0, \ldots, n_k)$ is a nonempty closed set with diameter less than 2^{-k}.
3. Map $(n_0, n_1, \ldots)$ to the unique point inside all $E(n_0, \ldots, n_k)$ for $k \geq 0$.

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Corollary

A nonempty Souslin set has the form $f(\mathbb{N}^{\mathbb{N}})$ with f continuous, where $\mathbb{N}^{\mathbb{N}}$ is equipped with the product topology and is a Polish space.

Remark

It proves the general structure of Souslin sets:

$$X = \bigcup_{(n_i)\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=0}^{\infty} E(n_0,\dots,n_k) \implies f(X) = \bigcup_{(n_i)\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k=0}^{\infty} \overline{f(E(n_0,\dots,n_k))}$$

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Unleash the power of Souslin spaces — step two

Luzin's Separation Theorem

Disjoint Souslin sets can be separated by Borel sets.

Proof.

Reduce by contradiction to the separation of two points using the structure of $\mathbb{N}^{\mathbb{N}}$.

Write $X := \cup_{j=0}^{\infty} E(j)$ and $Y := \cup_{i=0}^{\infty} F(k)$. If X and Y are not separated, then for some indices i and k, $E(j)$ and $F(k)$ are not separated. $\square$

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Luzin's Separation Theorem

Disjoint Souslin sets can be separated by Borel sets.

Corollary

If A and $X \setminus A$ are both Souslin sets, then A is a Borel set.

Corollary

Let (X, τ) be a Souslin space. If $\tau' \subset \tau$ is another topology on X, then (X, τ) and (X, τ') have the same Borel set.

Proof.

The map $\mathrm{Id} : (X, \tau) \to (X, \tau')$ is continuous. Apply the previous corollary. $\square$

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Borel measures and various topologies

Recall

Let X be a topological space.
A Borel measure is a non-negative and countably additive set function on $\mathcal{B}(X)$.

Question

Let μ be a finite Borel measure for (X, τ). Assume that τ' is a topology on X.

1. Is μ a Borel measure for (X, τ')?
2. Does μ give mass to compact sets with respect to τ or τ'?

Borel measures and various topologies

Remark

A finite Borel measure μ on a Souslin spaces is a Radon measure, meaning that

$$\forall A \in \mathcal{B}(X), \quad \mu(A) = \sup \{\mu(K) \mid K \subset A,\, K \text{ is compact}\}$$

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The question is trivial if $\tau' \subset \tau$, which does not need Souslin space theory.

Exemplar usage in my thesis

Background setting

Let X be a Polish space with a Borel measure ν. Denote by $\mathbb{A}$ the set of probability measures on X that are absolutely continuous with respect to ν.
Via the identification $f \cdot \nu \leftrightarrow f$, $\mathbb{A}$ is identified as the set of density functions

$$\left\{ f \in L^1(\nu) \,\middle|\, \int_X f \,\mathrm{d}\,\nu = 1,\ f \geq 0 \right\}.$$

Induced topologies on $\mathbb{A}$

1. Elements as measures: weak convergence, set-wise convergence, Wasserstein convergence, total variation convergence.
2. Elements as functions: weak topology induced by L^∞, L^1-norm convergence.

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Existence of jokes in measure theory [Ma, 2023]

Any Souslin set of $\mathbb{R}^n$ *is Lebesgue measurable.*

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Sketch of the proof

We prove this theorem in two steps.

1. Souslin sets of $\mathbb{R}^n$ are projections of measurable sets from $\mathbb{R}^{n+1}$ [Bogachev, 2006].

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Remark

The author is hesitant to conjecture the uniqueness.