Approximate and exact controllability criteria for linear 1D hyperbolic systems

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Network of 1D transport equations

$$(Hyp)\begin{cases} \partial_t R(t,x) + \Lambda(x)\partial_x R(t,x) + D(x)R(t,x) = 0, \; t>0, x\in(0,1), \\ \begin{pmatrix} R^+(t,0) \\ R^-(t,1) \end{pmatrix} = M \begin{pmatrix} R^+(t,1) \\ R^-(t,0) \end{pmatrix} + Bu(t), \qquad t \geq 0, \end{cases}$$

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- $\Lambda(x)$, $D(x)$ diagonal $n \times n$ matrices with nonzero diagonal entries whose sign is independent of x;
- R^+ (resp. R^-) gathers components of R whose corresponding diagonal element in $\Lambda(x)$ is positive (resp. negative);
- $u\colon \mathbb{R}_+ \to \mathbb{R}^m$ control law; B real $n \times m$ matrix; M real $n \times n$ matrix accounting for boundary conditions.

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Main goal: Determine (necessary/or sufficient) conditions in the frequency domain, i.e., Hautus tests for L^q approx. and/or exact controllability of (Hyp),

Previous work Tucsnak-Weiss (09); Bastin-Coron (16); Miller (05); Ramdani and al. (05); Coron-Nguyen (19).

Assumptions

- $\Lambda(x) = \mathrm{diag}\{\lambda_1(x), \dots, \lambda_n(x)\}$ and $\exists \tilde{n} \in \{0, \dots, n\}$ s.t.

$$\lambda_i(x) < 0 < \lambda_j(x) \qquad \forall i \in \{\tilde{n}+1, \dots, n\},\ j \in \{1, \dots, n\},$$

$$\lambda_i, \frac{1}{\lambda_i} \in L^\infty((0,1), \mathbb{R}) \quad \forall i \in \{1, \dots, n\};$$

- $D(x) = \mathrm{diag}\{d_1(x), \dots, d_n(x)\}$ with $d_i \in L^1((0,1), \mathbb{R})$ $\forall i \in \{1, \dots, n\}$;
- Solution R splits into positive and negative velocities, i.e.,

$$R = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} \quad \text{with} \begin{cases} R^+ & = & (R_1, \dots, R_{\tilde{n}})^T, \\ R^- & = & (R_{\tilde{n}+1}, \dots, R_n)^T, \end{cases}$$

Solution of (Hyp)

Definition (Solution)

$T > 0$, $u\colon [0,T] \to \mathbb{R}^m$, and $R_0\colon [0,1] \to \mathbb{R}^n$.
$R\colon [0,T] \times [0,1] \to \mathbb{R}^n$ *solution* of (Hyp) in $[0,T]$ with initial condition R_0 and control u if $R(0,x) = R_0(x)$ $\forall x \in [0,1]$, boundary equations satisfied $\forall t \geq 0$, and, $\forall i \in [\![1,n]\!]$, $t \in [0,T]$, and $x \in [0,1]$,

$$R_i\left(t + \int_x^{x+h} \frac{d\xi}{\lambda_i(\xi)}, x + h\right) = e^{-\int_x^{x+h} \frac{d_i(\xi)}{\lambda_i(\xi)} d\xi} R_i(t,x) \qquad (2)$$

$\forall h \in \mathbb{R}$ s.t. $t + \int_x^{x+h} \frac{d\xi}{\lambda_i(\xi)} \in [0,T]$ and $x + h \in [0,1]$.

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If R of class C^1 then it satisfies the PDE.
Cf. Concept of *broad solution* in Coron-Nguyen (19).

