# Carleman-based reconstruction algorithm on a wave network

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# Coefficient inverse problem in the wave equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_{tt} y(t,x) - \Delta_x y(t,x) + p^*(x) y(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

- Given data: Source terms $f, g$ ; initial data: $(y^0, y^1)$;
- Unknown: the potential $p^* = p^*(x)$;
- Additional measurement : the flux $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

# Motivation

- The determination in $\Omega$ of $p^*$ from an additional measurement are inverse problems for which *uniqueness* and *stability* are well-known and proved using Carleman estimates.
- Classical reconstruction : from the measurement $d^* = \partial_\nu y[p^*]$, calculate

$$\min J(p) = \frac{1}{2}\|\partial_\nu y[p] - d^*\|^2.$$

  But $J$ is not convex and may have several local minima, so that the solution will depend on the initialization $p_0$. Algorithms not guaranteed to converge to the global minimum.
- Klibanov, Beilina and co-authors have worked a lot on related questions...

# The Carleman-based reconstruction algorithm

- First goal : compute the PDE unknown coefficient with convergence estimates and no a priori first guess.
- Core idea : build a reconstruction algorithm (C-bRec)
  - from the appropriate Carleman estimates to build the cost functional;
  - using the structure of the proof of stability to prove the global convergence.
- Until now, the idea was applied to three reconstruction cases:
  - potential / wave speed in the wave equation ([Baudouin, de Buhan, Ervedoza 2013, 2017], [Baudouin, de Buhan, Ervedoza, Osses 2021]);
  - source term in a non linear heat equation by [Boulakia, de Buhan, Schwindt, 2020].

# Outline

1. Presentation of the C-bRec algorithm
2. C-bRec algorithm on a network

# Outline

# Determination of the potential in the wave equation

$$\begin{cases} \partial_{tt} y - \Delta y + p^* y = f, & (0,T)\times\Omega, \\ y = g, & (0,T)\times\partial\Omega \\ (y(0), \partial_t y(0)) = (y^0, y^1), & \Omega. \end{cases}$$

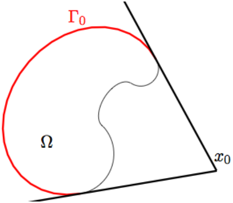
*Is it possible to retrieve the potential* $p^* = p^*(x)$, $x \in \Omega$ *from measurement of the flux* $d^* = \partial_\nu y[p^*](t,x)$ *on* $(0,T)\times\Gamma_0$ *?*

- Uniqueness: Given $p_1 \neq p_2$, can we guarantee $\partial_\nu y[p_1] \neq \partial_\nu y[p_2]$ ?
- Stability: If $\partial_\nu y[p_1] \simeq \partial_\nu y[p_2]$, can we guarantee that $p_1 \simeq p_2$ ?
- Reconstruction: Given $d^* = \partial_\nu y[p^*]$, can we compute $p^*$ ?

• Known results: Uniqueness ([Klibanov 92], stability ([Yamamoto 99], [Imanuvilov, Yamamoto 01]), using Carleman estimates.

• Main question: Reconstruction : how to compute the potential from the boundary measurement ?

# Stability Result ([Yamamoto 99], [Baudouin, Puel 01])




Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let $\Gamma_0$ and $T$ satisfy

$$\{x \in \partial\Omega,\ (x - x_0) \cdot \nu(x) > 0\} \subset \Gamma_0 \quad ; \quad T > \sup_{x \in \Omega}\{|x - x_0|\}.$$

Let the potential $p$, the initial data $y^0$ and the solution $y[p]$ s.t.

$$\|p\|_{L^\infty(\Omega)} \leq m, \quad \inf_{x \in \Omega}\{|y^0(x)|\} \geq \gamma > 0, \quad y[p] \in H^1(0, T; L^\infty(\Omega)).$$

Then, one can prove uniqueness and local Lipschitz stability of the inverse problem for the wave equation: $\forall q \in L^\infty_{\leq m}(\Omega)$,

$$\|p - q\|_{L^2(\Omega)} \leq C\|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))}.$$

# Towards a (re)constructive approach

The idea is considering $p^*$ as the fix point of a contracting application

$\rightsquigarrow$ construct a sequence $(q^k)_{k\in\mathbb{N}}$ converging towards $p^*$.

Based on the Bukhgeim-Klibanov method, it is easy to check that $Z = \partial_t \left(y[q^k] - y[p^*]\right)$ satisfies

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q^k(x) Z = (p^* - q^k)\partial_t y[p^*] =: h, & (t,x) \in (0,T)\times\Omega, \\ Z(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (p^* - q^k) y^0), & x\in\Omega. \end{cases}$$

One should notice that $Z$ was built to be the unique minimizer of the functional

$$J_h^k(z) = \int_0^T\int_\Omega e^{2s\varphi}|\partial_{tt} z - \Delta_x z + q^k(x) z - h|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu^k|^2,$$

where $\mu^k = \partial_t\left(\partial_\nu y[q^k] - \partial_\nu y[p^*]\right)$ on $\Gamma_0\times(0,T)$.Then

$$p^* = q^k + \frac{\partial_t Z(0)}{y^0}$$

Be careful: $h$ is unknown.

Idea: minimize another functional $J_0^k$ associated to $h = 0$.

