Lie brackets and interpolation for controllability.

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Small-time local controllability (STLC)

Let $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^n$, real-analytic, with $f_0(0) = 0$. Consider:

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Definition

We say that $(\star)$ is STLC when, for every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, there exists $u \in L^\infty((0,T);\mathbb{R})$ such that $x(T; u, 0) = x^*$ and $\|u\|_\infty \leq \eta$.

= Local surjectivity at $(0,0)$ of the input-output map

$$\left| \begin{array}{lccccc} \mathcal{F} : & \mathbb{R} & \times & L^\infty & \to & \mathbb{R}^n \\ & (T & , & u) & \mapsto & x(T; u, 0) \end{array} \right.$$

Goal: Find conditions on f_0 and f_1 for $(\star)$ to be STLC or not.

Some examples

Linear theory (Kalman rank condition):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \end{cases}$$

Quadratic theory (looks bad):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2. \end{cases}$$

Cubic theory (looks good):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3. \end{cases}$$

Why Lie brackets? (# 1)

Lie brackets measure the lack of commutativity between motions.
For vector fields $f, g \in C^\omega(\mathbb{R}^n; \mathbb{R}^n)$, $[f, g]$ is the vector field

$$[f, g](x) := Dg(x) \cdot f(x) - Df(x) \cdot g(x).$$

Example: If $\dot{x} = f_0(x) + u(t) f_1(x)$, $x(0) = 0$ and one uses

$$\begin{cases} u(t) = +\eta & \text{for } t \in (0, \tau), \\ u(t) = -\eta & \text{for } t \in (\tau, 2\tau), \end{cases}$$

then

$$x(2\tau; u, 0) = \tau^2 \eta [f_1, f_0](0) + \mathcal{O}(\tau^3).$$

For **all** systems, one can move towards **both** $\pm[f_1, f_0](0) \in \mathbb{R}^n$.
The underlying "abstract" Lie bracket $[X_1, X_0]$ is "good".

Algebraic foundations

- Let $X := \{X_0, X_1\}$ be non-commutative **indeterminates**
- Let $\mathcal{A}(X)$ be the **free algebra** over X, i.e. the vector space of non-commutative polynomials, e.g. $7X_0^2 + 3X_1X_0 + 2X_0X_1$
- Let $\mathcal{L}(X)$ the **free Lie algebra** over X, i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X_0, X_1, and stable by the Lie bracket (commutator) operation $[a, b] := ab - ba$
- One can "**evaluate**" (although not injective)

$$b \in \mathcal{L}(X) \hookrightarrow f_b \in C^\omega(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow f_b(0) \in \mathbb{R}^n$$

$$[X_1, X_0] = X_1X_0 - X_0X_1 \to [f_1, f_0] = (Df_0)f_1 - (Df_1)f_0 \to [f_1, f_0](0)$$

The Lie algebra rank condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Hermann 1963, Nagano 1966)

If $(\star)$ *is STLC, then it satisfies*

$$\mathit{Lie}(f_0, f_1)(0) := \operatorname{span}\{f_b(0);\ \ b \in \mathcal{L}(X)\} = \mathbb{R}^n. \qquad \text{(LARC)}$$

For non-zero drift $f_0 \neq 0$, (LARC) is not sufficient.

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^2, \end{cases}$$

has $f_{X_1}(0) = f_1(0) = e_1$ and $f_{W_1}(0) = [f_1, [f_1, f_0]](0) = 2e_2$.

The quadratic Lie bracket $W_1 := [X_1, [X_1, X_0]]$ looks like a "bad" bracket, associated with a signed motion in an oriented direction.

Why Lie brackets? (#2)

Consider

$$\dot{x} = f_0(x) + u(t) f_1(x) \quad \text{with} \quad f_0(0) = 0$$

$$\dot{y} = g_0(y) + u(t) g_1(y) \quad \text{with} \quad g_0(0) = 0.$$

Theorem (Krener 1973)

The two systems are diffeomorphic ***iff*** *same vectorial structure:*

$$\{b \in \mathcal{L}(X); f_b(0) = 0\} = \{b \in \mathcal{L}(X); g_b(0) = 0\}.$$

Hence, **the vectors $f_b(0)$ contain all the information for STLC**.