Equivalent Linear Difference Delay System

Set $K = M\,\mathrm{diag}\left\{e^{-\int_0^1 \frac{d_1(x)}{|\lambda_1(x)|}dx}, \ldots, e^{-\int_0^1 \frac{d_n(x)}{|\lambda_n(x)|}dx}\right\}$, and

$$\tau_i = \int_0^1 \frac{dx}{|\lambda_i(x)|}, \qquad i \in [\![1, n]\!].$$

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Consider the Linear Difference Delay System

$$(LDDS) \quad \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = K \begin{pmatrix} y_1(t-\tau_1) \\ \vdots \\ y_n(t-\tau_n) \end{pmatrix} + Bu(t), \qquad t \geq 0,$$

Claim: (Hyp) and (LDDS) are “equivalent”.

Existence of L^q-solutions of (Hyp)

For $q \in [1, \infty]$ and $t \geq 0$, 1-to-1 correspondence between

- solutions $R(t, \cdot)$ of (Hyp) defined on $L^q((0,1), \mathbb{R}^n)$;
- solutions $y_{[t]} := (y_i(t + \cdot))_{1 \leq i \leq n}$ of (LDDS) defined on

$$\Sigma^q = \prod_{i=1}^{n} L^q((-\tau_i, 0), \mathbb{R}).$$

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Proposition (Mazanti, Sigalotti, C. 2020)

Let $q \in [1, +\infty]$, $T > 0$, $R_0 \in L^q((0,1), \mathbb{R}^n)$, and $u \in L^q((0, T), \mathbb{R}^m)$. Then (Hyp) admits a unique solution $R \colon [0, T] \times [0, 1] \to \mathbb{R}^n$ in $[0, T]$ with initial condition R_0 and control u, which satisfies $R(t, \cdot) \in L^q((0,1), \mathbb{R}^n)$ for every $t \in [0, T]$.

Controllability notions

Definition

Let $q \in [1, +\infty]$. (Hyp) (resp. (LDDS)) is said to be

1) *L^q-approximately controllable* if, for every $\epsilon > 0$ and $\phi, \psi \in L^q([0,1], \mathbb{R}^n)$ (resp. Σ_q), there exists $u \in L^q([0, T], \mathbb{R}^m)$ such that the solution R (resp. y) of (Hyp) (resp. (LDDS)) with initial condition ϕ and control u satisfies $\|R(T, \cdot) - \psi\|_{[0,1], q} < \epsilon$ (resp. $\|y_{[T]} - \psi\|_{\Sigma^q} < \epsilon$).

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2) *L^q-exactly controllable* if, for every $\phi, \psi \in L^q([0,1], \mathbb{R}^n)$, there exists $u \in L^q([0, T], \mathbb{R}^m)$ such that the solution R (resp. y) of (Hyp) (resp. (LDDS)) with initial condition ϕ and control u satisfies $R(T, \cdot) = \psi$ (resp. $y_{[T]} = \psi$).

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In above definition, T depends on data ϵ, ϕ, ψ.
L^q-approximately (resp. exactly) controllable in time T if time T above does not depend on ϵ, ϕ, ψ.

Representation Formulas - 1 -

Systems of the form (LDDS) :

$$y(t) = \sum_{j=1}^{d} K e_j e_j^T y(t - \tau_j) + Bu(t), \; y \in \mathbb{R}^d, \; u \in \mathbb{R}^m, \; t \geq 0. \qquad (3)$$

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Definition

Consider family of matrices $\Xi_n \in \mathcal{M}_{d,d}(\mathbb{R})$, $n \in \mathbb{Z}^d$, defined by

$$\Xi_n = \begin{cases} 0 & \text{if } n \in \mathbb{Z}^d \setminus \mathbb{N}^d, \\ I_d & \text{if } n = 0, \\ \sum_{j=1}^{d} K e_j e_j^T \Xi_{n-s_j} & \text{if } n \in \mathbb{N}^d \text{ and } |n| > 0, \end{cases} \qquad (4)$$

where s_k = k-th canonical vector of $\mathbb{N}^d$.

