Different approaches to integral action in infinite-dimensional nonlinear dynamics

EDP, commande et observation des systèmes, LAAS-CNRS, Toulouse

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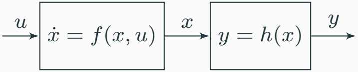
Orientation

Set-point output regulation/tracking problem

Consider a controlled plant with state $x \in X$, input $u \in U$ and output $y \in Y$.



- **(Set-point output tracking problem.)** Given $y_{\text{ref}} \in Y$, find a control law s.t.
  1. The state x remains bounded;
  2. (Asymptotic tracking.)

$$y(t) \to y_{\text{ref}}, \quad t \to +\infty.$$

- **(Robust regulation.)** Ensure that those properties hold "robustly":

  1. (Disturbance rejection.) In presence of some classes of exogeneous disturbances d;
  2. Under parameter uncertainties?..

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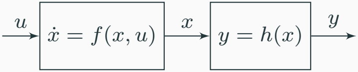

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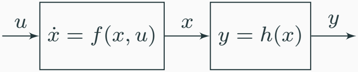

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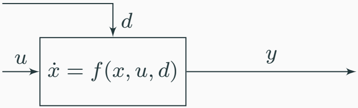

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Integral action

Standing assumption

A constant input $u^\star$ produces a unique steady state $x^\star$.

For linear systems of the form $\dot{x} = Ax + Bu$ where e^{tA} is exponentially stable, $0 \in \rho(A)$ and

$$x^\star = A^{-1} B u^\star$$

is globally exponentially stable equilibrium w.r.t. to $\dot{x} = Ax + Bu^\star$.

A nonlinear example

Minea system

Let $\delta > 0$. Consider the following control system on $X = \mathbb{R}^2$:

$$\dot{x}_1 + x_1 + \delta x_2^2 = u,$$
$$\dot{x}_2 + x_2 - \delta x_1 x_2 = 0.$$

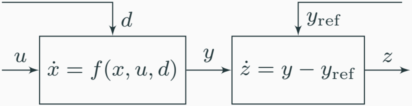
- We have
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x\|^2 = -\|x\|^2 + x_1 u \leqslant -\frac{1}{2}\|x\|^2 + \frac{1}{2}|u|^2$$
so that if $u = 0$, $x \to 0$ uniformly and exponentially, plus ISS property.
- **Nevertheless**, depending on δ, the constant input $u^\star$ produces up to 3 equilibria with heteroclinic curves.

Takeway point

Stability is not enough!

Integral action

Consider adding an output integrator to the loop.



- Given y_{ref} and a control law $u = k(x, z)$, at any equilibrium $(x^\star, z^\star)$,

$$y^\star = y_{\text{ref}}.$$

- **Control objective:** Find a feedback control for which the system possesses an attractive equilibrium.
- Integral control is robust w.r.t. whatever d that preserves existence of such points.

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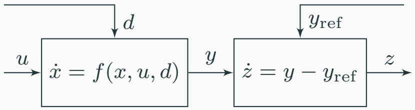

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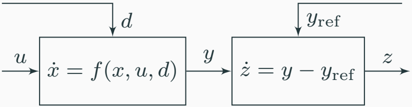

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Some possible strategies

Assume that 0 is an equilibrium for $\dot{x} = f(x,0,0)$ plus some suitable ISS w.r.t. u.

1. **(Perturbation approach.)** Stabilize the cascade

$$\dot{x} = f(x,u,0), \quad \dot{z} = y$$

around 0 and hope that for small (y_{ref}, d), there is an attractive equilibrium $(x^\star, z^\star)$ for

$$\dot{x} = f(x,u,d), \quad \dot{z} = y - y_{\text{ref}}, \quad u = k(x,z).$$

2. **(Change of variable.)** If we already know $u^\star$ s.t. $x^\star$ has output $y^\star = y_{\text{ref}}$, after setting $x \mapsto x - x^\star$, stabilize the cascade

$$\dot{x} = f(x + x^\star, u^\star + u) - f(x^\star, u^\star), \quad \dot{z} = y.$$

A question

Can we assume $y_{\text{ref}} = 0$ without loss of generality? *Yes and no...*

- Dependence on the feedback function k w.r.t. y_{ref}.