Goal of this talk

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad (\star)$$

- Prove sufficient/necessary conditions of STLC formulated in terms of Lie brackets of f_0 and f_1 evaluated at 0
- With a new strategy :
  - to go further on ODEs
  - to prepare the transfer to PDEs

Definition ($m \in [\![-1, \infty[\![$)

$(\star)$ is $W^{m,\infty}$-STLC when, $\forall T, \eta > 0$, $\exists \delta > 0$ st $\forall x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, $\exists u \in W^{m,\infty}(0,T)$ st $x(T; u, 0) = x^*$ and $\|u\|_{W^{m,\infty}} \leq \eta$.

$(W^{m,\infty}\text{-STLC}) \Rightarrow (L^\infty\text{-STLC}) \Rightarrow (W^{-1,\infty}\text{-STLC}) = (\text{small-state STLC})$

Computing the state using Lie brackets

$$\dot{x} = f_0(x) + u(t) f_1(x) \qquad\qquad x(0) = 0$$

Theorem (Beauchard, Le Borgne, Marbach 2020)

$$x(t;u) = \sum_b \eta_b(t,u) f_b(0) + O(\text{“\textit{remainders}”}) + o(x(t;u)).$$

The sum

- ranges over elements b of a basis of $\mathcal{L}(X)$
- involves system-dependent vectors $f_b(0) \in \mathbb{R}^n$
- universal functionals $\eta_b(t,u)$ homogeneous:
  $\eta_b(t, \epsilon u) = \epsilon^{n_1(b)} \eta_b(t,u) \qquad \eta_b(\epsilon, u(\frac{\cdot}{\epsilon})) = \epsilon^{|b|} \eta_b(1,u) \qquad \ldots$

Caution: The full sum does not converge, even with analyticity. One has to consider (possibly infinite) truncations (wrt t, or u, or a parameter). And well chosen bases of $\mathcal{L}(X)$. **This is not a Taylor expansion, but a csq of a Magnus-type formula.**

State-of-the-art about sufficient conditions

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Known sufficient conditions for STLC share a common structure:

Theorem

Assume (LARC) *and that, for every $b \in \mathfrak{B}$,*

$$f_b(0) \in \operatorname{span}\{f_g(0);\ \omega(g) < \omega(b)\}.$$

Then $(\star)$ *is STLC.*

- $\mathfrak{B} \subset \mathcal{L}(X)$ is a set of “potentially bad” brackets, which you do not know how to use with your current technology
- $\omega : \mathcal{L}(X) \to \mathbb{R}$ is a “weight” which sorts the brackets according to a small-parameter limit you are considering

Version #1: Linear test

An example:

$$\begin{cases} \dot{x}_1 = u + x_1^3, \\ \dot{x}_2 = x_1 + x_1^2 + x_3^5, \\ \dot{x}_3 = x_2 + x_2^4. \end{cases}$$

Version #1: Linear test

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Kalman 1960, Markus 1965)

The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ \ n_1(b) \geq 2\}$
- $\omega(b) := n_1(b)$

Indeed, use a control of the form $u(t) = \varepsilon \bar{u}(t)$, then

$$\begin{aligned} x(t; u, 0) &\approx \sum_b \varepsilon^{n_1(b)} \eta_b(t, \bar{u}) f_b(0) \\ &\approx \varepsilon \sum_{n_1(b)=1} \eta_b(t, \bar{u}) f_b(0) + \varepsilon^2 \sum_{n_1(b)\geq 2} \cdots \end{aligned}$$

When $n_1(b) = 1$, $b = \pm \operatorname{ad}^k_{X_0}(X_1)$ and $f_b(0) = \pm (D f_0(0))^k f_1(0)$.

Version #2: Hermes condition

An example:

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1^3 + x_1^4, \\ \dot{x}_3 = x_2^5 + x_1^{16}. \end{cases}$$

Version #2: Hermes condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1983)

The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ \ n_1(b)$ *is even*$\}$
- $\omega(b) := n_1(b)$

Indeed, use a control of the form $u(t) = \varepsilon \bar{u}(t)$, then

$$x(t; u, 0) \approx \sum_b \varepsilon^{n_1(b)} \eta_b(t, \bar{u}) f_b(0)$$

$$\approx \sum_{j \geq 0} \varepsilon^{2j+1} \left(\sum_{n_1(b)=2j+1} \eta_b(t, \bar{u}) f_b(0) + \varepsilon \sum_{n_1(b)=2j+2} \eta_b(t, \bar{u}) f_b(0) \right)$$

Key point: Odd terms aren't signed since $\eta_b(t, -u) = -\eta_b(t, u)$.