Representation Formulas - 2 -

Delay vector $\tau := (\tau_1, \cdots, \tau_d)$

1) (FLOW)
$\Upsilon_q(T) : \Sigma_q \longrightarrow \Sigma_q$ defined, for $\phi \in \Sigma^q$ and $i \in [\![1, d]\!]$, $s \in [-\tau_i, 0]$, by

$$(\Upsilon_q(T)\phi)_i(s) = e_i^T \sum_{\substack{(\ell,j) \in \mathbb{N}^d \times [\![1,d]\!] \\ -\tau_j \leq T+s-\tau\cdot\ell < 0}} \Xi_{\ell - e_j} K e_j \phi_j(T + s - \tau \cdot \ell).$$

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2) (End-Point MAP) $E_q(T) : L^q([0, T], \mathbb{R}^m) \longrightarrow \Sigma_q$ defined, for $u \in L^q([0, T], \mathbb{R}^m)$ and $i \in [\![1, d]\!]$, by

$$\left(E_q(T)u\right)_i(t) = e_i^T \sum_{\substack{\ell\in\mathbb{N}^d \\ \tau\cdot\ell\leq T+t}} \Xi_\ell B u(T+t-\tau\cdot\ell), \qquad t \in [-\tau_i, 0].$$

Representation Formulas - 3 -

Proposition (Mazanti, Sigalotti, C. 2020)

For $T \geq 0$, $q \in [1, +\infty]$, $u \in L^q([0, T], \mathbb{R}^m)$, *and* $\phi \in \Sigma^q$, *Unique Solution* y *of (LDDS) with initial condition* ϕ *and control* u *given by*

$$y_{[t]} = \Upsilon_q(t)\phi + E_q(t)u, \qquad t \in [0, T]. \tag{5}$$

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For later use, consider dual operator $E_q(T)^*$ of $E_q(T)$.

Proposition

$T \geq 0$, $q \in [1, +\infty)$, $\frac{1}{q} + \frac{1}{q}' = 1$.

$E_q(T)^* : \ \Sigma^{q'} \longrightarrow L^{q'}([0, T], \mathbb{R}^m)$, *for* $y \in \Sigma^{q'}$, $t \in [0, T]$

$$\left(E_q(T)^* y\right)_i(t) = e_i^T \sum_{\substack{(\ell, j) \in \mathbb{N}^d \times [\![1, n]\!] \\ -\tau_j \leq t - T + \tau \cdot \ell < 0}} B^* \Xi_\ell^* e_j y_j(t - T + \tau \cdot \ell). \tag{6}$$

Basic controllability properties

Let $T > 0$, $q \in [1, +\infty]$ and q' conjugate exponent of q.

Proposition

(Hyp) L^q-approximate (respectively, exactly) controllable in time T if and only if the same is true for (LDDS).

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Proposition

1. *The following assertions are equivalent:*
   - (1.a) *(LDDS) is L^q-approximately controllable in time T;*
   - (1.b) Ran $E_q(T)$ *is dense in* Σ^q;
   - (1.c) *The operator $E_q(T)^*$ is injective.*

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2. *The following assertions are equivalent:*
   - (2.a) *(LDDS) is L^q-exactly controllable in time T;*
   - (2.b) Ran $E_q(T) = \Sigma^q$;
   - (2.c) *$(q < \infty)$ $E_q(T)^*$ is bounded below: $\exists c_q > 0$*

$$(OBS) \qquad \|E_q(T)^* y\|_{[0,T],\, q'} \geq c_q \|y\|_{\Sigma^{q'}}, \qquad \forall y \in \Sigma^{q'}. \quad (7)$$

Available results

1. Delays $\tau_1, \ldots, \tau_d$ commensurable (i.e., all their pairwise ratios are rational): can reformulate (LDDS) as an equivalent difference equation with a single delay (up to state augmentation). Then L^q-approximate and exact controllability equivalent (and independent of q) and can be checked by a Kalman criterion.
2. Coron-Nguyen 2019: L^2-exact controllability in optimal time for specific systems (Hyp) (time-delay approach)
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3. Two delays in dimension 2 (Mazanti, Sigalotti, C. 2020): complete answers based on explicit (OBS).
   Case of 2 irrational delays approximate $\frac{\tau_1}{\tau_2}$ by sequences of rationals $(r_l)_{l\geq 0}$ and prove $(c_q^l)_{l\geq 0}$ in (OBS) uniform. lower bdd.