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Constrained integral control with monotone operators

Motivation

Let $\mathcal{K}$ be a nonempty closed convex subset of Y. How to use integral control under the set constraint

$$z \in \mathcal{K}?$$

- In case of pure integral output feedback, z is fed "as is" into the plant;
- **Pros:** controller satisfies operational constraints, anti-windup mechanism, etc.

A solution via projected dynamical systems

Replace the classical integrator with

$$\dot{z} = \Pi_{\mathcal{K}}(z, y - y_{\text{ref}})$$

where

$$\Pi_{\mathcal{K}}(z, y) = \text{argmin}_{w \in T_{\mathcal{K}}(z)} \|w - y\|$$

and $T_{\mathcal{K}}(z)$ is the tangent cone of $\mathcal{K}$ at z.

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Actual motivation

Well-posedness

How to guarantee well-posedness of the closed-loop?

- For projections of vector fields, existence & uniqueness results are available.
- In the "no vector field" case (e.g., most PDEs of interest), no such thing **but...**
    1. **(Constrained integrator as a subsystem.)** Investigate properties of the map

    $$w \in L^2(0,T;Y) \mapsto z, \quad \text{where} \quad \dot{z} = \Pi_{\mathcal{K}}(z,w), \quad z(0) = z_0,$$

    and then "close the loop" with a linear *well-posed* system via a fixed-point argument.
    2. Observe that

    $$\dot{z} = \Pi_{\mathcal{K}}(z,w) \iff \dot{z} + N_{\mathcal{K}}(z) \ni w,$$

    where the normal cone $N_{\mathcal{K}} : \mathcal{K} \rightrightarrows Y$ is maximal monotone.

Motives a direct argument for a special class of nonlinear systems!

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Monotone systems?

A suitable class of systems

What systems possess good monotonocity properties when coupled with the output integrator?

- Assume that $Y = U$. Linear impedance-passive systems satisfy

$$\frac{1}{2}\ \frac{\mathrm{d}}{\mathrm{d}t}\|x\|_X^2 \leqslant \langle u, y\rangle_Y,$$

  so adding the integrator $\dot{z} = y$ and choosing the output feedback $u = -z$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|x\|_X^2 + \|z\|_Y^2\right\} \leqslant 0.$$

  For impedance-passive systems, there is a energy-preserving coupling with the integrator!
- In the nonlinear setting, we seek incremental version of those properties.

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A prototype class of impedance-passive nonlinear systems

Let X and be Y be (real) Hilbert spaces. Assume that Y is finite-dimensional. Consider a control system of the form

$$\dot{x} + A(x) \ni Bu, \quad y = B^* u,$$

where:

- $A : \mathrm{dom}(A) \rightrightarrows X$ is maximal monotone, i.e.,

$$\langle a_1 - a_2, x_1 - x_2 \rangle \geqslant 0, \quad a_i \in A(x_i), \text{ and } \quad \mathrm{ran}(A + \lambda) = X, \quad \lambda > 0;$$

- $B \in \mathcal{L}(Y, X)$.

Remark (Generation of contraction semigroups)

Maximal monotone operators characterizes (strongly continuous) contraction semigroups on closed convex subsets of Hilbert spaces.

Given a nonempty closed convex subset $\mathcal{K}$ of Y, we close the loop with

$$\dot{z} + N_{\mathcal{K}}(z) \ni B^* x, \quad u = -z.$$

Lemma

1. *The closed-loop equations generate a contraction semigroup on* $\overline{\mathrm{dom}(A)} \times \mathcal{K}$.
2. *If A has compact resolvent $(A + \lambda)^{-1}$, then so has the closed-loop generator.*
3. *The same holds when adding $(d, -y_{\mathrm{ref}}) \in X \times Y$.*

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Equilibria and feasible references

Assumption

1. A^{-1} is well-defined and continuous.
2. $\ker(B) = \{0\}$.
3. $0 \in \langle A(x_1) - A(x_2), x_1 - x_2 \rangle_X$ implies $x_1 = x_2$.

Let $y_{\text{ref}} \in Y$ be feasible, i.e., there exists $z^\star \in \mathcal{K}^\circ$ s.t.

$$B^* x^\star = y_{\text{ref}}, \quad x^\star = -A^{-1}(Bu^\star), \quad u^\star = -z^\star.$$

Then, $(x^\star, z^\star) \in \text{dom}(A) \times \mathcal{K}^\circ$ is the unique equilibrium for the closed-loop.