Version #3: $\mathcal{S}(\theta)$ condition

An example (Jakubczyk, Sussmann 1983):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2^2 + x_1^3. \end{cases}$$

If u is oscillating (contains high frequencies), then x_1 and x_2 also.
Since $x_1 = \dot{x}_2$, one can have $|x_1^3| \gg x_2^2$.
Involves $W_2 = \mathrm{ad}^2_{[X_1, X_0]}(X_0)$ and $\mathrm{ad}^3_{X_1}(X_0)$.

Another (Stefani, 1985):

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_1^3 x_2. \end{cases}$$

The last one is quartic ... but good!
By time-reversal $\check{u}(t) = u(T - t)$ then $x_3(T) \leftarrow -x_3(T)$

Version #3: $\mathcal{S}(\theta)$ condition

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1987)

Let $\theta \in [0,1]$. The result holds with

- $\mathfrak{B} := \{b \in \mathcal{L}(X);\ \ n_1(b)$ *is even and* $n_0(b)$ *is odd*$\}$
- $\omega(b) := n_1(b) + \theta n_0(b)$

For controls of the form $u(t) = \varepsilon^{1-\theta}\bar{u}(t/\varepsilon^{\theta})$, then

$$x(t; u, 0) \approx \sum_b \varepsilon^{\omega(b)} \eta_b(t, \bar{u}) f_b(0)$$

Key point: Terms with $n_1(b)$ even and $n_0(b)$ even aren't signed since, by time reversal, $\eta_b(t, \check{u}) \approx -\eta_b(t, u)$.

Similar & stronger: Agrachev Gamkrelidze 1993, Krastanov 2009.

Sufficient conditions: conclusion

The brackets of $\mathcal{L}(X) \setminus \mathfrak{B}$ are good: we know how to use them for controllability. If all other brackets vanish, the system is STLC.

What about the brackets of $\mathfrak{B}$? Are they bad? Is some kind of compensation condition indeed **necessary** for STLC?

Necessary #1

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Sussmann 1983)

If $(\star)$ *is* $W^{-1,\infty}$*-STLC, then, with* $W_1 := [X_1, [X_1, X_0]]$,

$$f_{W_1}(0) \in \operatorname{span}\{f_b(0); n_1(b) = 1\}.$$

Idea:

$$x(t; u, 0) \approx \sum_{n_1(b)=1} \eta_b(t, u) f_b(0) + \frac{1}{2} \overbrace{\left(\int_0^t u_1^2 \right)}^{\text{coercive}} f_{W_1}(0) + \mathcal{O}(t \|u_1\|_{L^2}^2 + \|u_1\|_{L^3}^3)$$

where $u_1(s) := \int_0^s u$ and $\|u_1\|_{L^3}^3 \leq \|u_1\|_\infty \|u_1\|_{L^2}^2$.

Necessary #2

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Stefani 1986)

If $(\star)$ is $W^{-1,\infty}$-STLC, then, for each $\ell \in \mathbb{N}^*$,

$$f_{\mathrm{ad}_{X_1}^{2\ell}(X_0)}(0) \in \operatorname{span}\{f_b(0); n_1(b) \leq 2\ell - 1\}.$$

Idea:

$$x(t;u,0) \approx \sum_{n_1(b)<2\ell} \eta_b(t,u) f_b(0) + \frac{1}{(2\ell)!} \overbrace{\left(\int_0^t u_1^{2\ell}\right)}^{\text{coercive}} f_{\mathrm{ad}_{X_1}^{2\ell}(X_0)}(0)$$
$$+ \mathcal{O}(t\|u_1\|_{L^{2\ell}}^{2\ell} + \|u_1\|_{L^{2\ell+1}}^{2\ell+1}).$$

Necessary #3

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Kawski 1987)

If $(\star)$ is L^∞-STLC, then, for $W_2 := \mathrm{ad}^2_{[X_1,X_0]}(X_0)$

$$f_{W_2}(0) \in \operatorname{span}\{f_b(0); n_1(b) \leq 3^*\}.$$

Idea:

$$x(t;u,0) \approx \sum_{n_1(b)\leq 3^*} \eta_b(t,u) f_b(0) + \frac{1}{2}\left(\int_0^t u_2^2\right) f_{W_2}(0) + \mathcal{O}(\|u_1\|^4_{L^4}).$$

And the key point: $\|u_1\|^4_{L^4} \leq \|u\|^2_{L^\infty}\|u_2\|^2_{L^2}$.