Upper Bound on Controllability Time -1-

Theorem (Range saturation for End-Point Map E_q)

Set $T_* := \tau_1 + \cdots + \tau_d$. *Then*

$$\text{Ran}\, E_q(T) = \text{Ran}\, E_q(T_*), \qquad \forall T \geq T_*, \quad q \in [1, +\infty). \qquad (8)$$

Hence (LDDS) approx. (resp. exactly) controllable from the origin IFF (LDDS) approx. (resp. exactly) controllable in time T_*.

Upper Bound on Controllability Time -2-

SoP: Use representation formula of $E_q(T)$ and next lemma

Lemma (Generalized Cayley-Hamilton)

$\exists \alpha_k \in \mathbb{R}$ for $k \in \{0,1\}^d$ s. t. $\forall j \in [\![1, d]\!]$ and $\ell \in \{\ell' \in \mathbb{N}^d \mid \max_{i \in [\![1,d]\!]} \ell'_i \geq 2$ or $\ell'_j = 1\}$, we have

$$e_j^T \Xi_\ell = - \sum_{k \in \{0,1\}^d \setminus \{(0,\dots,0)\}} \alpha_k e_j^T \Xi_{\ell - k}. \tag{9}$$

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Proof of Lemma based on identity

$$\left(Id_d - t_1 K e_1 e_1^T - \cdots - t_d K e_d e_d^T \right)^{-1} = \sum_{\ell=(\ell_1,\cdots,\ell_d)\in\mathbb{N}^d} t_1^{\ell_1} \cdots t_d^{\ell_d} \Xi_\ell.$$

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- Initial state = Origin.
- $y(t) = \sum_{j=1}^{d} K e_j e_j^T y(t - \tau_j) + Bu(t),\ \ t \geq \inf supp(u)$.
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- Output $z(t) = (y_j(t - \tau_j))_{1 \leq j \leq d},\ t \geq 0$.

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- Output $z(t) = (y_j(t - \tau_j))_{1 \leq j \leq d}, \; t \geq 0$.

Write z using convolution operator with kernel in $\mathcal{R}(\mathbb{R}_+) =$ space of Radon measures supported in $\mathbb{R}_+$, i.e. find $A \in \mathcal{R}(\mathbb{R}_+)$ s.t.

$$z(t) = \int_{-\infty}^{+\infty} A(t - \tau) u(\tau) d\tau = (A * u)(t), \qquad t \geq 0. \qquad (10)$$

Realization theory (after Yamamoto) -2-

$$(IOS) \quad z(t) = (A * u)(t), \qquad t \geq 0 \quad \text{and} \quad A = Q^{-1} * P,$$

where

$$Q = \mathrm{diag}(\delta_{-\tau_1}, \ldots, \delta_{-\tau_n}) - K\delta_0, \quad P := B\delta_0,$$

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Realization theory (after Yamamoto) -3-

$\pi : \phi \to \phi|_{\mathbb{R}_+}$ truncation operator on $L^q(\mathbb{R}_+, \mathbb{R}^d)$.

State space of (IOS) in terms of distribution Q

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State space of (IOS) in space of distributions:

$$(\mathcal{D})^Q := \left\{ \pi\phi \mid \ \phi \in \left(\mathcal{D}'(\mathbb{R}_+)\right)^d \text{ and } \pi(Q * \pi\phi) = 0 \right\}. \tag{13}$$

Classical Definitions of Controllability

Set $X \in \{L^q, \mathcal{R}_c, \mathcal{D}_c\}$ ($\mathcal{R}_c$ for Radon with and $\mathcal{D}$ for distrib; "c" compact support).

