Null controllability of underactuated linear parabolic-transport system

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Summary

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Controllability of the transport/heat equation on the torus

ω non-empty open interval of $\mathbb{T} := \mathbb{R}\backslash 2\pi\mathbb{Z}$, $T > 0$.

Theorem

The heat equation is null-controllable in any time T*:* $\forall f_0 \in L^2(\mathbb{T})$, $\exists u \in L^2([0,T]\times\omega)$, *the solution* f *of*

$$\partial_t f - \partial_{xx}^2 f = 1_\omega u, \qquad f(0,\cdot) = f_0$$

satisfies $f(T,\cdot) = 0$ *on* $\mathbb{T}$.

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Theorem

ω *Let $c > 0$. The transport equation at speed c is exactly controllable in time T if $T > \frac{2\pi - |\omega|}{c}$: $\forall f_0, f_T \in L^2(\mathbb{T})$, $\exists u \in L^2((0,T)\times\omega)$, the solution f of*

$$\partial_t f - c\partial_x f = 1_\omega u, \qquad f(0) = f_0$$

satisfies $f(T,\cdot) = f_T$ on $\mathbb{T}$. But not controllable if $T < \frac{2\pi - |\omega|}{c}$.

Motivation

Investigate systems of PDEs that involve both parabolic and transport effects.

- Many models of interest can be written/transformed in this form.
- Coupling different dynamics, with different behaviours in control theory, is a challenging question: which dynamics “wins” ?
- Many difficulties that are specific to systems: influence of the coupling terms, regularity issues on the initial conditions...
- Emphasis here on underactuated systems: less controls than equations.

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Here, aim to work in a setting that might cover or generalize already known results, under strong technical restrictions:

- Work on the torus.
- Restrict to linear constant couplings.

Partial study by Beauchard-Koenig-Le Balc'h '20.

Parabolic-Transport Systems

The abstract system of $d = d_h + d_p$ equations and m controls

$$\partial_t f + A\partial_x f - B\partial_x^2 f + Kf = M1_\omega u, \quad (t,x) \in (0,+\infty)\times\mathbb{T}$$

$$f = \begin{pmatrix} f_h \\ f_p \end{pmatrix} \in \mathbb{C}^d = \mathbb{C}^{d_h+d_p};\ B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix},\ D + D^* \text{ positive definite };\quad K = \begin{pmatrix} K_{11} & \boldsymbol{K_{12}} \\ \boldsymbol{K_{21}} & K_{22} \end{pmatrix};$$

$$A = \begin{pmatrix} A_{11} & \boldsymbol{A_{12}} \\ \boldsymbol{A_{21}} & A_{22} \end{pmatrix},\quad A_{11} \text{ diagonalizable, } \mathrm{Sp}(A_{11}) \subset \mathbb{R}\setminus\{0\};$$

$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix} \in \mathcal{M}_{d,m}(\mathbb{C})$.

Coupling between parabolic and transport equations

$$f = \begin{pmatrix} f_h \\ f_p \end{pmatrix},\ \begin{cases} (\partial_t + A_{11}\partial_x + K_{11})f_h + (\boldsymbol{A_{12}}\partial_x + \boldsymbol{K_{12}})f_p = 1_\omega M_1 u, \\ (\partial_t - D\partial_x^2 + A_{22}\partial_x + K_{22})f_p + (\boldsymbol{A_{21}}\partial_x + \boldsymbol{K_{21}})f_h = 1_\omega M_2 u. \end{cases}$$

Question

For which f_0, T does there exist $u \in L^2((0,T)\times\omega, \mathbb{C}^m)$ such that $f(T,\cdot) = 0$?

Example I: Linearized compressible Navier-Stokes

Navier-Stokes

ρ: fluid density. v: fluid velocity. $a, \gamma, \mu > 0$.