*: More precisely, one also has to exclude W_2 itself.

Necessary #4

$$\dot{x} = f_0(x) + u(t) f_1(x). \qquad (\star)$$

Theorem (Beauchard, Marbach 2022)

If $(\star)$ *is* L^∞*-STLC, then, for each* $k \in \mathbb{N}$, $W_k := \mathrm{ad}^2_{\mathrm{ad}^{k-1}_{X_0}(X_1)}(X_0)$

$$f_{W_k}(0) \in \mathrm{span}\{f_b(0); n_1(b) \leq 2k-1^*\}.$$

Idea:

$$x(t;u,0) \approx \sum_{n_1(b) \leq 2k-1^*} \eta_b(t,u) f_b(0) + \frac{1}{2}\left(\int_0^t u_k^2\right) f_{W_k}(0) + \mathcal{O}(\|u_1\|^{2k}_{L^{2k}})$$

where, by **Gagliardo-Nirenberg** $\|u_1\|^{2k}_{L^{2k}} \leq \|u\|^{2k-2}_{L^\infty} \|u_k\|^2_{L^2}$.

- Kawski had conjectured this in 1986, and proved a weaker version where $n_1(b) \leq 2^k$.
- We prove an analogue result for $W^{m,\infty}$-STLC, $\forall m \in [\![-1, \infty[\![$.

Necessary #5

Theorem (KB FM 2020)

STLC $\Rightarrow$ $f_{W_j}(0) \in \mathcal{N}_j(f)(0)$ *for* $j = 2, 3$, *where*
$\mathcal{N}_2 = \{M_\nu, P_{1,1,\nu}; \nu \in \mathbb{N}\}$ *and*
$\mathcal{N}_3 =$
$\{M_\nu, P_{1,l,\nu}, Q_{1,1,1}, Q_{1,1,2,\nu}, Q^\flat_{1,0}, Q^\flat_{1,1}, Q^\flat_{1,2}, R_{1,1,1,1,\nu}, R^\sharp_{1,1,1,\mu,\nu}; l \in \mathbb{N}^*, \mu, \nu \in \mathbb{N}\}$.

where

$M_\nu := X_1 0^\nu$,
$W_{j,\nu} := (M_{j-1}, M_j)0^\nu$,
$P_{j,k,\nu} := (M_{k-1}, W_{j,0})0^\nu$,
$Q_{j,k,l,\nu} := (M_{l-1}, P_{j,k,0})0^\nu$, $Q^\sharp_{j,\mu,k,\nu} := (W_{j,\mu}, W_k)0^\nu$,
$Q^\flat_{j,\mu,\nu} := (W_{j,\mu}, W_{j,\mu+1})0^\nu$,
$R_{j,k,l,m,\nu} := (M_{m-1}, Q_{j,k,l,0})0^\nu$, $R^\sharp_{j,k,l,\mu,\nu} := (W_{l,\mu}, P_{j,k,0})0^\nu$

Necessary conditions: conclusion, perspectives

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

We have proposed methodology ingredients to prove NC for STLC:

- approximate formula for the state from the $f_b(0)$,
- interpolation inequalities to absorb the remainder by the coercive signed drift and the smallness of the control.

A long-standing problem is to "split" $\mathcal{L}(X)$ between good and bad brackets (for definitions to be found). For example:

- "good": a system involving only good ones should be STLC,
- "bad": if a system is STLC, then no bad one is alone.

A contribution: a new Hall basis $\mathcal{B}^\star$ of $\mathcal{L}(X)$, specifically designed for this purpose.