# Carleman estimate [Baudouin, de Buhan, Ervedoza 13]

Assuming $q \in L^\infty_{\leq m}(\Omega)$, $\quad L_q = \partial_{tt} - \Delta_x + q(x)$, $\quad \varphi(t,x) = e^{\lambda(|x-x_0|^2 - \beta t^2)}$

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0\ , \qquad \sup_{x\in\Omega} |x - x_0| < \beta T$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that

$$s\int_0^T\!\!\int_\Omega e^{2s\varphi}\left(|\partial_t w|^2 + |\nabla w|^2 + s^2|w|^2\right) dxdt + s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\, dx$$
$$\leq\ M\int_0^T\!\!\int_\Omega e^{2s\varphi}|L_q w|^2\, dxdt + Ms\int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi}\,|\partial_\nu w|^2\ d\sigma dt,$$

for all $s > s_0$ and $w \in L^2(-T,T;H^1_0(\Omega))$ satisfying

$$\begin{cases} L_q w \in L^2(\Omega\times(0,T)) \\ \partial_\nu w \in L^2((0,T)\times\Gamma_0), \\ w(0,x) = 0,\ \forall x\in\Omega. \end{cases}$$

$\rightsquigarrow$ but also Imanuvilov, Zhang, Klibanov,...

# Carleman based Reconstruction Algorithm

Initialization: $q^0 = 0$ or any initial guess.
Iteration: Given $q^k$,
1 - Compute $w[q^k]$ the solution of

$$\begin{cases} \partial_t^2 w - \Delta w + q^k w = f, & \text{in } \Omega \times (0,T), \\ w = g, & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0, \quad \partial_t w(0) = w_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu w[q^k] - \partial_\nu w[p^*]\right)$ on $\Gamma_0 \times (0,T)$.

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$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} z|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2,$$

on the space
$\mathcal{T}^k = \{z \in L^2(0,T; H_0^1(\Omega)), z(t=0) = 0,$
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## Theorem

*Assume some geometric and time conditions. Then, $\forall s > 0$ and $k \in \mathbb{N}$, the functional $J_0^k$ is continuous, strictly convex and coercive on $\mathcal{T}^k$ endowed with a suitable weighted norm.*

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3 - Let $Z^k$ be the unique minimizer of the functional $J_0^k$, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{w_0}$$

where $w_0$ is the initial condition.

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where $w_0$ is the initial condition.
4 - Finally, set $q^{k+1} = T_m(\tilde{q}^{k+1})$ where

$$T_m(q) = \begin{cases} q, & \text{if } |q| \le m, \\ \text{sign}(q) m, & \text{if } |q| \ge m. \end{cases}$$

# Algorithm's convergence [Baudouin, de Buhan & Ervedoza 13]

### Theorem

*Assuming the geometric and time conditions (among others), there exists a constant* $M > 0$ *such that* $\forall s \geq s_0(m)$ *and* $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)}(q^{k+1} - p^*)^2 \, dx \leq \frac{M}{\sqrt{s}} \int_\Omega e^{2s\varphi(0)}(q^k - p^*)^2 \, dx.$$

*In particular, when $s$ is large enough, the algorithm converges.*

Remark : Convergence to the global minimum from any initial guess.

# Proof

As proposed earlier, let us set $v^k = \partial_t \left(y[q^k] - y[p^*]\right)$ that solves

$$\begin{cases} \partial_t^2 v - \Delta v + q^k v = f^k, & \text{in } \Omega \times (0,T), \\ v = 0, & \text{on } \partial\Omega \times (0,T), \\ v(0) = 0, \quad \partial_t v(0) = (p^* - q^k) y^0, & \text{in } \Omega, \end{cases}$$

where $f^k = (p^* - q^k)\partial_t y[p^*]$.
By definition, $\mu^k = \partial_\nu v^k$ on $\Gamma_0 \times (0,T)$, and we notice that $v^k$ is the unique minimizer of the functional:

$$J_h^k(w) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} w - f^k|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu w - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{w \in L^2(0,T; H_0^1(\Omega)), w(t=0) = 0,$ $L_{q^k} w \in L^2(\Omega \times (0,T)), \partial_\nu w \in L^2(\Gamma_0 \times (0,T))\}.$

# Proof II

Let us write the Euler Lagrange equations satisfied by:
$Z^k$ minimizer of $J_0^k$

$$\int_0^T\int_\Omega e^{2s\varphi} L_{q^k} Z^k \, L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z^k - \mu^k)\partial_\nu w = 0,$$

and $v^k$ minimizer of $J_h^k$

$$\int_0^T\int_\Omega e^{2s\varphi}(L_{q^k} v^k - f^k) L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu v^k - \mu^k)\partial_\nu w = 0,$$

for all $w \in \mathcal{T}^k$. Applying these to $w = Z^k - v^k$ and subtracting the two identities, we obtain:

$$\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 = \int_0^T\int_\Omega e^{2s\varphi} f^k \, L_{q^k} w,$$

implying ($2ab \leq a^2 + b^2$)

$$\frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 \leq \frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|f^k|^2.$$

# Proof III

The *LHS* is precisely the *RHS* of the Carleman estimate. Hence:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\,dx \le M\int_0^T\int_\Omega e^{2s\varphi}|f^k|^2\,dxdt,$$

where $\partial_t w(0) = \partial_t Z^k(0) - \partial_t v^k(0)$. Moreover,

$$\partial_t Z^k(0) = (\tilde{q}^{k+1} - q^k)y^0, \quad \partial_t v^k(0) = (p^* - q^k)y^0, \quad f^k = (p^* - q^k)\partial_t y[p^*].$$