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) & = 0 \quad \text{on } [0, T] \times \mathbb{T}, \\ \rho(\partial_t v + v\partial_x v) + \partial_x(a\rho^\gamma) - \mu \partial_x^2 v & = 1_\omega u_2(t,x) \quad \text{on } [0, T] \times \mathbb{T}, \end{cases}$$

Linearization around a stationary state $(\bar{\rho}, \bar{v}) \in \mathbb{R}_+^* \times \mathbb{R}^*$:

$$\begin{cases} \partial_t \rho + \bar{v}\partial_x \rho + \bar{\rho}\partial_x v = & 0 \quad \text{in } [0, T] \times \mathbb{T}, \\ \partial_t v + \bar{v}\partial_x v + a\bar{\rho}^{\gamma-2}\partial_x \rho - \frac{\mu}{\bar{\rho}}\partial_x^2 v & = 1_\omega u_2(t,x) \quad \text{in } [0, T] \times \mathbb{T}. \end{cases}$$

- [Chowdhury-Mitra-Ramaswamy-Renardy 2014]: control in time $T > 2\pi/|\bar{v}|$ for initial conditions $(\rho_0, v_0) \in H^1_m \times L^2$. ($_m$: mean-value equal to 0).
- [Beauchard-Koenig-Le Balc'h '23]: control in time $T > (2\pi - |\omega|)/|\bar{v}|$ for initial conditions in $H^2_m \times H^2$, non-controllability in time $T < (2\pi - |\omega|)/|\bar{v}|$.
- [Koenig-Lissy 2023] control in time $T > (2\pi - |\omega|)/|\bar{v}|$ for initial conditions in $H^1_m \times L^2$, non-controllability in time $T < (2\pi - |\omega|)/|\bar{v}|$ in no Sobolev space.

Example II : wave equation with structural damping

Wave equation with structural damping and moving control

$$\partial_{tt} y - \partial_{xx} y - \partial_{txx} y + b \partial_t y = h \text{ in } [0, T] \times \mathbb{T},$$

where $b \in \mathbb{R}$ and $h(t, x)$ is a moving control at speed $c \geqslant 0$:
$h(t, x) = u(t, x) 1_{\omega + ct}(x)$.

- [Rosier-Rouchon 2007]: $c = 0$, controllability in no time.
- [Martin-Rosier-Rouchon 2014]: $c \neq 0$,controllability in time $T > 2\pi$ for $(y, \partial_t y) \in H^{s+2} \times H^s$, $s > 15/2$.
- [Beauchard-Koenig-Le Balc'h 2020]: $x \leftrightarrow x - ct$, $z = \partial_t y - \partial_{xx} y + (b-1) y$, with $f = \begin{pmatrix} \bar{z} \\ y \end{pmatrix}$, $A = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & \mu/\rho \end{pmatrix}$ and $K = \begin{pmatrix} 1 & 1-b \\ -1 & b-1 \end{pmatrix}$: controllable in time $T > (2\pi - |\omega|)/c$ for initial conditions in $H^1 \times L^2$, not controllable in this space if $T < (2\pi - |\omega|)/c$.
- [Koenig-Lissy 2023] : not controllable in no Sobolev spaces.

Fully actuated system

Theorem (Case $M = I$, Beauchard-Koenig-Le Balc'h 2020)

Introduce

$$T^* = \frac{2\pi - |\omega|}{\min_{\mu \in \mathrm{Sp}(A_{11})} |\mu|}.$$

Then

1. *the system is not null-controllable on ω in time $T < T^*$,*
2. *the system is null-controllable on ω in time $T > T^*$.*

Minimal time = minimal time for the transport equation

In the case

$$\partial_t f_h + A_{11} \partial_x f_h = u_h 1_\omega$$

Free solutions = sums of waves travelling at speed $\mu_k \in \mathrm{Sp}(A_{11})$.

Underactuated system

For $n \in \mathbb{Z}$, introduce $B_n = n^2 B + inA + K$ and $[B_n|M] = [M, B_n M, \dots, |B_n^{d-1} M \in \mathcal{M}_{d,md}(\mathbb{R})$.

Theorem (Underactuated system (Koenig-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ iff $rank([B_n|M]) = d$.