Remark

y_{ref} remains feasible under small (in norm) and matched disturbance $d \in \text{ran}(B)$.

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$$\omega(x_0, z_0) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} \tilde{S}_t(x_0, z_0)}, \quad \tilde{S}_t \text{ closed-loop semigroup.}$$

- $\omega(x_0, z_0)$ is nonempty, invariant and attracts the solution originating at (x_0, z_0).
- **Contraction semigroup:** $\omega(x_0, z_0) \subset \text{dom}(A) \times \mathcal{K}$ and $\tilde{S}_t$ are isometries on $\omega(x_0, z_0)$. This imposes

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1. **(Tracking with output feedback.)** *Closed-loop solutions (x, z) in $\overline{\text{dom}(A)} \times \mathcal{K}$ converge in $X \times Y$ to the unique steady state $(x^\star, z^\star)$. In particular,*

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Ccontrol of a nonlinear parabolic equation

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. Consider the control system:

$$\begin{aligned} &\frac{\partial w}{\partial t} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\right)^{p-1} + w^{p-1} = \sum_{j=1}^{m} u_j b_j && \text{in } \Omega \times (0,+\infty), \\ &w = 0 && \text{on } \partial\Omega \times (0,+\infty), \end{aligned}$$

where $p \in \mathbb{N}^*$ is even, b_j are some smooth functions and u_j are scalar control inputs.

- Let $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$ be given by

  $$\langle A(w), \varphi \rangle \triangleq \sum_{i=1}^{d} \int_\Omega \left(\frac{\partial w}{\partial x_i}\right)^{p-1} \frac{\partial \varphi}{\partial x_i}\,\mathrm{d}x + \int_\Omega w^{p-1}\varphi\,\mathrm{d}x, \quad w,\varphi \in W_0^{1,p}(\Omega).$$

  and consider

  $$\mathrm{dom}(A) \triangleq \{w \in W_0^{1,p}(\Omega) : A(w) \in L^2(\Omega)\}$$

  so that $A : \mathrm{dom}(A) \to L^2(\Omega)$ is maximal monotone with dense domain, compact resolvent and

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  We assume linear independance of the b_j.

Ccontrol of a nonlinear parabolic equation

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. Consider the control system:

$$\frac{\partial w}{\partial t} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_i}\right)^{p-1} + w^{p-1} = \sum_{j=1}^{m} u_j b_j \quad \text{in } \Omega \times (0,+\infty),$$

$$w = 0 \quad \text{on } \partial\Omega \times (0,+\infty),$$

where $p \in \mathbb{N}^*$ is even, b_j are some smooth functions and u_j are scalar control inputs.

- Let $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))'$ be given by

$$\langle A(w), \varphi \rangle \triangleq \sum_{i=1}^{d} \int_\Omega \left(\frac{\partial w}{\partial x_i}\right)^{p-1} \frac{\partial \varphi}{\partial x_i}\, \mathrm{d}x + \int_\Omega w^{p-1} \varphi \, \mathrm{d}x, \quad w, \varphi \in W_0^{1,p}(\Omega).$$

  and consider

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Extensions and limitations

- Finite-dimensional models of interest: impedance-passive systems with hysteresis.
- "Unbounded"/nonlinear B? Observability-type assumption? **Case by case basis.**
- **(Robustness to parameter uncertainties.)** Consider the following model:

$$\begin{aligned} &\frac{\partial w}{\partial t} - \Delta w + w^{p-1} + a_{p-2} w^p + \ldots + w = Bu && \text{in } \Omega \times (0, +\infty), \\ &w = 0 && \text{on } \partial\Omega \times (0, +\infty). \end{aligned}$$

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Forwarding approach

Motivation

A nonlinear coupled PDE-ODE model: Planar motion of an homogeneous Euler-Bernoulli beam of length L attached to a rotating joint.

- The deflection w in the beam frame and the rotation angle θ solve

$$
\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} + \lambda \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial \xi^4} + \xi \ddot{\theta} - \rho \dot{\theta}^2 w = 0,\\
&\ddot{\theta}(t) = \frac{\partial^2 w}{\partial \xi^2}(0,t) + \tau(t),\\
&w(0,t) = \frac{\partial w}{\partial \xi}(0,t) = 0,\\
&\frac{\partial^3 w}{\partial \xi^3}(L,t) = \frac{\partial^2 w}{\partial \xi^2}(L,t) = 0,
\end{aligned}
$$

  where $\lambda > 0$ is a viscous damping coefficient.