(IOS) is said to be:

1) X -approximately controllable (from the origin) if $\forall \phi \in \prod_{i=1}^{d} X((-\tau_i, 0), \mathbb{R})$, $\exists n \in \mathbb{N}$, $T_n > 0$ and $u_n \in X([0, T_n], \mathbb{R}^m)$ s. t.

$$z_(T_n + \cdot) \underset{n \to +\infty}{\longrightarrow} \phi(\cdot), \quad \text{in } X\text{-sense.}$$

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$$z_(T_n + \cdot) \underset{n\to+\infty}{\longrightarrow} \phi(\cdot), \quad \text{in } X\text{-sense.}$$

2) X-exactly controllable (from the origin) if $\forall \phi \in \prod_{i=1}^{d} X((-\tau_i, 0), \mathbb{R})$, $\exists T > 0$ and $u \in X([0, T], \mathbb{R})$ s. t. for every $j \in [\![1, d]\!]$

$$z_j(T + \theta) = \phi_j(\theta) \quad \text{for } \theta \in [-\tau_j, 0] \text{ in } X\text{-sense.}$$

Classical Definitions of Controllability

Set $X \in \{L^q, \mathcal{R}_c, \mathcal{D}_c\}$ ($\mathcal{R}_c$ for Radon with and $\mathcal{D}$ for distrib; "c" compact support).

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Remark: Similar definitions with uniform controllability time.

Controllability in terms of Realization Theory

$X \in \{L^q, \mathcal{R}_c, \mathcal{D}_c\}$. (What follows is a tautology!)

Realization (IOS) $z = A \star u$ is

1) X-approximately controllable if, for every $\forall \pi\phi \in (X)^Q$, $\exists$ a sequence of inputs $(u_n)_{n\in\mathbb{N}}$ (in "X") s. t.:

$$\pi(A * u_n) \underset{n\to+\infty}{\longrightarrow} \pi\phi \quad \text{in} \quad X\left(\mathbb{R}_+, \mathbb{R}^d\right);$$

2) X-exactly controllable if, $\forall \pi\phi \in (X)^Q$ $\exists$ u (in "X") s. t. the output z satisfies

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Similar definitions with uniform controllabilty time.

Approximate controllability -1-

Consider $\widehat{Q}\colon \mathbb{C} \to \mathcal{M}_{n,n}(\mathbb{C})$ defined by

$$\widehat{Q}(p) = \operatorname{diag}(e^{p\tau_1}, \dots, e^{p\tau_n}) - K, \qquad p \in \mathbb{C}. \tag{14}$$

Theorem (Fueyo, Mazanti, Sigalotti, C., 2023)

$q \in [1, +\infty)$. (Hyp) L^q-approx. contr. in time $T_* := \tau_1 + \cdots + \tau_n$ $\iff$ $\operatorname{rank}[K, B] = n$ and one of the following equivalent assertions holds true:

1. $\operatorname{rank}\left[\widehat{Q}(p), B\right] = n \quad \forall p \in \mathbb{C}$;
2. $\forall p \in \mathbb{C}$,

$$\inf\left\{\left\|g^T H(p)\right\| + \left\|g^T B\right\| \mid g \in \mathbb{C}^n,\ \|g^T\| = 1\right\} > 0;$$

3. $\forall p \in \mathbb{C}$, $\det\left(\widehat{Q}(p)\widehat{Q}(p)^* + BB^*\right) > 0$.