Coupling condition

n-th Fourier component of the parabolic-transport system:

$$X_n'(t) + (n^2 B + inA + K)X_n(t) = Mu_n(t)$$

Condition of the theorem $\Leftrightarrow$ the finite-dimensional system $X_n' + (n^2 B + inA + K)X_n = Mu_n$ is controllable.

If $T < T^*$, no controllability in no Sobolev Space by an appropriate WKB construction (not totally trivial).

Summary

Heuristic

First introduced in Coron'92 for ODEs and Coron-Lissy'14 for PDEs (Navier-Stokes 3D).

This method is useful to control systems of linear partial differential equations (PDEs) having n equations with m controls, $m < n$. There are roughly two steps:

- Firstly, control the system with a different control acting on each equation. (Analytic resolution).
- Secondly, try to find a way to get rid of the controls that should not appear. (Algebraic resolution).

The first step is easier than the original problem.

Question

How to perform the second point?

Algebraic solvability of differential systems

$\mathcal{L} : C^\infty(Q_0)^m \to C^\infty(Q_0)^k$ linear partial differential operator (LPDO) on an open set Q_0 of $\mathbb{R}^d$.

Goal

$$\text{Solve} \quad \mathcal{L}y = f. \tag{Gen-Dif-Syst}$$

Unknown: y. f is a source term.

Définition

Equation (Gen-Dif-Syst) *is algebraically solvable if there exists a LPDO* $\mathcal{M} : C^\infty(Q_0)^k \to C^\infty(Q_0)^m$ *such that, for every* $f \in C^\infty(Q_0)^k$, $\mathcal{M}f$ *is a solution of* (Gen-Dif-Syst), *i.e.* $\mathcal{L}(\mathcal{M}(f)) = f$, *i.e.*

$$\mathcal{L} \circ \mathcal{M} = Id. \tag{LcompM=I}$$

Formal adjoint

Consider a LPDO $\mathcal{M} = \sum_{|\alpha| \leqslant m} A_\alpha \partial^\alpha$, associate (formal) adjoint

$$\mathcal{M}^* : C^\infty(Q_0)^l \to C^\infty(Q_0)^k$$

defined by

$$\mathcal{M}^* \psi := \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \partial^\alpha (A_\alpha^{\text{tr}} \psi), \ \forall \psi \in C^\infty(Q_0)^l.$$

Basic facts

- $\mathcal{M}^{**} = \mathcal{M}$.
- If $\mathcal{N}$ is another LPDO of appropriate size, then $(\mathcal{N} \circ \mathcal{M})^* = \mathcal{M}^* \circ \mathcal{N}^*$.

Consequence

(LcompM=I) is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id.$$

Some remarks

- If $\mathcal{M}$ such that (LcompM=I) exists, the crucial point is that the solution $\mathcal{M}f$ depends locally on the source term f: if f is supported in ω, so is the solution $\mathcal{M}f$.

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- If $\mathcal{M}$ such that (LcompM=I) exists, the crucial point is that the solution $\mathcal{M}f$ depends locally on the source term f: if f is supported in ω, so is the solution $\mathcal{M}f$.
- For many PDEs, $\mathcal{M}$ does not exist (the inverse operator is a non-local operator : the solution does not necessarily have the same support as f).

$\Rightarrow$ the system (Gen-Dif-Syst) has to be underdetermined (less equations than unknowns). In this case the "adjoint" equation $\mathcal{L}^* z = 0$ is over-determined (more equations than unknowns).

What is the link with controllability? (1)

Consider some LPDO $\mathcal{A}$ and $\mathcal{B}$ a control operator which is also a LPDO, on $(0, T) \times \Omega$, Ω bounded domain of $\mathbb{R}^d$. Consider a system of PDEs with n equations and distributed control:

$$\begin{cases} y' = \mathcal{A}y + \mathcal{B}u1_\omega \text{ in } (0, T) \times \Omega, \\ y(0, \cdot) = y^0 \text{ in } \Omega. \end{cases} \tag{Cont-Syst}$$

that we want to bring to 0 (for example) at time $T > 0$.

u: the control, supposed to act only on m of the equations ($m < n$) for instance, supported on a subdomain ω.

