Stabilization of networks of hyperbolic systems with a chain structure

Jean Auriol
Joint work with D. Bresch-Pietri, F. Bribiesca-Argomedo, S. Niculescu, J. Redaud

L2S, CNRS, Université Paris-Saclay, UMR 8506

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Motivation

Why hyperbolic systems?

- **Conservation/balance** of scalar quantities when taking into account:
  - Evolution (e.g., **transport**) of conserved quantities in space and time
  - Finite **speed of propagation** (vs. heat equation)

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  - slow propagation speeds (e.g. traffic)
  - spatially dependent characteristics (e.g. composite materials)
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Mathematically, this may look something like:

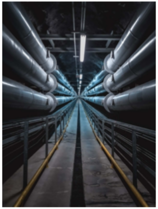
$$\partial_t \rho(t,x) = \nabla f(t,x) + S(t,x), \quad \forall (t,x) \in [0,T] \times \Omega,$$

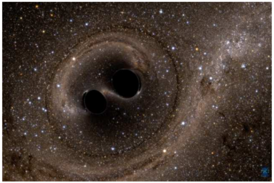
where ρ is the **quantity conserved**, f is a **flux density** and S is a **source term**.

Motivation

Many physical laws are **conservation/balance laws**, e.g. mass, charge, energy, momentum [Bastin, Coron; 2016]





Networks of hyperbolic systems

Why coupled and interconnected hyperbolic systems?

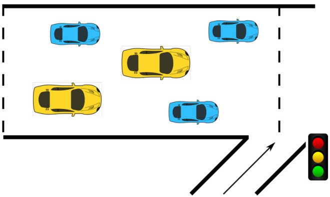
- Conservation/balance laws rarely appear isolated
  - Navier-Stokes $\rightarrow$ mass + energy + momentum
  - Propagation phenomena rarely occur in a single direction
- Systems modeled by hyperbolic PDEs do not exist in isolation, e.g.:
  - Electric transmission networks $\rightarrow$ interconnection of individual transmission lines
  - Mechanical vibrations in drilling devices $\rightarrow$ interconnection of different pipes
- Possible coupling with ODEs
  - actuator dynamics (e.g. pump, converter)
  - load dynamics (e.g. valve, motor)
  - sensor dynamics (e.g. flow-rate sensor, tachometer)

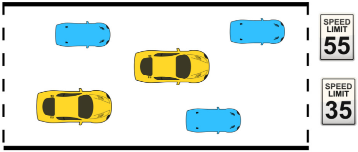
Example: Traffic congestion control [Hu, Krstic]

- Congested traffic $\rightarrow$ **Stop-and-go oscillations**

- **Macroscopic models:** hyperbolic PDEs that govern the evolution of density and velocity

- Different traffic control strategies
  1. **Ramp metering:** controls the traffic lights on a ramp
  2. **Varying speed limits (VSL):** driving velocities are time-varying, dependent on real-time traffic





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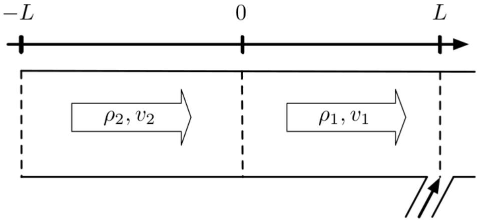
- **Simultaneous stabilization** of the trafic on two connected roads



Content of the presentation

What you will see (maybe learn!) in this presentation

- Backstepping stabilization of **elementary** systems of balance laws
  - Backstepping approach: integral change of coordinates
  - Time delay representation (Integral Difference Equation)
  - Scalar and non-scalar systems

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  - IDE with delayed actuation
  - Predictor design for IDEs
  - Explicit realization of the prediction using a TDS approach

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  - 1st approach: Successive backstepping transformations
  - 2nd approach: Recursive dynamics interconnection framework

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- Stabilization of interconnections with a **chain structure** actuated at the extremity
  - 1st approach: Successive backstepping transformations
  - 2nd approach: Recursive dynamics interconnection framework
- Stabilization at the **junction** of two scalar interconnected systems
  - IDE with delayed and distributed actuation
  - Controller obtained using Fredholm integral equations

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  - Only chains: no cycle, no tree
  - One and only one node of the chain is actuated
  - No generic methods for the stabilization of underactuated PDE systems