Approximate controllability -2-

SoP: Yamamoto's results + upper bound on controllability time

Proposition (Salamon, Manitius, Yamamoto 1989)

The following are equivalent:

a) L^q-*approximate controllability,* $q \in [1, \infty)$*;*

b) *Radon approximate controllability;*

c) *Distributional approximate controllability;*

d) $\exists$ *two sequences of distributions* $(S_n)_{n\in\mathbb{N}}$ *and* $(R_n)_{n\in\mathbb{N}}$ *compactly supported in* $\mathbb{R}_-$ *s.t.:*

$$Q * R_n + P * S_n \underset{n\to+\infty}{\longrightarrow} \delta_0 I_d, \quad \textit{in distributional sense;} \tag{15}$$

e) *(Hautus-Yamamoto criteria) The two conditions hold true:*

1) $\mathrm{rank}\left[\widehat{Q}(p), B\right] = d$ *for every* $p \in \mathbb{C}$*,*

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Fundamental ingredients: (a) algebraic approach with distributions; (b) Approximate Bézout identity (15).

Exact controllability

Theorem (Yamamoto 2011)

Distributional exact controllability $\Leftrightarrow \exists$ *two distributions* R *and* S *compactly supported in* $\mathbb{R}_-$ *s.t. the following Bézout Identity holds*

$$Q * R + P * S = \delta_0 I_d. \qquad \text{(Bézout-Dist.)}$$

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Remark: (Bézout-Radon) same statement as above with R and S Radon measures in $\mathcal{R}_c(\mathbb{R}_-)$.

Bézout and exact controllability

Realization $z = A * u$ $\mathcal{D}$-exact controllable $\Rightarrow$
since $\pi Q^{-1} \in (\mathcal{D})^Q$, $\exists$ distrib. S s.t.

$$\pi(A * S) = \pi(Q^{-1} * P * S) = \pi Q^{-1}.$$

Then $R := Q^{-1} * P * S - Q^{-1}$ has compact support in $\mathbb{R}_-$, hence
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Compute $Q^{-1} *$ Bézout-Dist. $* Q * \phi$.
It yields

$$\begin{aligned}
\phi &= Q^{-1} * (\delta_0 Id) * Q * \phi \\
&= Q^{-1} * (P * S + Q * R) * Q * \phi \\
&= Q^{-1} * P * S * Q * \phi + R * Q * \phi \\
&= A * S * Q * \phi + R * Q * \phi.
\end{aligned}$$

Bézout and exact controllability

Since (!!) $\pi(R * Q * \phi) = \pi(R * \pi(Q * (\pi\phi))) = 0$, we get:

$$\pi\phi = \pi\left(A * S * Q * \phi\right) = \pi\left(A * \omega\right).$$

$$\omega := S * Q * \phi \tag{16}$$

Here ω is a control steering target $\pi\phi$ along $(\Sigma)^Q$.

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Remarks:

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- Assume Q, P with coeffs. in $\mathcal{R}_c(\mathbb{R}_-)$ as for LDDS.
  - (Bézout-Radon) $\iff$ Radon exact controllability;
  - (Bézout-Radon) $\Rightarrow L^q$ exact controllability, $q \in [1, \infty]$.

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Consider $\widehat{Q}\colon \mathbb{C} \to \mathcal{M}_{n,n}(\mathbb{C})$ defined by

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(Hyp) L^1*-exactly controllable in time* $\tau_1 + \cdots + \tau_n \iff$ *one of the following assertions holds true:*

1. $\mathrm{rank}\,[M, B] = n \quad \forall M \in \overline{\widehat{Q}(\mathbb{C})}$;
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$$\inf\left\{\left\|g^T \widehat{Q}(p)\right\| + \left\|g^T B\right\| \mid g \in \mathbb{C}^n,\ \|g^T\| = 1\right\} \geq \alpha;$$

3. $\exists \alpha > 0$ *s.t.* $\forall p \in \mathbb{C}$, $\det\left(\widehat{Q}(p)\widehat{Q}(p)^* + BB^*\right) \geq \alpha$.

SoP: Resolution of Corona Problem in space of Radon measures compactly supported.

Conjecture: same holds for L^q exact controllability, $q \in (1, \infty)$.