Therefore, since $\varphi(t) \le \varphi(0)$ for all $t \in (0,T)$ we have:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|y^0|^2|\tilde{q}^{k+1} - p^*|^2\,dx \le M\|\partial_t y[p^*]\|^2_{L^2(0,T;L^\infty(\Omega))}\int_\Omega e^{2s\varphi(0)}|q^k - p^*|^2\,dx.$$

Using the positivity condition on $y^0$ and the fact that

$$|q^{k+1} - p^*| = |T_m(\tilde{q}^{k+1}) - T_m(p^*)| \le |\tilde{q}^{k+1} - p^*|$$

because $T_m$ is Lipschitz and $T_m(p^*) = p^*$, we can deduce

$$\int_\Omega e^{2s\varphi(0)}(q^{k+1} - p^*)^2\,dx \le \left(\frac{M}{\sqrt{s}}\right)^{k+1}\int_\Omega e^{2s\varphi(0)}(q^0 - p^*)^2\,dx. \qquad \square$$

# In theory, it works. But in practice ?

Two remarks:

- Discretizing the wave equation brings numerical artefacts...
- Minimizing a strictly convex and coercive quadratic functional based on a Carleman estimate means dealing with $e^{2se^{\lambda\psi}}$ for large parameters $s$ and $\lambda$...

New goal: propose a numerically efficient algorithm.

*Ideas:* We need an algorithm constructed with at least

- a regularization term in the cost functional,
- a single parameter Carleman estimate.

$\rightsquigarrow$ [Baudouin, de Buhan, Ervedoza 2017]

# Convergence of the discrete inverse problems

*Remarks:*

- Natural question for all inverse problems in infinite dimensions: Finding a source term, a conductivity...
- Depends *a priori* on the numerical scheme employed.

*Main difficulty:*

- Different dynamics for the continuous wave equation versus its discrete approximations, cf [Ervedoza, Zuazua 2011]:
  $\rightsquigarrow$ Numerical artefacts: High-frequency spurious waves.

*Convergence results for the inverse problem:*

- Penalization of high-frequencies with a regularization term in the discrete Carleman estimates.
- 1D [Baudouin, Ervedoza 2013] and 2D [Baudouin, Ervedoza, Osses 2015]

# New C-bRec algorithm [Baudouin, de Buhan, Ervedoza 2017]

The algorithm is also modified according to the following items :

- Single parameter Carleman estimate ;

  $\rightsquigarrow$ presence of an additional term on the right
  $$s^3 \int\int_{\mathcal{O}} e^{2s\varphi}|z|^2$$

- Preconditioning of the cost functional ;

  $\rightsquigarrow$ introduce the conjugate variable $y = e^{s\varphi}z$

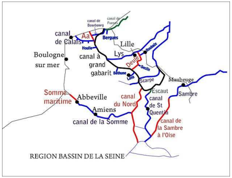
- Splitting of the observations by cut-off ;

  $$\rightsquigarrow v^k = \eta^\varphi \partial_t(y[q^k] - y[p^*])$$

... and the convergence result remains the same.

# Outline

# PDE on networks




Applications :

- control or stabilize the vibrations of elastic structures (as bridges, cranes,...),
- regulate the height of water in networks of irrigation canals,
- find the topography of the bottom in a network of irrigation canals,
- detect water losses by measurements in nodes,
- control gas flow in pipelines through compressors,
- determine the blood pressure leaving the heart with a finger pressure measurement,
- control road traffic on a network of roads or the flow of blood in a network of arteries,...

# PDE on networks

On networks, the state is represented by several components

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix}$$

and the components are coupled together by boundary conditions.
If $p < N$ is the number of controls/observations, it is therefore necessary to pass the information on the remaining $N - p$ branches.
Goals:

- minimize the number of observations, feedbacks or controls,
- choice of placement of observations, feedback mechanisms or controls based on network topology and branch lengths.

# An inverse problem on network

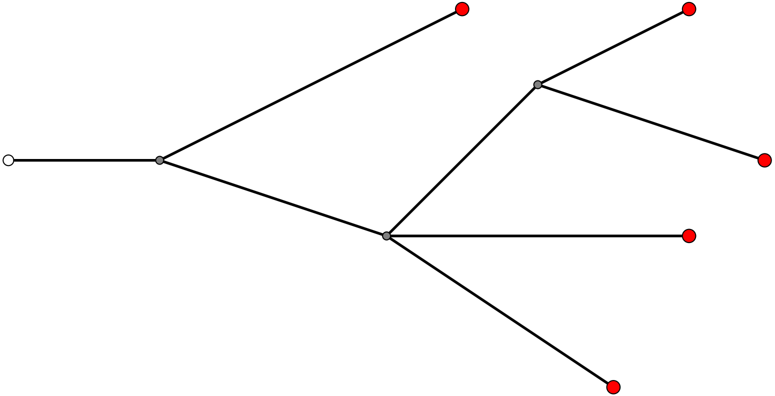


Figure: An $8$ branches tree-shaped network $\mathcal{R}$, with an unobserved root node and $5$ observed leaf nodes •.

# Notations

Let us thus consider a finite tree-shaped network $\mathcal{R}$.