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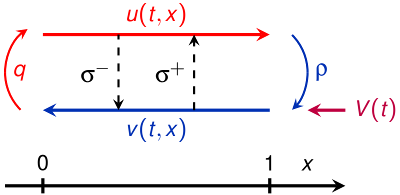
  - Only chains: no cycle, no tree
  - One and only one node of the chain is actuated
  - No generic methods for the stabilization of underactuated PDE systems
- No ugly computations (ok, maybe I'm lying for this one)

System under consideration

System of **scalar balance laws** $\rightarrow$ simple test case to present generic concepts

$$u_t(t,x) + \lambda(x)u_x(t,x) = \sigma^{++}(x)u(t,x) + \sigma^{+}(x)v(t,x),$$
$$v_t(t,x) - \mu(x)v_x(t,x) = \sigma^{-}(x)u(t,x) + \sigma^{--}(x)v(t,x),$$
$$u(t,0) = qv(t,0), \quad v(t,1) = \rho u(t,1) + V(t).$$



- Diagonal terms can be removed with exp. change of coordinates
- Distributed states and boundary control
- Initial conditions in H^1 with appropriate compatibility conditions $\rightarrow$ **well-posedness**
- Stabilization in the sense of the L^2-norm

System under consideration: well-posedness and stabilization objective

$$
\begin{aligned}
&u_t(t,x)+\lambda(x)u_x(t,x)=\sigma^+(x)v(t,x),\\
&v_t(t,x)-\mu(x)v_x(t,x)=\sigma^-(x)u(t,x),\\
&u(t,0)=qv(t,0), \quad v(t,1)=\rho u(t,1)+V(t).
\end{aligned}
$$

Well-posedness in open-loop

For every initial condition $(u_0,v_0)\in H^1([0,1],\mathbb{R}^2)$ that verifies the compatibility conditions

$$u_0(0)=Qv_0(0), \quad v_0(1)=Ru_0(1)$$

there exists one and one only

$$(u,v)\in \mathcal{C}^1([0,\infty),L^2([0,1],\mathbb{R}^2))\cap \mathcal{C}^0([0,\infty),H^1([0,1],\mathbb{R}^2)),$$

which is a **solution to the open-loop Cauchy problem** (i.e., $V\equiv 0$).
Moreover, there exists $\kappa_0>0$ such that for every $(u_0,v_0)\in H^1([0,1],\mathbb{R}^2)$ satisfying the compatibility conditions, the unique solution verifies

$$||(u(t,\cdot),v(t,\cdot))||_{L^2}\leq \kappa_0 e^{\kappa_0 t}||(u_0,v_0)||_{L^2}, \quad \forall t\in[0,\infty).$$

In closed-loop (continuous control-input) $\rightarrow$ no problem (invertibility of the transformations)

System under consideration: well-posedness and stabilization objective

$$u_t(t,x)+\lambda(x)u_x(t,x)=\sigma^+(x)v(t,x),$$
$$v_t(t,x)-\mu(x)v_x(t,x)=\sigma^-(x)u(t,x),$$
$$u(t,0)=qv(t,0), \quad v(t,1)=\rho u(t,1)+V(t).$$

Stabilization objective

Design a continuous control input that **exponentially stabilizes** the system in the sense of the L^2-norm, i.e. there exist κ_0 and $\nu>0$ such that for any initial condition $(u_0,v_0)\in L^2([0,1],\mathbb{R}^2)$, we have

$$||(u(t,\cdot),v(t,\cdot))||_{L^2}\le\kappa_0 e^{-\nu t}||(u_0,v_0)||_{L^2},\ 0\le t$$

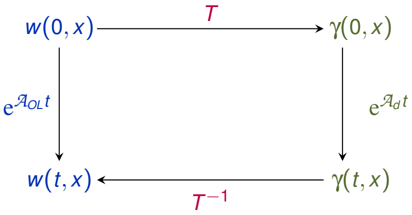
Backstepping methodology

- Map the original system to a *target system* for which the stability analysis is easier.
- Variable change: integral transformation, classically Volterra transform of the *second kind*

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi) + K^{uv}(x,\xi)v(t,\xi)d\xi,$$

$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi) + K^{vv}(x,\xi)v(t,\xi)d\xi,$$

Condensed form: $\gamma(t,x) = w(t,x) - \int_0^x K(x,y)w(t,y)dy.$

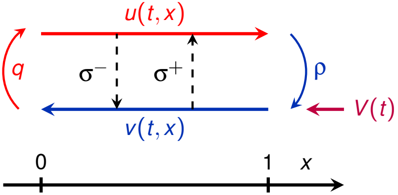


Limitations

- Choice of an adequate target system.
- Proof of existence and invertibility of an adequate backstepping transform.