Proof of Exact controllability result -1-

Proposition (Fueyo, Mazanti, Sigalotti, C. 2023)

Realization $z = A * u$ *is* L^1 *exact. contr.* $\iff$ *(Bézout-Radon), i.e.,* $\exists\, R, S$ *with entries in* $\mathcal{R}_c(\mathbb{R}_-)$ *s.t.*

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Sketch of proof: ($\implies$) If L^1 exactly controllable, $\exists$ sequence (S_n) in $L^1(\mathbb{R}, \mathbb{R}^{m\times d})$, compactly supported in $[-T_*, 0]$ s.t.

$$\pi(Q^{-1} * P * S_n) \underset{n\to+\infty}{\longrightarrow} \pi Q^{-1}, \quad \text{in distribution sense.} \tag{19}$$

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From Open Mapping Theorem, $\|S_n\|_1 \leq C$ for $C > 0$ independent of n. Weak compactness in $\mathcal{R}_c(\mathbb{R}_-)(\mathbb{R}_-) \Rightarrow \exists\, S \in \mathcal{R}_c(\mathbb{R}_-)(\mathbb{R}_-)$ s.t. $S_n \underset{n \to +\infty}{\rightharpoonup} S$.

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Conclude with $\pi(Q^{-1} * P * S) = \pi Q^{-1}$.

Proof of Exact controllability result -2-

Proposition (Fueyo, C., 2023)

(Bézout-Radon) for Realization $z = A * u \iff$ $\operatorname{rank}[M, B] = n \quad \forall M \in \overline{\widehat{Q}(\mathbb{C})}$.

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$\operatorname{rank}[M, B] = n \quad \forall M \in \overline{\widehat{Q}(\mathbb{C})}$.

Sketch of proof: $(\Rightarrow)$ easy.

$(\Leftarrow)$ This is a Corona Problem in $\mathcal{R}(\mathbb{R}_-)$:

Laplace transform of (Bézout-Radon) yields

$$\widehat{Q}(p)\widehat{R}(p) + B\widehat{S}(p) = I_d, \quad p \in \mathbb{C}. \tag{20}$$

Corona Problem -1-

For $T > 0$,

$$\Omega_-^T := \{h \in \mathcal{R}(\mathbb{R}_-) \mid h = \sum_{j=0}^{N} h_j \delta_{-\lambda_j},\ \lambda_j \in [0, T],\ h_j \in \mathbb{R},\ N \in \mathbb{N}\}.$$

Proposition (Corona problem in $\mathcal{R}(\mathbb{R}_-)$)

K positive integer and $T > 0$. Consider $f_i \in \Omega_-^T$ for $i = 1, \ldots, K$. Assume $\exists \alpha > 0$ s.t.

$$\sum_{i=1}^{K} \left| \hat{f}_i(s) \right| \geq \alpha, \quad \forall s \in \mathbb{C}, \tag{21}$$

then $\exists g_i \in \mathcal{R}_c(\mathbb{R}_-)$ for $i = 1, \ldots, K$ satisfying

$$\sum_{i=1}^{K} f_i * g_i = \delta_0. \tag{22}$$

Corona Problem -2-

SoP: (Classical strategy.) First notice that $(\mathcal{R}_c(\mathbb{R}_-), +, *)$ commutative algebra. Hence

$$\sum_{i=1}^{K} f_i * g_i = \delta_0 \;\Leftrightarrow\; \text{Two sided ideal } (f_1, \cdots, f_K) = \mathcal{R}_c(\mathbb{R}_-).$$

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(1) replace $\mathcal{R}_c(\mathbb{R}_-), +, *)$ by quotient algebra $\mathcal{A} = \mathcal{R}_c(\mathbb{R}_-), +, *)/(f_K)$ which can be shown to be a commutative unital Banach algebra with $[\delta_0]$ as unit.

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(2) describe the homomorphisms of $\mathcal{A}$, i.e., continuous linear mappings $\psi : \mathcal{A} \to \mathbb{C}$.

Gelfand theory says that every maximal ideal lies in the kernel of a homomorphism of $\mathcal{A}$. Thanks (2), can contradict (21).