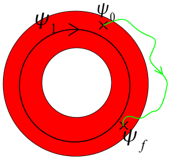
New interpolation inequalities are needed, for instance

$$\|D^j \varphi\|_{L^p}^p \leq \|D^{j+1}\varphi\|_{L^\infty}^q \int_{\mathbb{R}} |D^{j_1}\varphi|^{p_1} |D^{j_2}\varphi|^{p_2} \dots |D^{j_k}\varphi|^{p_k}$$

Example of transfer to Schrödinger PDE

$$i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi \qquad \psi(t,0) = \psi(t,1) = 0$$

Ground state:
$\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$



Depending on the assumption on μ:

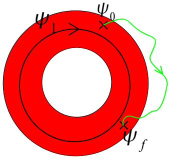
- linear test + smoothing effect [KB-Laurent 2010]
- 1 direction lost on the linearized syst and [Bournissou 2022]
  - quadratic obstruction in some regimes
  - STLC in other regimes : $A_3 \int_0^T u_3^2 dt + C \int_0^T u_1^2 u_2$
    This is the first positive STLC result for a PDE with a nonlinear competition.

Perspectives: Does it work for KdV?
How behave the high order terms for multi-input syst? [Gherdaoui]

Part 2: Schrödinger PDE

$$i\partial_t \psi = -\partial_x^2 \psi - u(t)\mu(x)\psi \qquad \psi(t,0) = \psi(t,1) = 0$$

Ground state:
$\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$



Qu: Small-time local controllability around the ground state?

1. What can be proved with the linear test?
2. When the linearized system around the ground state is not controllable:
   - quadratic obstructions to STLC
   - STLC with nonlinear competitions

Schrödinger: local exact control around the ground state

$$i\partial_t\psi = -\partial_x^2\psi - u(t)\mu(x)\psi \qquad \psi(t,.)|_{\{0,1\}} = 0$$

Theorem (KB-Laurent 2010)

Let $\mu \in H^3((0,1),\mathbb{R})$ *such that*

$$\exists c>0, \forall j \in \mathbb{N}^* \quad \frac{c}{j^3} \leqslant |\langle \mu\varphi_1, \varphi_j\rangle| = \left| \int_0^1 \mu(x)\sin(\pi x)\sin(j\pi x)dx \right|.$$

The Schrödinger equation is STLC in $H^3_{(0)}(0,1)$ *with controls in* L^2*:*
$\forall T>0$, $\exists \eta>0$ *st* $\forall \psi_f \in \mathcal{S} \cap H^3_{(0)}(0,1)$ *with* $\|\psi_f - \psi_1(T)\|_{H^3} < \eta$,
$\exists u \in L^2((0,T),\mathbb{R})$ *st* $\psi(T;u,\varphi_1) = \psi_f$ *and* $\|u\|_{L^2} \leq \|\psi_f - \psi_1(T)\|_{H^3}$

Rk: $\langle \mu\varphi_1, \varphi_j\rangle = \frac{\pm\mu'(1)-\mu'(0)}{j^3} + o(\frac{1}{j^3})$

Proof: linear test + smoothing effect, flexible

[Bournissou 2021]: When the first p odd derivatives of μ vanish on $\{0,1\}$ and $\forall j \in J$, $\frac{c}{j^{2p+3}} \leq |\langle \mu\varphi_1, \varphi_j\rangle|$ then the projection on Span$\{\varphi_j; j \in J\}$ is STLC in $H^{2(p+k)+3}_{(0)}(0,1)$ with controls in $H^k_0(0,T)$ and $\forall m \in \{-k,\dots,k\}$, $\|u\|_{H^m_0(0,T)} \leq C\|\psi_f - \mathbb{P}_J\psi_1(T)\|_{H^{2(p+m)+3}_{(0)}}$.

Schrödinger: quadratic obstructions

$$i\partial_t \psi = (-\partial_x^2 - u(t)\mu(x))\psi \qquad \psi(t,0) = \psi(t,1) = 0$$

[Coron 2006 / KB-Morancey 2014]: no L^∞/L^2-STLC when $n = 1$

Theorem (Bournissou 2021)

Let $n, K \in \mathbb{N}^*$, $\mu \in H^{2n}(0,1)$ such that

- $\langle \mu\varphi_1, \varphi_K \rangle = 0$, *:1 lost direction*
- *the first $(n-1)$ odd derivatives of μ vanish at $x = 0, 1$,*
- *n is the minimal value st $A_n = \langle [\mathrm{ad}_\Delta^{n-1}(\mu), \mathrm{ad}_\Delta^n(\mu)]\varphi_1, \varphi_K \rangle \neq 0$.*