- **Control problem.** Regulate θ.

Forwarding design for stabilization of nonlinear systems

Consider a system of the form

$$\dot{x} = f(x) + g(x)u, \quad x \in X,$$
$$\dot{z} = h(x), \quad z \in Y.$$

where $f(0) = 0$, $h(0) = 0$. Suppose that

1. **(ISS of the x-subsystem.)** There exist a Lyapunov functional V and $\alpha, \beta > 0$ s.t. solutions to $\dot{x} = f(x) + g(x)u$ satisfy

$$\dot{V} \leqslant -\alpha V + \beta \|u\|.$$

2. **(Invariant graph.)** There exists a map $M : X \to Y$ with $M(0) = 0$ s.t.

The graph of M is invariant under $\dot{x} = f(x)$, $\dot{z} = h(x)$,

i.e.,

$$\mathrm{d}M(x)f(x) = h(x), \quad x \in X, \quad \text{i.e.,} \quad \mathcal{L}_f M = h$$

Expression of M

Integrating along the flow Φ_t generated by f yields

$$M(x) = -\int_0^{+\infty} h(\Phi_t x)\,\mathrm{d}t, \quad x \in X.$$

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Construction of a Lyapunov function

Let

$$W(x,z) \triangleq V(x) + \beta\|z - M(x)\|^2, \quad x \in X, \quad z \in Y.$$

Then,

$$\begin{aligned}\dot{W} &= \dot{V} + 2\beta\langle z - M(x), \dot{z} - \mathrm{d}M(x)\dot{x}\rangle \\ &= \dot{V} - 2\beta\langle z - M(x), \mathrm{d}M(x)g(x)u\rangle\end{aligned}$$

Letting

Nonlinear full-state feedback law

$$u = g(x)^* \mathrm{d}M(x)^*[z - M(x)]$$

yields, thanks to the ISS property,

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Model under consideration

Consider a control system of the form

Semilinear system

$$\dot{x} = Ax + f(x) + g(x)u,$$
$$\dot{z} = Sz + Cx + h(x),$$

where

- $A : \mathcal{D}(A) \to X$ generates a C_0-semigroup $\{e^{tA}\}_{t \geqslant 0}$;
- $C \in \mathcal{L}(\mathcal{D}(A), Y)$ is A-bounded;
- $S \in \mathcal{L}(Y)$ is skew-adjoint, i.e., $S^* = -S$;

and

- $f : X \to X$
- $g : X \to \mathcal{L}(U, X)$
- $h : X \to Y$

are locally Lipschitz with $f(0) = 0$ and $h(0) = 0$.

Additional assumption

1. f and h are Fréchet differentiable with locally Lipschitz differential.
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Semiglobal ISS

Assumption

There exist:

1. A (quadratic-like) Lyapunov function $V \in \mathcal{C}^1(X, \mathbb{R})$ and $\beta > 0$ s.t., along solutions to $\dot{x} = Ax + f(x) + Bu$,

$$\dot{V} \leqslant \beta \|u\|^2;$$

2. For every open bounded set $\mathcal{B} \subset X$, a Lyapunov function $V_{\mathcal{B}} \in \mathcal{C}^1(X, \mathbb{R})$ and $\alpha_{\mathcal{B}}, \beta_{\mathcal{B}} > 0$ s.t. if x remains in $\mathcal{B}$ then

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Semiglobal exponential stability of the x-dynamics. With reference to the semigroup Φ_t associated with $\dot{x} = Ax + f(x)$, the equilibrium 0 uniformly attract bounded sets of X (with exponential rate depending on the bounded set).

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There exists $P \in \mathcal{L}(X)$ coercice self-adjoint and $\mu > 0$ s.t.

