Systems subject to input saturation: from ODEs to PDEs

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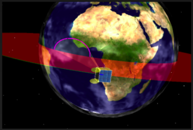
Outline

- Presence of constraints
  Any dynamical system is subject to constraints due to physical, safety or technological reasons
- Motivation
  - ▷ Take into account these constraints when studying desirable properties of the system (stability, performance, convergence)

Constraints affecting the actuators and/or sensors

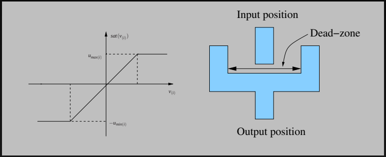


  - ▷ Need to develop adequate methodologies

Different types of isolated nonlinearities

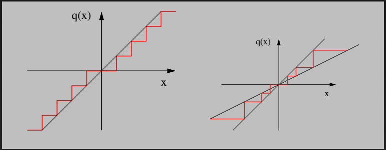
- Actuators/sensors practical limitations: saturation, hysteresis, dead-zone, discontinuities, . . .

Refs: E. Sontag, A. Teel, M. Turner, L. Zaccarian, J.M. Gomes da Silva, Z. Lin, J.M. Biannic, B. Jayawardhana, H. Logemann, P.O. Gutman, ...



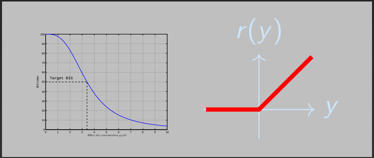
- Communication channels or information capacity: quantizer, coding, sampling . . .

Refs: D. Liberzon, C. De Persis, D. F. Delchamps, S. K. Mitter, F. Ferrante, ...



- Problem under study: positivity, ReLu or ramp, gradient $\nabla f(x)$, ...

Refs: M. Ait-Rami, Y. Ebihara, P. Seiler, M. Arcak, M.Korda, L. Lessard, R. Sanfelice, ...



Main objectives

- The main objectives are the ability to develop certificates of some properties (stability, performance, robustness, safety, tolerance, algorithmic convergence, ...), which are difficult or impossible to check analytically.
  - ▷ The solutions consist in using suitable abstractions sufficiently representative.
  - ▷ Very simple example. Vertical position of the baby.

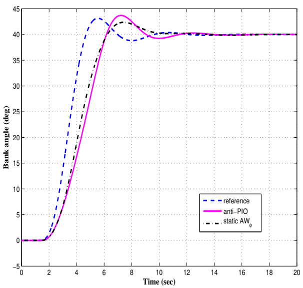
Indeed, the baby tries to control its position (from the ground to stand up)

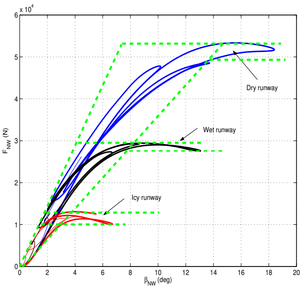
- baby = inverted pendulum
- diaper = a sort of uncertainty
- muscle strength = constraint

Particular case: saturation.
Class of nonlinearities apparently simple but difficult to manage

- Limitations in magnitude, rate, acceleration ... leading to saturations. Examples: PIO (Pilot-Induced-Oscillations) in aircrafts and problem of formation flight (satellites)

- The saturation function allows to approximate other types of nonlinearities. Examples: robust landing and on-ground control for civil aircraft - Approx. of ground forces (nose wheel force)





Main idea: embed the nonlinearity

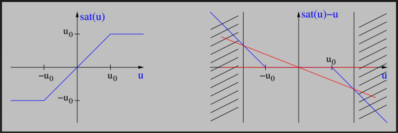
- Consider for example a continuous-time plant

$$\dot{x} = f(x) + g(x)\psi(u(x)) \quad (1)$$

- One can provide **sector conditions** on $\phi(u(x)) = \psi(u(x)) - u(x)$:

$$(\psi(u(x)) - u(x))^\top h(x) \geq 0, x \in \Omega \quad (2)$$

▷ Used to handle different problems (stability analysis, optimization of the region of stability, anti-windup schemes, delay, sampling, event-triggered control, ...)