What is the link with controllability? (2)

- First step (analytic part): we control the system on each equation. Assume there exists a solution $\widehat{y}$ and a control $\widehat{u}$ verifying

$$\begin{cases} \widehat{y}' = \mathcal{A}\widehat{y} + \widehat{u}1_\omega, \\ y(0,\cdot) = y^0. \end{cases} \qquad \text{(An-Syst)}$$

$\widehat{u}1_\omega$ is supposed to be regular enough, and to vanish at some order at time $t = 0$ and time $T = 0$ and on $\partial\omega$ (for example $\widehat{u}$ compactly supported in $(0, T) \times \omega$).

What is the link with controllability? (3)

- Second step (algebraic part): we now consider $\widehat{u}$ as a source term, and we work **locally** on $(0, T) \times \omega$. We want to prove the algebraic solvability of the following system:

$$\{\tilde{y}' = \mathcal{A}\tilde{y} + \mathcal{B}\tilde{u} + \widehat{u}1_{\omega}, \qquad \text{(Alg-Sys)}$$

Can be rewritten under the form

$$\mathcal{L}(\tilde{y}, \tilde{u}) = f,$$

where $\mathcal{L}(\tilde{y}, \tilde{u}) := \tilde{y}' - \mathcal{A}\tilde{y} - \mathcal{B}\tilde{u}$. Underdetermined. If we assume the algebraic solvability, then there exists a solution $(\tilde{y}, \tilde{u})$ which has the same support as $\widehat{u}1_{\omega}$, and hence vanishes outside ω and at $t = 0$ and $t = T$.

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- $y(T,.) = \widehat{y}(T,.) - \tilde{y}(T,.) = 0$ because $\widehat{y}$ is controlled to 0 and $\tilde{y}$ vanishes at time $t = T$.
- $\tilde{u}$ is supported in space on ω since it involves linear combinations of derivatives of $\widehat{u}$.

A remark

$\tilde{y}$ modifies $\widehat{y}$ only locally on ω.

Summary

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Kalman condition

Consider the system of n ODEs controlled by m controls

$$\begin{cases} \partial_t y = Ay + Bu, \\ y(0) = y^0, \end{cases} \qquad \text{(ODE-Cont)}$$

where $y^0 \in \mathbb{R}^n$, $u \in L^2((0,T);\mathbb{R}^m)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

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Kalman-Ho-Narendra'63 CDE

System (ODE-Cont) is controllable at time $T > 0$ if and only if

$$\text{Rank}[A|B] = n,$$

where $[A|B] := (B|AB|...|A^{n-1}B)$.

Kalman implies null controllability

Analytic Problem:
Find $(\widehat{y}, \widehat{u})$ with $\widehat{v} \in C_c^\infty(0, T)$ such that

$$\begin{cases} \partial_t \widehat{y} = A\widehat{y} + \widehat{u}, \\ \widehat{y}(0) = y^0, \ \widehat{y}(T) = 0 \end{cases} .$$

Algebraic Problem:
Find $(\tilde{y}, \tilde{u}) \in C_c^\infty(0, T)$ such that

$$\partial_t \tilde{y} = A\tilde{y} + B\tilde{u} + \widehat{u} \ \text{ in } \ (0, T).$$

Conclusion:
The couple $(y, u) := (\tilde{y} - \widehat{y}, \tilde{u})$ is solution to system (ODE-Cont) satisfying $y(T) = 0$.

Resolution of the analytic problem

Consider $\eta \in C^\infty([0,T],\mathbb{R})$ with $\eta = 1$ on $[0, T/3]$ and $\eta = 0$ on $[2T/3, T]$, and consider y_F solution of

$$\begin{cases} y_F' = Ay_F \\ y_F(0) = y^0 \end{cases}$$

Then $\widehat{y} = \eta y_F$ solution to

$$\begin{cases} \widehat{y}' = & A\widehat{y} + \widehat{u}, \\ y(\cdot,0) = & y^0, \end{cases}$$

for $\widehat{u} = \widehat{y}' - A\widehat{y}$.