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On Sylvester equations

The nonlinear operator equations for the existence of a suitable invariant graph for $\dot{x} = Ax + f(x)$, $\dot{z} = Sx + Cx + h(x)$ are

$$\begin{aligned} &\mathrm{d}M(x)(A+f)(x) = SM(x) + (C+h)(x), \quad x \in \mathrm{dom}(A), \\ &M(0) = 0. \end{aligned}$$

Linear Sylvester equation

We seek $M_0 \in \mathcal{L}(X, Y)$ s.t.

$$M_0 A = SM_0 + C.$$

- Operator or matrix equations of the form $PA + BP = C$ are used in different contexts:
  1. Lyapunov equations when A is a semigroup generator, $B = A^*$ and $C = -\mathrm{id}$;
  2. Internal-model based control when S is a signal generator;
  3. Existence of bounded solutions to $\dot{x} = Ax + f$ with $f \in L^p(0, +\infty; X)$.
- Here, exponentially stability of e^{tA} and boundedness of e^{tS} backward in time:

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Perturbation argument

We search a solution M of the form

$$M(x) = M_0 + F(x)$$

where $F : X \to Y$, $F(0) = 0$, is Fréchet differentiable.

1. M solves the equation iff F solves

$$M_0 f(x) + \mathrm{d}F(x)(A+f)(x) = SF(x) + h(x), \quad x \in \mathrm{dom}(A),$$

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$$M_0 f(\Phi_t x) + \mathrm{d}F(\Phi_t x)(A+f)(\Phi_t x) = SF(\Phi_t x) + h(\Phi_t x), \quad x \in \mathrm{dom}(A), \quad t \geqslant 0.$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-tS} F(\Phi_t x) = -e^{-tS}[M_0 f(\Phi_t x) - h(\Phi_t x)], \quad x \in \mathrm{dom}(A), \quad t \geqslant 0.$$

3. Thus, if M is a solution, then because e^{-tS} is uniformly bounded and $\|\Phi_t x\|_X \to 0$ as $t \to +\infty$ at exponential rate:

$$F(x) = \int_0^{+\infty} e^{-tS}[M_0 f(\Phi_t x) - h(\Phi_t x)]\,\mathrm{d}t, \quad x \in \mathrm{dom}(A).$$

Sketch of the proof, continued

Now, assume that F given by the integral is Fréchet differentiable.

1. First, write

$$e^{-tS}F(\Phi_t x) - F(x) = -\int_0^t e^{-sS}[M_0 f(\Phi_s x) - h(\Phi_s x)]\,\mathrm{d}s, \quad x \in \mathrm{dom}(A), \quad t \geqslant 0.$$

2. Dividing by $t > 0$ and letting $t \to 0$ yield

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-tS} F(\Phi_t x)\Big|_{t=0} = h(x) - M_0 f(x), \quad x \in \mathrm{dom}(A),$$

that is, by the chain rule,

$$-SF(x) + \mathrm{d}F(x)(A+f)(x) = h(x) - M_0 f(x), \quad x \in \mathrm{dom}(A).$$

Theorem (Existence of M)

There exists a unique Fréchet differentiable solution $M : X \to Y$ given by

$$M(x) = M_0 x + \int_0^{+\infty} e^{-tS}[M_0 f(\Phi_t x) - h(\Phi_t x)]\,\mathrm{d}t$$

Furthermore, both M and $\mathrm{d}M$ are locally Lipschitz, and

$$\mathrm{d}M(x)h = M_0 h + \int_0^{+\infty} e^{-tS}[M_0 \mathrm{d}f(\Phi_t x) - \mathrm{d}h(\Phi_t x)]\mathrm{d}\Phi_t(x) h\,\mathrm{d}t, \quad x, h \in X.$$

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Stability of the closed-loop

We go back to the stabilized cascade with state feedback:

$$\dot{x} = Ax + f(x) + g(x)u, \quad \dot{z} = Sz + Cx + h(x), \quad u = g(x)^* \mathrm{d}M(x)^*[z - M(x)].$$

Remark

- Local (in time) well-posedness on $X \times Y$ follows from results on Lipschitz perturbations of linear equations.
- Our prior Lyapunov analysis indicates that solutions are global.

Assumption

1. Y is finite-dimensional.
2. **(Nonresonance condition.)**

$$\mathrm{ran}(M_0 g(0)) = Y.$$

   If $S = 0$, this reads as $\mathrm{ran}(CA^{-1}g(0)) = Y$.

This condition serves two purposes:

- **(GAS via LaSalle.)** Prove that a certain Lyapunov function W is zero only for $(x, z) = 0$.
- **(LES.)** Around zero, get "strict dissipation in the z-variable" too!

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Stability result

With reference to the closed-loop equations

$$\dot{x} = Ax + f(x) + g(x)u, \quad \dot{z} = Sz + Cx + h(x), \quad u = g(x)^* \mathrm{d}M(x)^*[z - M(x)],$$

we have the following stability properties.