- $\mathcal{J}$: the set of names of all branches of the network.
- We define the name of the branches by recurrence:
  - To the root branch, named $1$, we associate its $N_1$ children branches denoted by $1_i \in \mathbb{N}$ for $i = 1..N_1$.
  - From a branch named $j \in \mathcal{J}$ we define the names of its $N_j$ children branches by $j_i$ for $i = 1..N_j$.
- $\ell_j$: the length of the branch $j$.
- $\mathcal{J}_{ext} = \{j \in \mathcal{J}, N_j = 0\}$.
- $\mathcal{J}_{int} = \{j \in \mathcal{J}, N_j > 0\}$.
- $f_j$: the restriction of the function $f$ on $\mathcal{R}$ to the branche $j$.
- $\displaystyle\int_{\mathcal{R}} f(x)dx := \sum_{j\in\mathcal{J}} \int_0^{\ell_j} f_j(x)dx,$
- $\displaystyle [f]_j := f_j(\ell_j) - \sum_{i=1}^{N_j} f_{j_i}(0), \quad \forall j \in \mathcal{J}_{int}.$

# An inverse problem on network

On each branch $j \in \mathcal{J}$ of the network, we consider the one-dimensional wave equation system

$$\begin{cases} \partial_{tt} u_j(t,x) - \partial_{xx} u_j(t,x) + p_j(x) u_j(t,x) = g_j(t,x), & (t,x) \in (0,T) \times (0,\ell_j), \\ u_j(0,x) = u_j^0(x), \qquad \partial_t u_j(0,x) = u_j^1(x), & x \in (0,\ell_j), \end{cases}$$

with

$$\begin{cases} \text{for } j = 1, & u_1(t,0) = h_1(t), \\ \text{if } j \in \mathcal{J}_{ext}, & u_j(t,\ell_j) = h_j(t), \\ \text{if } j \in \mathcal{J}_{int}, & u_j(t,\ell_j) = u_{j_i}(t,0), \ \forall i \in \{1, \cdots, N_j\}, \\ & [\partial_x u]_j(t) = 0, \end{cases}$$

# Inverse problem on a network

## Inverse problem

Knowing, for each branch $j \in \mathcal{J}$, the source term $g_j$ and the initial data $(u_j^0, u_j^1)$, for the root and for each leaf $j \in \{1\} \cup \mathcal{J}_{ext}$ the boundary source term $h_j$, is it possible to identify the unknown potentials $p_j^*(x)$ for any $x \in (0, \ell_j)$, from the only extra knowledge of the flux of the solutions through the leaf nodes of the network, meaning:

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \quad \text{for } i \in \mathcal{J}_{ext} \text{ and } t \in (0, T),$$

where $u_i^*$ is the solution associated to potential $p_i^*$?

# Lipschitz stability result [Baudouin, Crépeau, V. 2011]

### Theorem

*There exist a time* $T_0 > 0$ *and a scalar* $\alpha_0 > 0$ *such that if*

1. *Time condition:* $T > T_0$,
2. *Regularity condition:* $u \in H^1(0, T; L^\infty(\mathcal{R}))$,
3. *Sign condition:* $|u^0| \geq \alpha^0 > 0$ *on the whole network* $\mathcal{R}$,

*then for a fixed* $m > 0$, *there exists a positive constant* $C = C(\mathcal{R}, T, m)$ *such that, if* $p$ *and* $p^*$ *belong to* $L^\infty_m(\mathcal{R}) = \{p \in L^\infty(\mathcal{R}), \|p\|_{L^\infty(\mathcal{R})} \leq m\}$, *we have*

$$\|p - p^*\|^2_{L^2(\mathcal{R})} \leq C \sum_{i \in \mathcal{J}_{ext}} \|\partial_x u_i(\cdot, \ell_i) - \partial_x u_i^*(\cdot, \ell_i)\|^2_{H^1(0,T)}.$$

Proof: based on the Bukhgeim and Klibanov method and a two parameters Carleman estimate.

# Carleman weight function $\varphi$

$$\forall j \in \mathcal{J},\ \varphi_j(t,x) = (x - x_j)^2 - \beta t^2 + M_j, \quad (t,x) \in \mathbb{R} \times (0, \ell_j).$$

There exist $(x_j)_{j\in\mathcal{J}} \in \mathbb{R}^-$, $(M_j)_{j\in\mathcal{J}} \in \mathbb{R}^+$, $\beta \in (0,1)$ and $T > 0$ satisfying

$$\beta T > \sup_{j\in\mathcal{J}} (\ell_j - x_j)$$

such that it holds
(i) $\forall j \in \mathcal{J}_{int}$, $\varphi_{j_i}(t,0) = \varphi_j(t,\ell_j), \quad \forall i \in \{1, \cdots N_j\}$.
(ii) The matrices $A_j^\varphi(t)$ satisfy for any $j \in \mathcal{J}_{int}$: $\exists \alpha_j^0 > 0, \beta_j > 0$, $\forall \xi \in \mathbb{R}^{N_j+1}$,

$$\begin{array}{lll} (A_j^\varphi(t)\xi,\xi) \geq \alpha_j^0 \|\xi\|^2, & \forall t, & |t| \leq T_j := \frac{\ell_j - x_j}{\beta}; \\ (A_j^\varphi(t)\xi,\xi) \geq \alpha_j^0 \|\xi\|^2 - \beta_j |\xi_{N_j+1}|^2, & \forall t, & T_j \leq |t| \leq T; \end{array}$$

where $A_j^\varphi(t)$ are $(N_j+1)\times(N_j+1)$ symmetric matrices defined by

$$A_j^\varphi(t) := \begin{pmatrix} \phi_{j_1}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j) & \cdots & -\phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & -\phi_j(\ell_j) & \vdots \\ & & & \phi_{j_{N_j}}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & & & & a_j(t) \end{pmatrix}$$

with $\phi(x) := \partial_x \varphi(t,x)$ and $a_j(t) = -\phi_j(\ell_j)[\phi]_j^2 + \left[(|\partial_t \varphi(t)|^2 - |\phi|^2)\phi\right]_j$.