- Other methods: via differential inclusions, PWA, IQC (Khalil, Lin, Alamo, Rantzer, Scherer, Valmorbida, ...) but may reveal to be "more complex/conservative" in the control design context

Main idea: embed the nonlinearity (cont'd)

- System (1) can be re-written as

$$\dot{x} = (f(x) + g(x)u(x)) + g(x)(\psi(u(x)) - u(x)) \tag{3}$$

  - ▷ The closed loop $f(x) + g(x)u(x)$ is assumed to satisfy the desired property (stability).

- One can build a Lyapunov function V using the abstraction on the nonlinearity to guarantee its decreasing along the closed-loop trajectories:

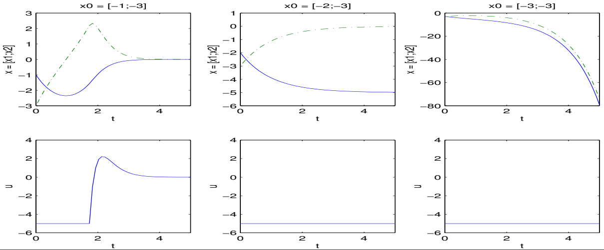
$$\begin{array}{l} V(x) > 0, x \neq 0, x \in \Omega \\ \dot{V}(x) + \tau(x)(\psi(u(x)) - u(x))^\top h(x) < 0 \end{array} \tag{4}$$

Focus: linear systems with saturating input

- Consider the following system:

$$\begin{array}{rcl} \dot{x} & = & Ax + Bsat(u) \\ u & = & Kx \end{array} \qquad (5)$$

- Indeed, saturation is an *abrupt* nonlinearity [Tarbouriech et al., 2011], [Teel and Zaccarian, 2011]
  - ▷ Small signal (around the origin): $sat(u) = u$ and no effect on the system trajectories
  - ▷ Large signal (far from the origin): $sat(u)$ is uniformly bounded and there is a severe effect on the system



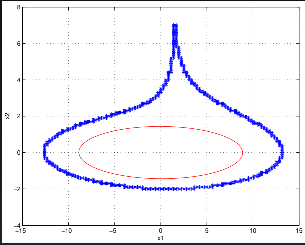
- In general, $\exists x(0)$ such that the trajectories converge to the origin, i.e. $x(t) \to 0$ as $t \to \infty$, but also initial conditions leading to diverging trajectories, i.e. $x(t) \to \infty$ as $t \to \infty$.

- Region of attraction RA of the origin = the set of all points $x(0) \in \mathbb{R}^n$ leading to solutions that converge asymptotically to the origin.

▷ $RA(0)$ = the exact stability region of the saturated system.

▷ Global stability: $RA(0) = \mathbb{R}^n$

▷ Local stability: $RA(0) \neq \mathbb{R}^n$



Objective: Approximate the RA

Stability may be local or global

Quick overview

- Seminal works but with different saturation maps as L^2 saturation:
  [Slemrod, 1989], [Lasiecka and Seidman, 2003] (See also [Curtain and Zwart, 2020] for the case of Lipschitz nonlinearity)
- Recent works dealing with cone bounded nonlinearly/saturation for abstract systems, hyperbolic systems, reaction-diffusion systems:
  [Prieur et al., 2016], [Marx et al., 2017], [Prieur and Tarbouriech, 2019], [Chitour et al., 2020], [Mironchenko et al., 2021], [Vanspranghe et al., 2021], [Gauvrit et al., 2023], [Lhachemi and Prieur, 2023], ...

Questions

- What happens in the context of PDE in presence of saturation?
- Distributed or boundary control subject to saturation?
- Is it possible to use the same framework (quadratic abstraction + Lyapunov-based conditions)?