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for $\widehat{u} = \widehat{y}' - A\widehat{y}$.

- $\widehat{y}, \widehat{u}$ are in C^∞,
- $\widehat{y}(0) = y^0$ since $\eta = 1$ on $0, T/3]$,$\widehat{y}(T) = 0$ since $\eta = 0$ on $[2T/3, T]$,
- $\widehat{u}$ is compactly supported since $\eta = 1$ on $[0, T/3]$ (so $\widehat{u} = y_F' - Ay_F = 0$) and $\eta = 0$ on $[2T/3, T]$.

Resolution of the algebraic problem

Find $(\tilde{z}, \tilde{v})$ compactly supported such that

$$\mathcal{L}(\tilde{z}, \tilde{v}) = \widehat{v},$$

where

$$\mathcal{L}(\tilde{z}, \tilde{v}) := \partial_t \tilde{z} - A\tilde{z} - B\widehat{v}.$$

It suffices to find a differential operator $\mathcal{M}$ s.t.

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$$\mathcal{L} \circ \mathcal{M} = Id.$$

The last equality is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id,$$

where $\mathcal{L}^*$ is given by

$$\mathcal{L}^* \varphi = \begin{pmatrix} -\partial_t \varphi - A^* \varphi \\ -B^* \varphi \end{pmatrix}.$$

Heuristics

Remind $\mathcal{L}^*\varphi = \begin{pmatrix} -\partial_t\varphi - A^*\varphi \\ -B^*\varphi \end{pmatrix}$. Call $\mathcal{L}_1^* = -\partial_t - A^*$ and $\mathcal{L}_2^* = -B^*$.

By induction :

- $-\mathcal{L}_2^* = B^*$.
- Take $-\mathcal{L}_2^*$. Compose by ∂_t. Substract $B^*\mathcal{L}_1^*$. We obtain $\mathcal{L}_3^* := \partial_t B^* + -B^*\partial_t - B^*A^* = B^*A^*$.
- Take $\mathcal{L}_3^*$. Compose by ∂_t. Substract $B^*A^*\mathcal{L}_1^*$. We obtain $\mathcal{L}_4^* := \partial_t B^*A^* + -B^*A^*\partial_t - B^*(A^*)^2 = B^*(A^*)^2$.

By induction, we recover $B^*, B^*A^*, \ldots B^*(A^*)^{n-1}$.

More rigorously

Let $\mathcal{S} := (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ given for all $(x_1, x_2) \in \mathcal{C}^\infty(\Omega; \mathbb{R}^{n+m})$ by

$$\begin{cases} \mathcal{S}_1(x_1, x_2) := -x_2, \\ \mathcal{S}_2(x_1, x_2) := x_2' - B^* x_1, \\ \mathcal{S}_k(x_1, x_2) := \mathcal{S}_{k-1}(x_1, x_2)' - B^*(A^*)^{k-2} x_1, \quad \forall\, k \in \{3, \ldots, n\}. \end{cases}$$

Then, we obtain

$$\mathcal{S} \circ \mathcal{L}^* = [A, B]^*.$$

Since the rank of $[A|B] := (B|AB|\ldots|A^{n-1}B)$ is equal to n, there exists $[A|B]^{-1} \in \mathcal{M}_{nm,n}(\mathbb{R})$ such that $[A|B][A|B]^{-1} = I_n$. The operator

$$\mathcal{M} := \mathcal{S}^*[A|B]^{-1}$$

is a differential operator of order $n-1$ in time and is a solution of our problem.