# First tool: one-parameter Carleman estimate [Baudouin, de Buhan, Crépeau, V.]

## Theorem

*There exist* $C > 0$, $s_0 > 0$ *such that for all* $s \geq s_0$ , *for all* $p \in L^\infty_m(\mathcal{R})$,

$$s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0,x)} |\partial_t z(0,x)|^2 dx + s \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} \left( |\partial_t z|^2 + |\partial_x z|^2 + s^2 |z|^2 \right) dx dt$$

$$\leq C \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + pz|^2 dx dt$$

$$+ Cs \sum_{i \in \mathcal{J}_{ext}} \int_{-T}^{T} e^{2s\varphi_i(t,\ell_i)} |\partial_x z_i(t,\ell_i)|^2 dt + Cs^3 \mathcal{I}(z,z),$$

*satisfied by all* $z \in H^1((-T,T); H^1_0(\mathcal{R}))$ *s.t.* $\partial_{tt} z - \partial_{xx} z \in L^2((0,T) \times \mathcal{R})$, *under Kirchhoff node condition and* $z(0,\cdot) = 0$ *in* $\mathcal{R}$, *and where*

$$\mathcal{I}(z,z) = \iint_{(|t|,x) \in \mathcal{O}} e^{2s\varphi} |z|^2 dx dt + \sum_{j \in \mathcal{J}_{int}} \int_{|t| \in \mathcal{O}_{T_j}} e^{2s\varphi_j(t,\ell_j)} |z_j(t,\ell_j)|^2 dt$$

*with* $\mathcal{O} = \cup_{j \in \mathcal{J}} \mathcal{O}_j$ *where* $\mathcal{O}_j = \{(t,x) \in (0,T) \times (0,\ell_j), |x - x_j| - \beta|t| < 0\}$ *and* $\mathcal{O}_{T_j} = \{t \in (0,T), |\ell_j - x_j| - \beta|t| < 0\}$ *defined only for* $x = \ell_j$, $j \in \mathcal{J}_{int}$.

# The domains $\mathcal{O}_j$ and $\mathcal{O}_{T_j}$

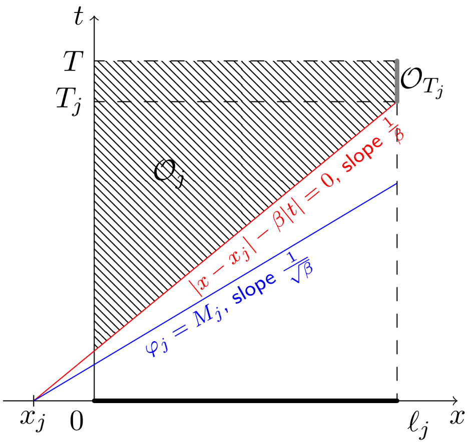



Figure: Illustration of domains $\mathcal{O}_j$ and $\mathcal{O}_{T_j}$ for the branch $(0, \ell_j)$, denoting $T_j = |l_j - x_j|/\beta$.

# Some ideas of the proof

We set $y = ze^{s\varphi}$ on $(-T,T)\times(0,\ell)$ and the conjugate operator $L_s(y) = e^{s\varphi}(\partial_{tt}-\partial_{xx})(e^{-s\varphi}y)$. Easy calculations bring

$$L_s(y) = \underbrace{(\partial_{tt}y - \partial_{xx}y + s^2(|\partial_t\varphi|^2 - |\partial_x\varphi|^2)y)}_{P_1y} + \underbrace{2s\partial_x\varphi\partial_x y - 2s\partial_t\varphi\partial_t y}_{P_2y} - \underbrace{s(\partial_{tt}\varphi - \partial_{xx}\varphi)y}_{Ry},$$

and

$$\int_{-T}^{T}\int_{\mathcal{R}} |L_s(y) - Ry|^2 dxdt = \int_{-T}^{T}\int_{\mathcal{R}}(|P_1y|^2 + |P_2y|^2)dxdt + 2\int_{-T}^{T}\int_{\mathcal{R}} P_1yP_2y dxdt.$$

# Some ideas of the proof

The main work of the proof consists in the computation and bound from below of the cross-term

$$I = \int_{-T}^{T} \int_0^{\ell} P_1 y \, P_2 y dx dt,$$

by:

- positive and dominant terms as

$$s \int_{-T}^{T} \int_{\mathcal{R}} \left( |\partial_t y|^2 + |\partial_x y|^2 + s^2 |y|^2 \right) dx dt$$

- negative boundary terms (measured)

$$-s \sum_{i \in \mathcal{J}_{ext}} \int_{-T}^{T} |\partial_x y_i(t, \ell_i)|^2 dt$$

- negative boundary terms

$$-s^3 \iint_{(|t|,x) \in \mathcal{O}} |y|^2 dx dt, \quad -s^3 \sum_{j \in \mathcal{J}_{int}} \int_{|t| \in \mathcal{O}_{T_j}} |y_j(t, \ell_j)|^2 dt.$$