Objective: Move the in-domain coupling terms at the actuated boundary.

$$u_t(t,x) + \lambda u_x(t,x) = \sigma^+ v(t,x),$$
$$v_t(t,x) - \mu v_x(t,x) = \sigma^- u(t,x).$$



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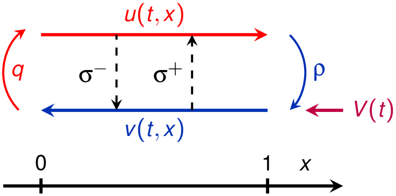

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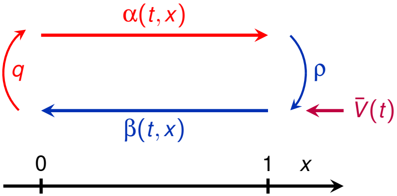

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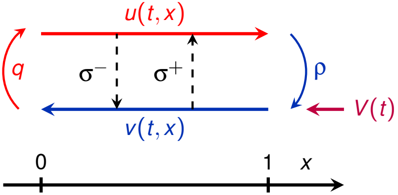

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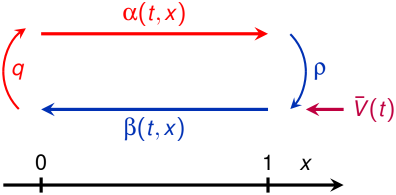

$$\alpha(t,0)=q\beta(t,0)$$
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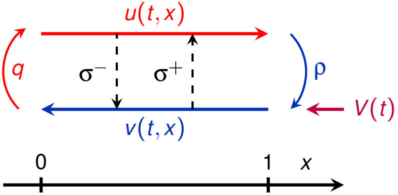

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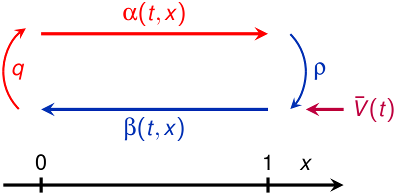

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Natural control law

$$V(t) = -\rho\alpha(t,1) + \int_0^1 \left(N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)\right)d\xi.$$

Finite-time stabilization $\to$ lack of robustness

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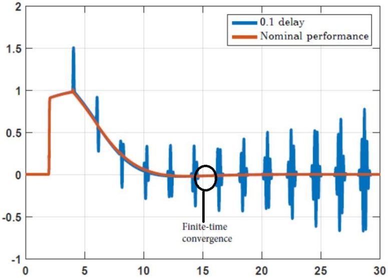

Lack of robustness

The control law is not strictly proper $\to$ no/poor robustness margins.

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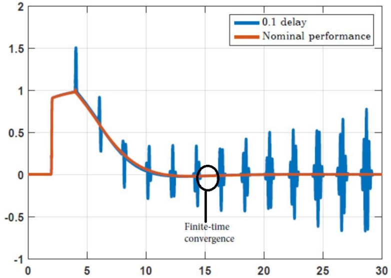

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Solutions for a robust controller

1. Cancel a part of the reflection: $V(t) = -\tilde{\rho}\alpha(t,1) + \int_0^1 \left(N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)\right) d\xi.$
2. Low-pass filter the control law.

Time-delay representation

$$\begin{aligned}\alpha_t(t,x)+\lambda\alpha_x(t,x) &= 0\\ \beta_t(t,x)-\mu\beta_x(t,x) &= 0\end{aligned}$$

$$\alpha(t,0)=q\beta(t,0)$$

$$\beta(t,1)=\rho\alpha(t,1)-\int_0^1\left(N^{\alpha}(\xi)\alpha(t,\xi)+N^{\beta}(\xi)\beta(t,\xi)\right)d\xi+V(t)$$

Time-delay representation

$$\alpha_t(t,x) + \lambda\alpha_x(t,x) = 0 \rightarrow \text{Transport equation}$$
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Integral Difference Equation (IDE) satisfied by $\beta(t,1)$

$$\beta(t,1) = \rho q\beta(t-\tau,1) - \int_0^\tau N(\xi)\beta(t-\xi,1)d\xi + V(t), \quad t > \frac{1}{\lambda} + \frac{1}{\mu} = \tau$$

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Necessary condition for delay-robustness

The product ρq verifies $|\rho q| < 1 \rightarrow$ Stability of the **principal part**.

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Stability analysis

The PDE system and the time-delay system have equivalent stability properties.