Summary

Parabolic Components, Hyperbolic Components

Fourier components

$$(-B\partial_x^2 + A\partial_x + K)Xe^{inx} = n^2\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)Xe^{inx}.$$

Spectrum of $-B\partial_x^2 + A\partial_x + K$

$$\mathrm{Sp}(-B\partial_x^2 + A\partial_x + K) = \left\{n^2\,\mathrm{Sp}\left(B + \frac{i}{n}A - \frac{1}{n^2}K\right)\right\}.$$

Perturbation theory

λ_{nk} eigenvalue of $B + \frac{i}{n}A - \frac{1}{n^2}K$. λ_k eigenvalue of B: $\lambda_{nk} \to \lambda_k \in \mathrm{Sp}(B)$

- If $\lambda_k \neq 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} n^2\lambda_k$: parabolic frequencies
- If $\lambda_k = 0$, $n^2\lambda_{nk} \underset{n\to+\infty}{\sim} in\mu_k$: hyperbolic frequencies
- Free solutions: $= \sum X_{nk} e^{inx - n^2\lambda_{nk}t} \approx \underbrace{\sum X_{nk} e^{inx-n^2\lambda_k t}}_{\text{parabolic}} + \underbrace{\sum X_{nk} e^{inx - in\mu_k t}}_{\text{hyperbolic}}$

Analytic resolution $M = Id$

The control problem under study

$$\partial_t f + A\partial_x f - B\partial_x^2 f + Kf = 1_\omega u, \quad (t,x) \in (0,+\infty) \times \mathbb{T}. \qquad \text{(Anal-Prob)}$$

The result we aim to obtain

For any $T > T^*$, for any $s \in \mathbb{N}^*$ and any $f_0 \in H^s(\mathbb{T})$, there exists a control $u \in H_0^s((0,T) \times \omega)$ such that the solution f of (Anal-Prob) with initial condition $f(0,\cdot) = f_0$ verifies $f(T,\cdot) = 0$.

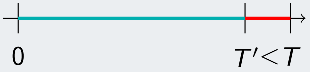
Follows from a well-known principle :

- For parabolic equations with smooth coefficients, one can create C_0^∞ controls even for rough initial condition (Lebeau-Robbiano'95).
- For groups of operators, more regular initial conditions allow more regular controls (Dehman-Lebeau'09, Ervedoza-Zuazua'10).

Here mix dynamics, but we adapt the arguments of Beauchard-Koenig-Le Balc'h'20 (which themselves are inspired by Lebeau'Zuazua'98).

Analytic resolution $M = Id$ II

Decouple and control

- 

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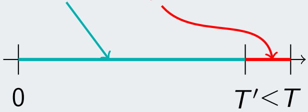
- 

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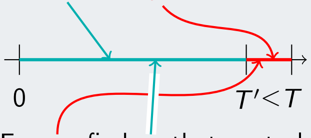
- 

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Analytic resolution $M = Id$ II

Decouple and control

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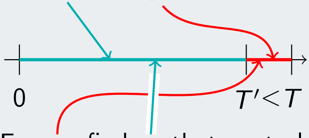
- 

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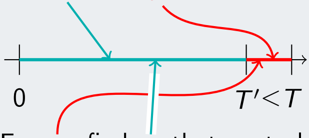
- 

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- If both steps agree, OK.
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- Deal with the finite dimensional subspaces that are left: compactness-uniqueness.

Fourier projection and algebraic solvability

Let $B_n = n^2 B + inA + K$ and $[B_n|M] = [M, B_n M, \dots, |B_n^{d-1} M] \in \mathcal{M}_{d,md}(\mathbb{R})$.

Theorem

If $\mathrm{rank}[B_n|M] = d$*, for every* $X_0 \in \mathbb{C}^d$*, there exists* $u \in H_0^k(0,T)$ *such that the solution* X *of*

$$X' = B_n X + Mu, \quad X(0) = X_0$$

satisfies $X(T) = 0$*.*

Proof by algebraic solvability:

- Analytic part: already done, take the projection on the n-th mode of the control u of the previous slide.
- Then perform the algebraic solvability.