Then, the Schrödinger equation is not H^{2n-3}-STLC

$$\langle \text{Quad}(T), \varphi_K \rangle \overset{IBP}{=} \sum_{p=1}^{n} A_p \int_0^T u_p(t)^2 e^{i(\lambda_K - \lambda_1)(T-t)} dt + \dots \underset{T\to 0}{\approx} A_n \int_0^T u_n^2$$

$$\langle \text{Order} \geq 3, \varphi_K \rangle \overset{aux}{=} \mathcal{O}(\|u_1\|_{L^2}^3 + |u_1(T)|^3)$$

$$\|u_1\|_{L^2}^3 \leq C_T \|u\|_{H^{2n-3}} \|u_n\|_{L^2}^2$$

$$\pm \Im \langle \psi(T), \varphi_K e^{-i\lambda_1 T} \rangle \geq |A_n|^- \int_0^T u_n(t)^2 dt - C\|\psi(T) - \varphi_1 e^{-i\lambda_1 T}\|^2$$

Schrödinger: STLC recovery thanks to cubic terms

$$i\partial_t\psi = (-\partial_x^2 - u(t)\mu(x))\psi \qquad \psi(t,0)=\psi(t,1)=0$$

Theorem (Bournissou 2022)

Let $K \geq 2$ and $\mu \in H^{11}(0,1)$ such that

- $\langle \mu\varphi_1, \varphi_K\rangle = 0$ *:1 lost direction*
- $\mu' = \mu^{(3)} = 0$ *on* $\{0,1\}$ *and* $\frac{c}{j^7} \leq |\langle \mu\varphi_1, \varphi_j\rangle|, \quad \forall j \in \mathbb{N}^* \setminus \{K\}$
- $n = 3$ *is smallest value of* $n \in \mathbb{N}^*$ *for which* $A_n := \langle [\mathrm{ad}_\Delta^{n-1}(\mu), \mathrm{ad}_\Delta^n(\mu)]\varphi_1, \varphi_K\rangle$ *does not vanish* *:quadratic drift*
- $C := \langle [[\mu,\Delta], \mathrm{ad}_\mu^2(\Delta)]\varphi_1, \varphi_K\rangle \neq 0.$ *:cubic term*

Then the Schrödinger equation is H_0^2-STLC with targets in $H_{(0)}^{11}(0,1)$.

Rk: Not H_0^3-STLC

$$\langle \psi(T) - \varphi_1 e^{-i\lambda_1 T}, \varphi_K\rangle \approx A_3 \int_0^T u_3^2 dt + C\int_0^T u_1^2 u_2 + \dots$$

Schrödinger: quadratic-cubic competition

$$\langle \psi(T) - \varphi_1 e^{-i\lambda_1 T}, \varphi_K \rangle \approx A_3 \int_0^T u_3^2 dt + C \int_0^T u_1^2 u_2 + ...$$

Proof: For a given target $\psi_f \in H^{11}_{(0)}(0,1)$

1. Use oscillating controls for which the cubic term dominates the quadratic one, to get the expected component along φ_K.
2. Correct the other components **in infinite number**, thanks to the local controllability in projection [Bournissou 2021].

Then $\psi(T; u, \varphi_1) = \psi_f + c\varphi_K$ with $|c| << \|\psi_f - \psi_1(T)\|$ + Brouwer.

Key point: The second step does not affect much the component along φ_K. Sharp estimates on this correction are needed. They involve the H^{-k}-norms of the control used and are proved in [Bournissou 2021].

Bilinear control of Schrödinger: conclusion, perspectives

- **A proof on the bilinear Schrödinger PDE of all the quadratic drifts known in finite dimension.**
  Other examples of quadratic drifts: Burgers [Marbach 2017], parabolic eqs [KB-Marbach 2020], KdV and St Venant [Coron, Koenig, Nguyen 2020 & 2022]
- **The first positive STLC result for a PDE with a nonlinear competition** [Bournissou 2022]
- Does it work for KdV?
- How behave the high order terms for a multi-input Schrödinger PDE? [Gherdaoui]

Thanks !