$V(t) = \int_0^\tau N(\xi)\beta(t-\xi)d\xi$.

Non-scalar systems of balance laws

$$
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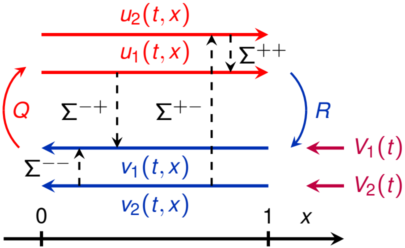
\end{aligned}
$$

where $\Lambda^+=\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, $\Lambda^-=\mathrm{diag}(\mu_1,\ldots,\mu_p)$ with

$$
-\mu_p<\ldots<-\mu_1<0,\quad 0<\lambda_1<\ldots<\lambda_n
$$



One boundary of the system is completely actuated.

Backstepping transformation and time-delay formulation

- Target system

$$\alpha_t(t,x) + \Lambda^+ \alpha_x(t,x) = G_1(x)\beta(t,0),$$
$$\beta_t(t,x) - \Lambda^- \beta_x(t,x) = G_2(x)\beta(t,1),$$
$$\alpha(t,0) = Q\beta(t,0) \quad \beta(t,1) = R\alpha(t,1) + \int_0^1 L_1(\xi)\alpha(t,\xi) + L_2(\xi)\beta(t,\xi)d\xi + V(t)$$

- Stabilizing control law: $V(t) = -R\alpha(t,1) - \int_0^1 L_1(\xi)\alpha(t,\xi) + L_2(\xi)\beta(t,\xi)d\xi$.

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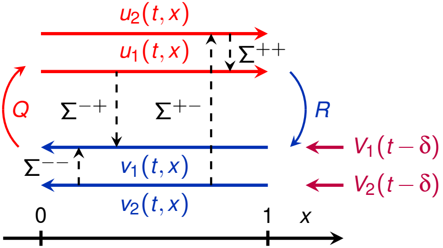
Time-delay formulation

Integral difference equation (IDE) for $z(t) = \beta(t,1)$

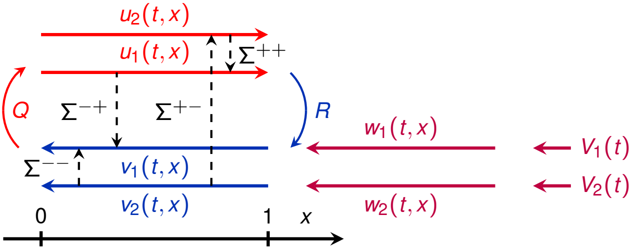
$$z(t) = \sum_{k=1}^{M} A_k z(t-\tau_k) + \int_0^{\tau_M} N(\nu)z(t-\nu)d\nu + V(t).$$

$$u_t(t,x)+\Lambda^+ u_x(t,x)=\Sigma^{++}(x)u(t,x)+\Sigma^{+-}(x)v(t,x),$$
$$v_t(t,x)-\Lambda^- v_x(t,x)=\Sigma^{--}(x)v(t,x)+\Sigma^{-}(-x)u(t,x),$$
$$u(t,0)=Qv(t,0) \quad v(t,1)=Ru(t,1)+V(t-\delta) \quad \text{with } \delta>0.$$



Non-scalar systems of balance laws with an input delay

$$
\begin{aligned}
&u_t(t,x)+\Lambda^{+}u_x(t,x)=\Sigma^{++}(x)u(t,x)+\Sigma^{+-}(x)v(t,x),\\
&v_t(t,x)-\Lambda^{-}v_x(t,x)=\Sigma^{--}(x)v(t,x)+\Sigma^{-}(-x)u(t,x),\\
&w_t(t,x)-\frac{1}{\delta}w_x(t,x)=0,\\
&u(t,0)=Qv(t,0)\quad v(t,1)=Ru(t,1)+w(t,0),\ w(t,1)=V(t).
\end{aligned}
$$



The boundaries of the system are not completely actuated $\rightarrow$ **under-actuated system**.

Predictors for IDEs

Backstepping + method of characteristics $\rightarrow$ IDE with equivalent stability properties.

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu) X(t-\nu) d\nu + V(t-\delta), \quad t \geq 0$$

Exponential stabilization using a predictor

The control law

$$V_{\text{pred}}(t) = -\int_0^{\tau_M} N(\nu) P(t, t-\nu) d\nu,$$

in which the **prediction** $P(t,s)$ is implicitly defined as

$$P(t,s) = \sum_{k=1}^{M} A_k P(t, s-\tau_k) + \int_0^{\tau_M} N(\nu) P(t, s-\nu) d\nu + V(s), \quad t-\delta \leq s \leq t$$

with initial condition $P(t,s) = X(t+\delta)$ if $s < t-\delta$, exponentially stabilizes the system.