PDE + saturation

PDEs

- Wave equation
- Beam equation

Control law

- Static control law
- Dynamic control law

Nonlinearity

- Saturation
- Cone bounded

Common tools

- Quadratic abstraction to embed the nonlinearity
- Lyapunov function to ensure the stability

Main objectives

- Well-posedness
- Stability guarantees
- Characterization of the basin of attraction (Estimate)

Two kinds of PDE

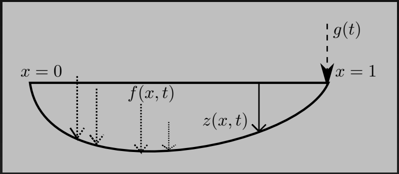
- Wave equation (with boundary conditions)

$$\begin{aligned} z_{tt}(x,t) &= z_{xx}(x,t) + f(t) \\ z(0,t) &= 0 \\ z_x(1,t) + g(t) &= 0 \end{aligned} \quad (6)$$

with the following initial condition, for all x in $(0,1)$,

$$\begin{aligned} z(x,0) &= z^0(x) \\ z_t(x,0) &= z^1(x) \end{aligned} \quad (7)$$

where z^0 and z^1 stand respectively for the initial deflection of the slope and the initial deflection speed.



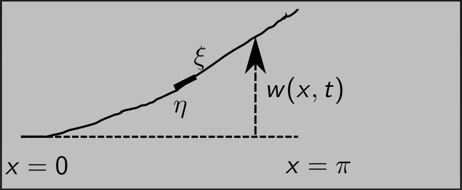
Vibrating slope subject to a external distributed action $f(x,t)$ and to a boundary action $g(t)$

Two kinds of PDE (cont'd)

- Beam equation

$$
\begin{cases}
w_{tt}(x,t) + w_{xxxx}(x,t) = u(t)\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)],\\
w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0,\\
w(x,0) = w^0(x),\\
w_t(x,0) = w^1(x)
\end{cases}
\tag{8}
$$

with $w(x,t)$ the deflection of the beam with respect to the rest position, at point x in $[0,\pi]$ and at time t, $u(t)$ the voltage applied on a actuator located between on the interval $[\eta,\xi]$.



A clamped-free beam subject to a piezoelectric actuator

Two kinds of control law

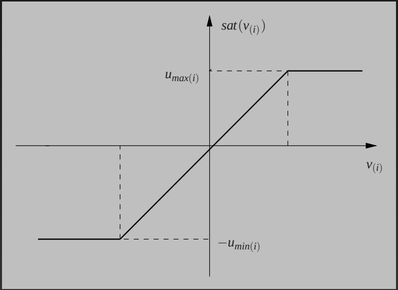
- Static nonlinear control: saturating static control

▷ Wave equation: $g(t) = sat(dz_t(1,t)), \forall t \geq 0$ yields the boundary conditions become:

$$z(0,t) = 0 \ , \quad z_x(1,t) + sat(dz_t(1,t)) = 0$$

where *sat* is the localized saturated map [Prieur et al., 2016].

$$sat(v_{(i)}) = \begin{cases} u_{max(i)} & \text{if } v_{(i)} > u_{max(i)} \\ v_{(i)} & \text{if } -u_{min(i)} \leq v_{(i)} \leq u_{max(i)} \\ -u_{min(i)} & \text{if } v_{(i)} < -u_{min(i)} \end{cases}$$



Two kinds of control law (cont'd)

- Static nonlinear control: saturating static control

▷ Beam equation: $u(t) = sat(k(w_{xt}(\eta) - w_{xt}(\xi))),\ \forall t \geq 0$ yields:

$$\begin{array}{l} w_{tt}(x,t) + w_{xxxx}(x,t) = sat(k(w_{xt}(\eta) - w_{xt}(\xi)))\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \\ w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0, \\ w(x,0) = w^0(x),\ \ w_t(x,0) = w^1(x) \end{array} \tag{9}$$

where *sat* is the localized saturated map [Prieur and Tarbouriech, 2019].

Two kinds of control law (cont'd)

- Dynamic nonlinear control:

▷ Wave equation: $g(t) = sat(Dz_t(1,t) + Cw(t)), \forall t \geq 0$ yields the boundary conditions:

$$\begin{aligned} & z(0,t) = 0 \ , \\ & z_x(1,t) + sat(Dz_t(1,t) + Cw(t)) = 0 \ , \\ & \dot{w} = Aw + Bz_t(1,t) \end{aligned} \tag{10}$$

where $w(t) \in \mathbb{R}^n$ and *sat* is the localized saturated map [Gauvrit et al., 2023].