# Terms at the internal nodes

The terms at the internal nodes are

$$B \geq \sum_{j\in\mathcal{J}_{int}} s \int_{-T}^{T} \langle A_j^{\varphi}(t) W_j(t), W_j(t)\rangle \, dt - Cs^2 \sum_{j\in\mathcal{J}_{int}} \int_{-T}^{T} |y_j(t,\ell_j)|^2 dt,$$

where $W_j(t) \in \mathbb{R}^{N_j+1}$ is defined by

$$W_j(t) = \begin{pmatrix} \partial_x y_{j_1}(t,0) & \dots & \partial_x y_{j_{N_j}}(t,0) & s y_j(t,\ell_j) \end{pmatrix}^{\top}.$$

Moreover, by assumption on $A_j^{\varphi}$, we have

$$\sum_{j\in\mathcal{J}_{int}} s \int_{-T}^{T} \langle A_j^{\varphi}(t) W_j(t), W_j(t)\rangle \, dt$$

$$\geq Cs^3 \sum_{j\in\mathcal{J}_{int}} \int_{|t|<T_j} |y_j(t,\ell_j)|^2 dt - Cs^3 \sum_{j\in\mathcal{J}_{int}} \int_{|t|>T_j} |y_j(t,\ell_j)|^2 dt.$$

# Second tool: properties of the cut-off function $\eta^\varphi$

$v^k = \eta^\varphi \partial_t \left(u^k - u^*\right)$ (with $\eta^\varphi \in C^2((0,T)\times\mathcal{R})$) is solution of

$$\begin{cases} \partial_{tt} v^k(t,x) - \partial_{xx} v^k(t,x) + p^k(x) v^k(t,x) = f^k(t,x), & \text{in } (0,T)\times\mathcal{R}, \\ v^k(0,x) = 0, \quad \partial_t v^k(0,x) = \eta^\varphi(0,x)(p^*(x) - p^k(x))u^0(x), & \text{in } \mathcal{R}, \end{cases}$$

where $f^k := \eta^\varphi(p^* - p^k)\partial_t u^* - [\eta^\varphi, \partial_{tt} - \partial_{xx}]\partial_t \left(u^k - u^*\right)$.
$v^k$ satisfies also the continuity and the Kirchhoff law at the internal nodes, and the Dirichlet boundary condition at the external nodes.

$v^k$ is built to be the unique minimizer of the functional

$$F_s[p^k, f^k, \mu^k](z) = \frac{1}{2}\int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + p^k z - f^k|^2 \, dxdt$$
$$+ \frac{s}{2} \sum_{i\in\mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t,\ell_i)} |\partial_x z_i(t,\ell_i) - \mu_i^k(t)|^2 dt + \frac{s^3}{2}\mathcal{I}(z,z),$$

where we set, for all $i \in \mathcal{J}_{ext}$, $\mu_i^k(t) = \eta_i^\varphi(t,\ell_i)\partial_t \left(\partial_x u_i^k(t,\ell_i) - d_i^*(t)\right)$.

# Properties expected from $v^k$

- Encoding $(p^k - p^*)$, which is the information we seek, through the initial speed data $\partial_t v^k(0,\cdot) = \eta^\varphi(0,\cdot)(p^* - p^k)u^0$

$$\rightsquigarrow \eta_j^\varphi(0,\cdot) = 1.$$

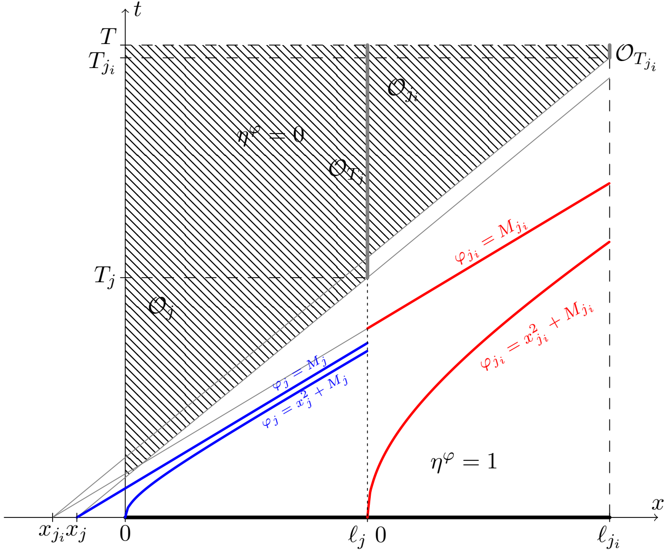
- Vanishing in the domains $\mathcal{O}$ and $\mathcal{O}_{T_j}$ so that $\mathcal{I}(v^k, v^k) = 0$

  $\rightsquigarrow \eta_j^\varphi = 0$ on some domain greater than $\mathcal{O} \cup \left(\cup_{j\in\mathcal{J}_{int}} \mathcal{O}_{T_j} \times \{\ell_j\}\right)$.

- Allowing the source term $f^k$ solved by $v^k$ to be manageable. We will ask for $\eta^\varphi$ to vary (between $0$ and $1$) only in a small region of $(0,T) \times \mathcal{R}$. Actually, on each $(0,T) \times (0,\ell_j)$, it will be specifically possible (meaning *manageable*) where $M_j < \varphi_j < x_j^2 + M_j$.

- But it also has to be done properly across each internal node to ensure continuity and Kirchhoff law for $v^k$ at those nodes.