- Integral relation of Volterra type $\rightarrow$ Prediction well-defined.
- Possible to explicitly compute this predictor?

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu) X(t-\nu) d\nu + V(t-\delta), \quad t \geq 0$$

The τ_k are increasing and $\delta = \tau_M$ (not restrictive).
We consider the following candidate control law

$$V(t) = \int_0^{\delta} [f(\nu) X(t-\nu) + g(\nu) V(t-\nu)] \, d\nu,$$

with f and g **piecewise continuous matrix-valued functions**

Objective

Find f and g such that the control law V stabilizes the system.

Explicit realization of the predictor

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu) X(t-\nu) d\nu + V(t-\delta), \quad t \geq 0$$

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$$X(t) - \int_0^{\delta} g(\nu)X(t-\nu)d\nu = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\delta} (N(\nu)X(t-\nu) - g(\nu)V(t-\nu-\delta))d\nu$$

$$+ V(t-\delta) - \sum_{k=1}^{M} \int_0^{\delta} g(\nu) A_k X(t-\nu-\tau_k) d\nu - \int_0^{\delta}\int_0^{\delta} g(\nu)N(\eta)X(t-\nu-\eta) d\eta d\nu,$$

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$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu) X(t-\nu) d\nu + V(t-\delta), \quad t \geq 0$$

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$$+ V(t-\delta) - \sum_{k=1}^{M} \int_0^{\delta} g(\nu)A_k X(t-\nu-\tau_k)d\nu - \int_0^{\delta}\int_0^{s} g(s+\eta)N(\eta)d\eta X(t-s)ds$$
$$- \int_{\delta}^{2\delta}\int_{s-\delta}^{\delta} g(s-\eta)N(\eta)d\eta X(t-s)ds,$$

Explicit realization of the predictor

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu)X(t-\nu)d\nu + V(t-\delta), \quad t \geq 0$$

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Find f and g such that the control law V stabilizes the system.

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$$- \sum_{k=1}^{M} \int_0^{\delta} g(\nu) A_k X(t-\nu-\tau_k) d\nu - \int_0^{\delta} \int_s^{0} g(s+\eta) N(\eta) d\eta X(t-s) ds$$

$$- \int_{\delta}^{2\delta} \int_{s-\delta}^{\delta} g(s-\eta) N(\eta) d\eta X(t-s) ds + \int_{\delta}^{2\delta} f(\nu-\delta) X(t-\nu) d\nu,$$

Explicit realization of the predictor

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\tau_M} N(\nu) X(t-\nu) d\nu + V(t-\delta), \quad t \geq 0$$

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with f and g **piecewise continuous matrix-valued functions**

Objective

Find f and g such that the control law V stabilizes the system.

$$X(t) - \int_0^{\delta} g(\nu) X(t-\nu) d\nu = \sum_{k=1}^{M} A_k X(t-\tau_k) + \int_0^{\delta} N(\nu) X(t-\nu) d\nu$$
$$- \sum_{k=1}^{M} \int_0^{\delta} g(\nu) A_k X(t-\nu-\tau_k) d\nu - \int_0^{\delta} [\int_0^{s} g(s-\eta) N(\eta) d\eta] X(t-s) ds,$$
$$- \int_{\delta}^{2\delta} [\int_{s-\delta}^{\delta} g(s-\eta) N(\eta) d\eta] X(t-s) ds + \int_{\delta}^{2\delta} f(\nu-\delta) X(t+\nu) d\nu,$$

Volterra equations and explicit realization of the predictor

We have $X(t)=\sum_{k=1}^{M} A_k X(t-\tau_k)$ if

$$0=g(\nu)+N(\nu)-\int_0^{\nu} g(\nu-\eta)N(\eta)d\eta-\sum_{k=1}^{M}\mathbb{1}_{[\tau_k,\delta]}(\nu)g(\nu-\tau_k)A_k, \quad (1)$$

$$0=f(\nu-\delta)-\int_{\nu-\delta}^{\delta} g(\nu-\eta)N(\eta)d\eta-\sum_{k=1}^{M}\mathbb{1}_{[\delta,\tau_k+\delta]}(\nu)g(\nu-\tau_k)A_k, \quad (2)$$

Volterra equations and explicit realization of the predictor

We have $X(t) = \sum_{k=1}^{M} A_k X(t - \tau_k)$ if

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$$0 = f(\nu - \delta) - \int_{\nu-\delta}^{\delta} g(\nu - \eta) N(\eta) d\eta - \sum_{k=1}^{M} \mathbb{1}_{[\delta, \tau_k + \delta]}(\nu) g(\nu - \tau_k) A_k, \quad (2)$$

Existence of the functions f and g

There exist two unique piecewise continuous functions (f, g) that are solutions of (1)-(2).