Two kinds of control law (cont'd)

- Dynamic nonlinear control:

▷ Beam equation: $u(t)$ is the output of a first order dynamical system and u_e is the new control law to design

$$\begin{array}{l} w_{tt}(x,t) + w_{xxxx}(x,t) = sat(u(t))\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \\ \dot{u}(t) = -\frac{1}{\tau}u(t) + \frac{1}{\tau}u_e(t) \\ w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0, \\ w(x,0) = w^0(x),\ \ w_t(x,0) = w^1(x) \end{array} \quad (11)$$

where *sat* is the localized saturated map [Prieur and Tarbouriech, 2019].

Focus on wave equation + dynamic nonlinear control

- We are interested in a PDE coupled at the boundary with an ODE.
- For all $0 < x < 1$ and for all $t \geq 0$, one gets

$$z_{tt}(x,t) = z_{xx}(x,t) \ , \tag{12}$$

$$\dot{w} = Aw + Bz_t(1,t) \ , \tag{13}$$

$$z(0,t) = 0 \ , \tag{14}$$

$$z_x(1,t) + sat(Dz_t(1,t) + Cw(t)) = 0 \ , \tag{15}$$

  - ▷ with the initial condition $z(x,0) = z^0(x)$ and $z_t(x,0) = z^1(x)$
  - ▷ $z(x,t)$ is the amplitude of the wave dynamics with respect to the rest position, at point x in $[0,1]$ and at time $t \geq 0$,
  - ▷ $w(t)$ is a dynamical state (in $\mathbb{R}^n$) solving a linear finite-dimensional differential equation,
  - ▷ A, B and C are matrices of appropriate dimensions.
- Objectives. Well-posedness + stability

Well-posedness

Well-posedness without saturation

Let us use the following notation $H^1_{(0)}(0,1) = \{z \in H^1(0,1),\ z(0) = 0\}$, $\mathcal{H} = H^1_{(0)}(0,1) \times L^2(0,1)$ and $\mathfrak{H} = \mathcal{H} \times \mathbb{R}^n$. The linear system

$$z_{tt}(x,t) = z_{xx}(x,t)\ , \tag{16}$$
$$\dot{w} = Aw + Bz_t(1,t)\ , \tag{17}$$
$$z(0,t) = 0\ , \tag{18}$$
$$z_x(1,t) + Dz_t(1,t) + Cw(t) = 0\ , \tag{19}$$

is well-posed **if and only if $D \neq -1$**.

- The proof of this well-posedness results from the classical Lumer-Philips theorem (see e.g., Chapter 1 in [Pazy, 1983]).

Well-posedness with saturation

The saturated system (12)-(15)

$$
\begin{aligned}
&z_{tt}(x,t) = z_{xx}(x,t)\ ,\\
&\dot{w} = Aw + Bz_t(1,t)\ ,\\
&z(0,t) = 0\ ,\\
&z_x(1,t) + sat(Dz_t(1,t) + Cw(t)) = 0\ ,
\end{aligned}
$$

is well-posed **if $D > -1$**.

- The proof proposed in [Gauvrit et al., 2023] is based on the use of semigroups+quasi-dissipativity [Miyadera, 1992]:

$$
(z, z_t, w) \text{ in } D(\mathcal{A}) = \Big\{(u,v,w) \in H^1_{(0)}(0,1) \times L^2(0,1) \times \mathbb{R}^n,\quad u \in H^2(0,1),\ v \in H^1_{(0)}(0,1),\ u'(1) + sat(Dv(1) + Cw) = 0\Big\}
$$

Stability

- Recall the saturated system (12)-(15)

$$
\begin{aligned}
& z_{tt}(x,t) = z_{xx}(x,t) \ , \\
& \dot{w} = Aw + Bz_t(1,t) \ , \\
& z(0,t) = 0 \ , \\
& z_x(1,t) + sat(Dz_t(1,t) + Cw(t)) = 0 \ ,
\end{aligned}
$$

- The state of the system is constituted from z (PDE) and w (ODE).
  - ▷ Preliminary result: stability for the linear case (without saturation)

Stability without saturation

The linear system (12)-(15) recalled below

$$
\begin{aligned}
&z_{tt}(x,t) = z_{xx}(x,t)\ ,\\
&\dot{w} = Aw + Bz_t(1,t)\ ,\\
&z(0,t) = 0\ ,\\
&z_x(1,t) + Dz_t(1,t) + Cw(t) = 0\ ,
\end{aligned}
$$

is exponentially stable if and only if the spectrum of A is in the strict left part of the plane, $\sigma(A) \subset \mathbb{C}_-$, and if $D > 0$.