# Context of application of the cut-off functions $\eta^\varphi$ over two consecutive branches $j$ and $j_i$.



# Third tool: properties of the cost functional $F_s$

## Lemma

*For all $s > 0$ large enough, $p \in L^\infty(\mathcal{R})$, $f \in L^2(0,T;L^2(\mathcal{R}))$ and $\mu \in L^2(0,T)$, the functional $F_s[p,f,\mu]$ recalled here*

$$F_s[p,f,\mu](z) = \frac{1}{2}\int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + pz - f|^2 \, dxdt + \frac{s}{2} \sum_{i \in \mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t,\ell_i)} |\partial_x z_i(t,\ell_i) - \mu_i(t)|^2 dt + \frac{s^3}{2} \mathcal{I}(z,z),$$

*is continuous, strictly convex and coercive on $\mathcal{T}$ defined by*

$$\mathcal{T} = \Big\{ z \in C^0([0,T]; H_0^1(\mathcal{R})) \cap C^1([0,T]; L^2(\mathcal{R})), \partial_{tt} z - \partial_{xx} z \in L^2((0,T)\times\mathcal{R}), z(0,\cdot) = 0 \text{ in } \mathcal{R}, \text{ and } [\partial_x z]_j(t) = 0, \ \forall j \in \mathcal{J}_{int}, t \in (0,T) \Big\}$$

*and equipped with an appropriate weighed norm.*
*Thenceforth, the functional $F_s[p,f,\mu]$ admits a unique minimizer on the set $\mathcal{T}$.*

# The C-bRec algorithm on a network

Knowing, for each branch $j \in \mathcal{J}$, $g_j$, $h_j$ and $(u_j^0, u_j^1)$, we have the extra measured information at the leaves of the network $\mathcal{R}$:

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \text{ for } i \in \mathcal{J}_{ext} \text{ and } t \in (0,T).$$

Initialisation: Choose any initial guess $p^0 \in L_m^\infty(\mathcal{R})$.
Iteration: Knowing $p^k \in L_m^\infty(\mathcal{R})$,

1. Calculate the solution $u^k$ associated to $p^k$, and set

$$\forall i \in \mathcal{J}_{ext}, \forall t \in (0,T), \quad \mu_i^k(t) = \eta_i^\varphi(t, \ell_i)\partial_t \left(\partial_x u_i^k(t, \ell_i) - d_i^*(t)\right).$$

2. Minimize the functional $F_s[p^k, 0, \mu^k]$ defined by on the space $\mathcal{T}$ and denote $w^k$ its unique minimizer.
3. Then set

$$\tilde{p}^{k+1} = p^k + \frac{\partial_t w^k(0,\cdot)}{u^0}, \quad \text{on } \mathcal{R}.$$

4. Finally, construct

$$p^{k+1} = T_m(\tilde{p}^{k+1}) := \begin{cases} \tilde{p}^{k+1}, & \text{if } |\tilde{p}^{k+1}| \leq m, \\ \text{sign}(\tilde{p}^{k+1})m, & \text{if } |\tilde{p}^{k+1}| > m. \end{cases}$$

Stopping criterion: Choose $\epsilon > 0$ and $K \in \mathbb{N}^*$ and stop the iterative loop as soon as

$$\sup_{j \in \mathcal{J}_{ext}} \frac{\left\| \partial_x u_i^k(t, \ell_i) - d_j^* \right\|_2}{\|d_j^*\|_2} \leq \epsilon, \quad \text{or} \quad \sup_{j \in \mathcal{J}} \frac{\|p_j^{k+1} - p_j^k\|_\infty}{m} \leq \epsilon,$$

or when the maximal number of iterations $K$ is reached.

# Convergence result [Baudouin, de Buhan, Crépeau, V.]

### Theorem

*Assume that* $p^* \in L^\infty_m(\mathcal{R})$*. Then there exists a constant* $C > 0$ *such that for all* $s$ *large enough and for all* $k \in \mathbb{N}$*, it holds*

$$\int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 \, dx \leq \left( \frac{C}{s^{1/2}} \right)^k \int_{\mathcal{R}} e^{2s\varphi(0)} |p^0 - p^*|^2 \, dx.$$

*In particular, if* $s$ *is large enough, the sequence* $(p^k)_{k \in \mathbb{N}}$ *given by the algorithm converges towards* $p^*$ *when* $k$ *tends to infinity.*

# Discretization of the algorithm

- Discretization of the system: finite differences (explicit centered scheme) in space and time.
- Minimization of $F_s[p^k, 0, \mu^k]$: resolution of a variational formulation
  - approximation of the integrals using rectangle quadrature rules and standard centered finite differences,
  - attention must be paid to the discretization process of $\mathcal{T}$,
  - add viscosity terms to guarantee coercivity property uniformly with respect to discretization parameters (to handle high frequency spurious waves).
- Presence of large exponential factors in $F_s[p^k, 0, \mu^k]$:
  - to work on the conjugate variable $(y_j^k)_i^n = (w_j^k)_i^n e^{s\varphi_j(t^n, x_i)}$ that acts as a preconditioner of the linear system,
  - there are still exponential factors in the right hand side vector $\rightsquigarrow$ develop a progressive process to compute the solution as the aggregation of several problems localized in subdomains in which the exponential factors are all of the same order.