Theorem (Stability & convergence)

0 *is locally exponentially stable and globally asymptotically stable, i.e., for all initial data* $(x_0, z_0) \in X \times Y$,

$$\|(x(t), z(t))\|_{X\times Y} \to 0, \quad t \to +\infty$$

and there exist $K, \lambda > 0$ *and a neighborhood* $\mathcal{V}$ *of* 0 *in* $X \times Y$ *s.t.*

$$\|(x(t), z(t))\|_{X\times Y} \leqslant Ke^{-\lambda t}\|(x_0, z_0)\|_{X\times Y}, \quad t \geqslant 0, \quad (x_0, z_0) \in \mathcal{V}.$$

The proof relies on Lyapunov functions of the form

$$W(x, z) = V(x) + \frac{\rho}{2}\|z - M(x)\|_Y^2.$$

Going back to the beam model

Steady state input and new coordinates: Having set

$$\tau = -\theta + \theta_{\text{ref}} + \tilde{\tau}, \quad \phi(t) = \theta(t) - \theta_{\text{ref}}, \quad v(\xi, t) = w(\xi, t) + \xi\phi(t),$$

the plant equations become

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} + \lambda \frac{\partial v}{\partial t} + \frac{\partial^4 v}{\partial \xi^4} - \lambda \xi \dot{\phi} - \dot{\phi}^2 (v - \xi\phi) = 0, \\
&\ddot{\phi}(t) = \frac{\partial^2 v}{\partial \xi^2}(0, t) - \phi(t) + \tilde{\tau}(t), \\
&\frac{\partial^3 v}{\partial \xi^3}(L, t) = \frac{\partial^2 v}{\partial \xi^2}(L, t) = v(0, t) = 0, \\
&\frac{\partial v}{\partial \xi}(0, t) = \phi(t).
\end{aligned}$$

Remark

The (v, ϕ)-equations do not depend on θ_{ref}!

Prestabilization and forwarding

1. **Nonlinear prestabilization.** Choose

$$\tilde{\tau} = -\dot{\phi}\int_\Omega v\frac{\partial v}{\partial t}\,\mathrm{d}\xi + (\phi\dot{\phi} - \lambda)\int_\Omega \xi\frac{\partial v}{\partial t}\,\mathrm{d}\xi - \dot{\phi} + u,$$

so that, letting

$$V = \frac{1}{2}\left(\int_\Omega \left|\frac{\partial v}{\partial t}\right|^2 + \left|\frac{\partial^2 v}{\partial \xi^2}\right|^2 \mathrm{d}\xi + |\dot{\phi}|^2 + |\phi|^2\right),$$

we have

$$\dot{V} = -\lambda\int_\Omega \left|\frac{\partial v}{\partial t}\right|^2 \mathrm{d}\xi - |\dot{\phi}|^2 + u\dot{\phi} \leqslant \frac{1}{2}|u|^2.$$

2. **Strict control Lyapunov function on bounded set.** Let

$$V_\varepsilon = V + \varepsilon \cdot \text{ Multiplier term.}$$

Lemma

For any open bounded set $\mathcal{B}$ of the energy space X $(\simeq H^2(\Omega)\times L^2(\Omega)\times\mathbb{R}^2$ with coupling and some BC.), there exists $\varepsilon,\alpha,\beta > 0$ s.t.

$$\dot{V}_\varepsilon \leqslant -\alpha V_\varepsilon + \beta\|u\|^2 \quad \text{in } \mathcal{B}.$$

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Result for the beam model

We are now in position to add the output integrator

$$\dot{z} = \phi = \theta - \theta_{\mathrm{ref}}$$

and implement the forwarding feedback law

$$u = B^* \mathrm{d}M(x)^* [z - M(x)]$$

to stabilize the cascade.

Theorem

1. **(Output tracking.)** *Let $\theta_{\mathrm{ref}} \in \mathbb{R}$. With reference to the $(w, \dot{w}, \theta, \dot{\theta}, z)$-dynamics, the unique equilibrium at which $\theta = \theta_{\mathrm{ref}}$ is LES and GAS (in energy norm). In particular,*

$$\theta(t) \to \theta_{\mathrm{ref}}, \quad t \to +\infty.$$

2. **(Robustness.)** *Existence of a LES equilibrium is preserved under small matched disturbance $d \in \mathbb{R}$.*

Merci pour votre attention !