- This result could be proven by a spectral analysis of the linear operator describing (12)-(15).
- We consider the following assumption $\boxed{\sigma(A) \subset \mathbb{C}_- \text{ and } D > 0.}$

Two cases of study

- Assumption: $\sigma(A) \subset \mathbb{C}_-$ and $D > 0.$
- We consider two cases in order to be able to prove exponential stability
  - PDE-to-ODE case.
    That corresponds to consider
    $$C = 0$$
  - ODE-to-PDE case.
    That corresponds to consider
    $$B = 0$$

Warning: For the moment no solution for $B \neq 0$ and $C \neq 0$

Case 1: $C = 0$

- PDE-to-ODE case. Consider the case where the PDE and the ODE are in cascade form in this order, that is when $C = 0$, namely:

$$z_{tt}(x,t) = z_{xx}(x,t)\ , \tag{20}$$
$$\dot{w} = Aw + Bz_t(1,t)\ , \tag{21}$$
$$z(0,t) = 0\ , \tag{22}$$
$$z_x(1,t) + sat(Dz_t(1,t)) = 0\ , \tag{23}$$

▷ Remark. The ODE dynamics do not have any impact on the PDE, but the boundary value $z_t(1,t)$ is the input of the ODE.

- The necessary and sufficient condition for the asymptotic stability of the linear system (12)-(15) is also a sufficient condition for the asymptotic stability of the nonlinear system (20)-(23).
- Assumption: $\sigma(A) \subset \mathbb{C}_-$ and $D > 0$.

Stability - Case 1

System (20)-(23) is globally asymptotically stable, that is, there exists a symmetric definite positive matrix P in $\mathbb{R}^{n\times n}$ such that the following stability condition

$$\begin{aligned}\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} + w(t)^\top P w(t) \\ \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} + w(0)^\top P w(0),\ \forall t \geq 0\ ,\end{aligned} \tag{24}$$

holds, together with the attractivity property (convergence property)

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} + \|w(t)\| \to_{t\to\infty} 0\ . \tag{25}$$

Main ingredients of the proof

- The proof is inspired by the proof of Thm 2 in [Prieur et al., 2016] (see, also, the proof of Thm 2.2 in [Marx et al., 2017]): LES+GA
- The following Lyapunov function candidate

$$V(z,w) = \frac{1}{2}\Big(\int_0^1 e^{\mu x}(z_t+z_x)^2 dx + \int_0^1 e^{-\mu x}(z_t-z_x)^2 dx\Big) + w^\top P w$$

with $\mu > 0$ and $P = P^\top > 0$.

- $\dot{V} = -\mu V + \frac{e^{\mu}}{2}(z_t(1,t) - sat(Dz_t(1,t)))^2 - \frac{e^{-\mu}}{2}(z_t(1,t) + sat(Dz_t(1,t)))^2 + w^\top(A^\top P + PA + \mu P)w + 2w^\top PBz_t(1,t)$

$$\dot{V} = -\mu V + \xi^\top \underbrace{\begin{pmatrix} A^\top P + PA + \mu P & PB & 0 \\ B^\top P & \frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} & -\frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} \\ 0 & -\frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} & \frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} \end{pmatrix}}_{\text{Not sign definite}} \xi$$

with $\xi = \begin{pmatrix} w \\ z_t(1,t) \\ sat(Dz_t(1,t)) \end{pmatrix}$

Main ingredients of the proof (cont'd)