# Numerical example

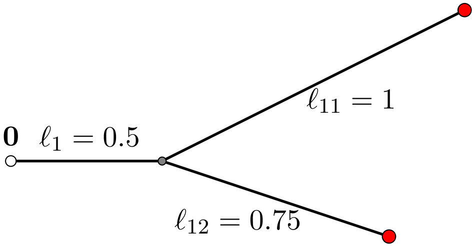


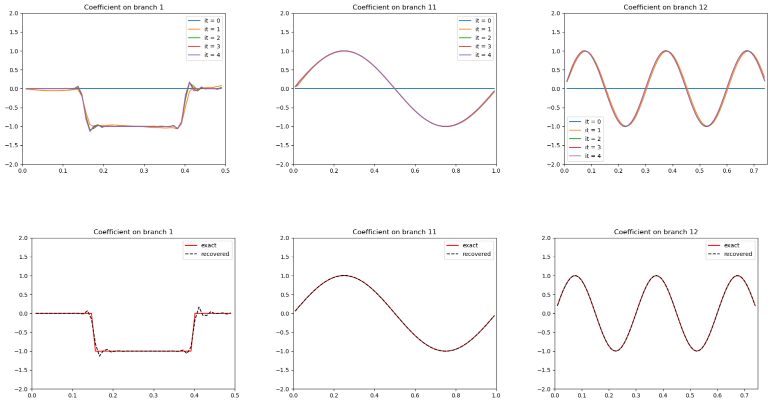

Figure: First setting - a $3$ branches network, with observations at $\bullet$.

# Numerical values

| $u_0$ | $u_1$ | $g$ | $h$ | $m$ |
|---|---|---|---|---|
| (2,2,2) | (0,0,0) | (0,0,0) | (2,2,2) | 2 |
| $\ell_j$ | $\beta$ | $s$ | $\epsilon_1$ | $\epsilon_2$ |
| (0.5,1,0.75) | 0.99 | 1 | $10^{-3}$ | $10^{-2}$ |
| $x_j$ | $M_j$ | $T$ | $N_{xj}$ | $N_t$ |
| (-0.3,-2.89,-2.89) | (7.71,0,0) | 3.9 | $100 * \ell_j$ | $110 * T$ |

Table: Numerical values of the variables used for all the numerical examples.

# Simulations from data without noise



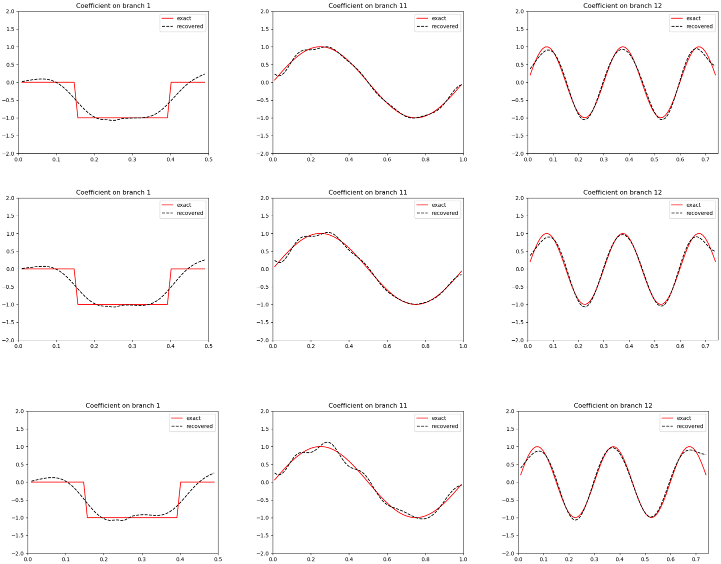

(a) $p_1^*(x) = -1_{[0.3,0.8]}(x/\ell_1)$

(b) $p_{11}^*(x) = \sin(2\pi x/\ell_{11})$

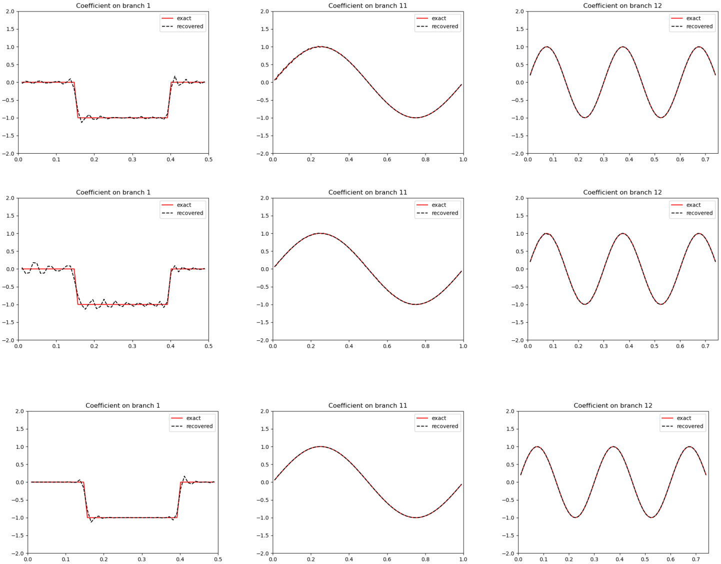
(c) $p_{12}^*(x) = \sin(5\pi x/\ell_{12})$

Figure: Top line: Convergence history of the reconstruction process. Bottom line: final reconstruction result (dotted black line) and exact coefficient (red line) for the three branches.

# Simulations with several levels of noise: $\theta = 1\%$, $\theta = 2\%$, $\theta = 5\%$ noise in the data



# Wrong choices of the parameters: $T = 1.5$, $T = 1.25$, without projection



# Conclusion

- Reconstruction of potentials on networks of wave equations.
- The C-bRec approach seems quite adaptable, even if it is to the price of appropriate one-parameter Carleman estimates.
- Other numerical examples of network?
- Other equations? KdV equation? Elasticity?