Fictitious control for parabolic-transport system

Theorem (Underactuated system (Koenig-Lissy 2023))

Null-controllability of every $H^{4d(d-1)}$ initial condition in time $T > T^*$ if

$$\forall n \in \mathbb{Z},\ \mathrm{Rank}([B_n|M]) = d.$$

Algebraic solvability on each Fourier components?

$$(\partial_t - B\partial_x^2 + A\partial_x + K)f = 1_\omega v$$

$$\xrightarrow{\text{Fourier}} X_n' = B_n X_n + v_n$$

$$\xrightarrow{\text{Kalman condition}} X_n' = B_n X_n + [B_n|M]w_n$$

$$\xrightarrow{\text{Algebraic Solvability}} X_n' = B_n X_n + M u_n$$

$$\xrightarrow{\text{Inverse Fourier}} (\partial_t - B\partial_x^2 + A\partial_x + K)f = Mu$$

$u = R(\partial_t, \partial_x)v$ with $R(\tau, n) = P(\tau, n)/Q(n)$ (rational function): , because of the equation $v_n = [B_n|M]w_n$, no guarantee on $\mathrm{Supp}(u)$

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$$(\partial_t - B\partial_x^2 + A\partial_x + K)f = 1_\omega Q(\partial_x)v \quad (v \text{ controls } Q(\partial_x)^{-1}f_0)$$

$$\xrightarrow{\text{Fourier}} X_n' = B_nX_n + Q(-in)v_n$$

$$\xrightarrow{\text{Kalman condition}} X_n' = B_nX_n + [B_n|M][B_n|M]^{-1}Q(-in)v_n$$

$$\xrightarrow{\text{Algebraic Solvability}} X_n' = B_nX_n + MP(\partial_t, -in)v_n$$

$$\xrightarrow{\text{Inverse Fourier}} (\partial_t - B\partial_x^2 + A\partial_x + K)f = MP(\partial_t, \partial_x)v$$

$u = R(\partial_t, \partial_x)Q(\partial_x)v$ with $R(\tau, n) = P(\tau, n)/Q(n)$ (rational function):
$\mathrm{Supp}(u) \subset \mathrm{Supp}(v)$

Refinement on the loss of regularity

Loss of regularity

- Null-controllability of every $H^{4d(d-1)}(\mathbb{T})^d$ initial condition: very crude regularity assumption...
- But *some* regularity assumption is needed in general<;
- $$\begin{cases} (\partial_t + \partial_x) f_{\mathrm{h}} + \partial_x f_{\mathrm{p}} + f_{\mathrm{p}} = 0 \\ (\partial_t - \partial_x^2) f_{\mathrm{p}} = 1_\omega u_{\mathrm{p}} \end{cases}$$
  Smoothing: if $f_{0,\mathrm{h}} \notin H^1$, we cannot steer f_0 to 0 with L^2 controls.
- Can be refined easily to $H^{4d(d-1)}(\mathbb{T})^{d_h} \times H^{4d(d-1)-1}(\mathbb{T})^{d_p}$ by parabolic regularity, and even a little bit more in some specific cases.
- Almost optimal in the case of systems of 2 equations.

Refinement on the Kalman condition

Equations with invariants

$$\begin{cases} (\partial_t + \partial_x) f_h + \partial_x f_p = 0 \\ (\partial_t - \partial_x^2) f_p = 1_\omega u_p \end{cases}$$ not null-controllable:

for $n = 0$, $\mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathrm{Vect}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \neq \mathbb{C}^d$.

The average of the hyperbolic component is conserved. Maybe null-controllability of every initial condition with zero hyperbolic-average?

Theorem (Koenig-Lissy 2023)

Assume $T > T_*$ and

- $\forall |n|$ *large enough*, $\mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\} = \mathbb{C}^d$
- $f_0 \in H^{4d(d-1)}(\mathbb{T})^d$
- $\forall n \in \mathbb{Z}$, $\widehat{f_0}(n) \in \mathrm{Vect}\{(n^2 B + inA + K)^i M v, i \in \mathbb{N}, v \in \mathbb{C}^d\}$

There exists a control in $L^2((0, T) \times \omega)$ that steers f_0 to 0 in time T.

Enables to treat (amongst others) the previous example.

Conclusion

Open problems

- Stabilization ?
- Domain other than $\mathbb{T}$? First $\mathbb{T}^n$, then other domains ?
- Sharp results in terms of regularity ?
- non-constant coefficients?
- ...

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Thank you for your attention.