- We then need to use more information about the nonlinearity $sat(Dz_t(1,t))$.
- Quadratic abstraction for the nonlinearity $\phi_1 = sat(Dz_t(1,t)) - Dz_t(1,t)$

$$\eta\phi_1(sat(Dz_t(1,t)) + Gz_t(1,t)) \leq 0 \iff \eta\phi_1(\phi_1 + (D+G)z_t(1,t)) \leq 0$$

$$\forall z_t(1,t) \in \{v; \phi_1(Gv) = 0\}, \iff |Gz_t(1,t)| \leq |G||z_t(1,t)| \leq u_0$$

with $\eta > 0$

- Condition: $\dot{V} \leq \dot{V} - 2\eta\phi_1(\phi_1 + Dz_t(1,t)) < 0$ along the trajectories of the closed loop $\Rightarrow \dot{V} \leq -\mu V$

$$\dot{V} \leq -\mu V + \zeta^\top \underbrace{\begin{pmatrix} A^\top P + PA + \mu P & PB & 0 \\ B^\top P & (1-D)^2\frac{e^{\mu}}{2} - (1+D)^2\frac{e^{-\mu}}{2} & \star \\ 0 & -(1-D)\frac{e^{\mu}}{2} - (1+D)\frac{e^{-\mu}}{2} - \eta(D+G) & \frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} - 2\eta \end{pmatrix}}_{\exists \eta,\mu,G,P \text{ such that negative definite}} \zeta$$

$$\text{with } \zeta = \begin{pmatrix} w \\ z_t(1,t) \\ \phi_1 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -D & 1 \end{pmatrix} \xi$$

Main ingredients of the proof (cont'd)

- Initial condition: (z, z_t, w) in
$$D(\mathcal{A}) = \Big\{(u, v, w) \in H^1_{(0)}(0,1) \times L^2(0,1) \times \mathbb{R}^n,$$
$$u \in H^2(0,1),\ v \in H^1_{(0)}(0,1),\ u'(1) + sat(Dv(1) + Cw) = 0\Big\}$$
  - ▷ Since $z_t(0,t) = 0$, it holds $|z_t(1,t)|^2 = |\int_0^1 z_{xt}(.,t)dx|^2 \leq \int_0^1 |z_{xt}(.,t)|^2 dx = \|z_t(.,t)\|^2_{H^1_{(0)}(0,1)}$.
  - ▷ Thus,
$$\begin{aligned} |z_t(1,t)| &\leq \|z(.,t)\|_{H^1_0(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} + w(t)^\top P w(t) \\ &\leq \|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)} + w(0)^\top P w(0) \end{aligned}$$
- Then for any initial condition satisfying
$$|G|(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)} + w(0)^\top P w(0)) \leq u_0$$
one gets
$$|Gz_t(1,t)| \leq |G||z_t(1,t)| \leq u_0$$

Case 2: $B = 0$

- ODE-to-PDE case. Let us now consider the case where the ODE and the PDE are in cascade form in this order, that is when $B = 0$:

$$z_{tt}(x,t) = z_{xx}(x,t) \ , \tag{26}$$

$$\dot{w} = Aw \ , \tag{27}$$

$$z(0,t) = 0 \ , \tag{28}$$

$$z_x(1,t) + sat(Dz_t(1,t) + Cw(t)) = 0 \ , \tag{29}$$

▷ Remark. The ODE dynamics has an impact on the PDE, since the boundary value $z_x(1,t)$ depends on w, but the stability of the dynamics of w only depends on A.

- Assumption: $\sigma(A) \subset \mathbb{C}_-$ and $D > 0$.

Stability - Case 2

System (20)-(23) is globally asymptotically stable, that is, there exists a symmetric definite positive matrix P in $\mathbb{R}^{n\times n}$ such that the following stability condition

$$\begin{aligned}&\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} + w(t)^\top P w(t)\\ &\quad \leq \|z^0\|_{H_0^1(0,1)} + \|z^1\|_{L^2(0,1)} + w(0)^\top P w(0),\ \forall t \geq 0\ ,\end{aligned} \tag{30}$$

holds, together with the attractivity property (convergence property)

$$\|z(.,t)\|_{H_0^1(0,1)} + \|z_t(.,t)\|_{L^2(0,1)} + \|w(t)\| \rightarrow_{t\rightarrow\infty} 0\ . \tag{31}$$

Main ingredients of the proof

- The following Lyapunov function candidate

$$V(z,w) = \frac{1}{2}\left(\int_0^1 e^{\mu x}(z_t + z_x)^2 dx + \int_0^1 e^{-\mu x}(z_t - z_x)^2 dx\right) + w^\top P w$$

  with $\mu > 0$ and $P = P^\top > 0$.

- From the assumption, there exists $P = P^\top > 0$ such that

$$A^\top P + PA + \frac{D^{-1}}{2} C^\top C = -Q \quad \text{with } Q = Q^\top > 0$$

- $\dot{V} = -\mu V + \frac{e^{\mu}}{2}(z_t(1,t) - sat(Dz_t(1,t) + Cw(t)))^2 - \frac{e^{-\mu}}{2}(z_t(1,t) + sat(Dz_t(1,t) + Cw(t)))^2 + w^\top(A^\top P + PA + \mu P)w$

Main ingredients of the proof (cont'd)

- We then need to use more information about the nonlineartity $sat(Dz_t(1,t)+Cw(t))$.
- Quadratic abstraction for the nonlinearity $\phi_2 = sat(Dz_t(1,t)+Cw(t)) - Dz_t(1,t) - Cw(t)$

$$\eta\phi_2(sat(Dz_t(1,t)+Cw(t)) + Gz_t(1,t) + G_2 w(t)) \leq 0$$
$$\iff \eta\phi_2(\phi_2 + (D+G_1)z_t(1,t) + (C+G_2)w(t)) \leq 0$$

$$\forall z_t(1,t), w(t) \in \{v_1, v_2; \phi_1(G_1 v + G_2 v_2) = 0\},$$
$$\iff |G_1 z_t(1,t) + G_2 w(t)| \leq u_0$$

with $\eta > 0$

- $\dot{V} \leq -\mu V + \zeta_2^\top M \zeta_2 < -\mu V$ with $\zeta = \begin{pmatrix} w \\ z_t(1,t) \\ \phi_2 \end{pmatrix}$

Main ingredients of the proof (cont'd)

- M is defined as follows

$$M = \begin{pmatrix} A^\top P + PA + \mu P + sh(\mu)C^\top C & \star & \star \\ -ch(\mu) + sh(\mu)DC & (1-D)^2\frac{e^{\mu}}{2} - (1+D)^2\frac{e^{-\mu}}{2} & \star \\ sh(\mu)C - \eta(G_2 + C) & -(1-D)\frac{e^{\mu}}{2} - (1+D)\frac{e^{-\mu}}{2} - \eta(D+G_1) & \frac{e^{\mu}}{2} - \frac{e^{-\mu}}{2} - 2\eta \end{pmatrix}$$

- $\exists \eta, \mu, G_1, G_2, P$ such that $M < 0$
- For example one can choose $G_1 = (\ell - 1)D$ and $G_2 = (\ell - 1)C$
- Initial condition: Due to the generalized sector condition (see e.g.,Lemma 1.5 in [Tarbouriech et al., 2011]), for all $1 > \ell > 0$, and for all $\eta > 0$, for any initial condition (z^0, z^1, w_0) in $D(\mathcal{A})$ such that

$$(1-\ell)V(z^0, w_0)) \leq u_0 \ , \tag{32}$$

it holds

$$\eta\phi_2(\phi_2 + \ell D z_t(1,t) + \ell C w(t)) \leq 0 \ .$$

Concluding remarks

- Context. Presence of constraint on the input (as magnitude saturation)
- Main topic: Stability analysis/stabilization via static or dynamic controller
  - ▷ ODE+saturation
  - ▷ PDE+saturation (wave equation, beam equation). Other results in the literature (reaction-diffusion systems, KdV for example)
- Focus on a case of dynamic controller: wave equation in closed loop with a dynamic boundary control
  - ▷ Two particular cases ($C = 0$ and $B = 0$)
  - ▷ Main tools: Lyapunov function and generalized sector condition

Prospectives

- What happens with respect to the focus when $B \neq 0$ and $C \neq 0$?
- Design of A, B, C, D
- Presence of constraint on the input (as magnitude and rate saturation)
- Extension to other nonlinearities
  - Beam with nonlinear piezoelectric control (Joint work with A. Mattioni and C. Prieur)
- Extension to other PDE (Schrödinger)
- Regulation problem: exosystem $\dot{\rho} = S\rho$, $r = E\rho$ (Joint work with J.M. Gomes da Silva Jr and C. Prieur)

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