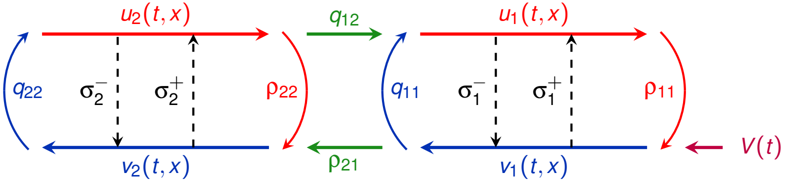
Closed-loop exponential stability

The control law $V(t) = \int_{-\delta}^{0} [f(-\nu) X(t+\nu) + g(\nu) V(t+\nu)] \, d\nu$, where f and g are solutions of (1)-(2) **exponentially stabilizes** the original system in the sense of the L^2-norm. Moreover, the control law is **strictly proper and exponentially converges to zero**.

This control law corresponds to an **explicit realization of the predictor**.

$$\partial_t u_i + \lambda_i \partial_x u_i = \sigma_i^+(x) v_i, \quad \partial_t v_i - \mu_i \partial_x v_i = \sigma_i^-(x) u_i,$$

$$u_2(t,0) = q_{22} v_2(t,0),\ v_2(t,1) = \rho_{22} u_2(t,1) + \rho_{21} u_1(t,0),$$
$$u_1(t,0) = q_{11} v_1(t,0) + q_{12} u_2(t,1),\ v_1(t,1) = \rho_{11} u_1(t,1) + V(t)$$



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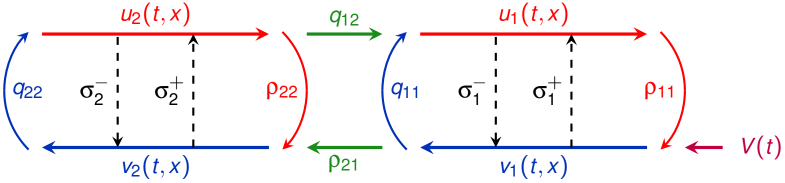

Assumption 1 : controllability

The coefficient ρ_{21} verifies $\rho_{21} \neq 0$

Necessary to act on the second subsystem.

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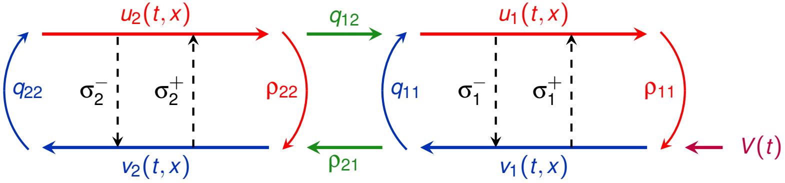

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Necessary to act on the second subsystem.

Assumption 2 : delay robustness

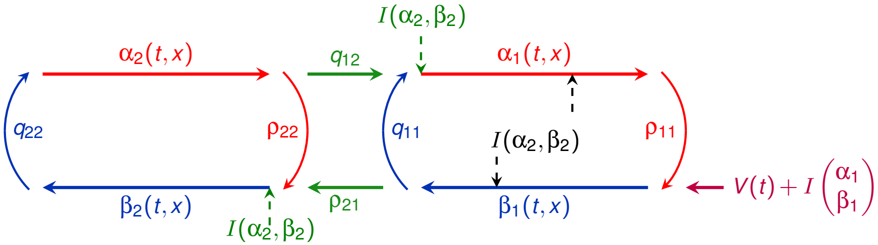
The open-loop system without in-domain couplings is exp. stable

This assumption implies $|\rho_{11} q_{11}| < 1$ and $|\rho_{22} q_{22}| < 1$.

Objective

Using successive backstepping transformations we want to move the in-domain couplings at the actuated boundary.

- Classical backstepping transformations for each subsystem



Due to couplings from syst. (2) to syst. (1), some undesired terms appear in syst. (1).

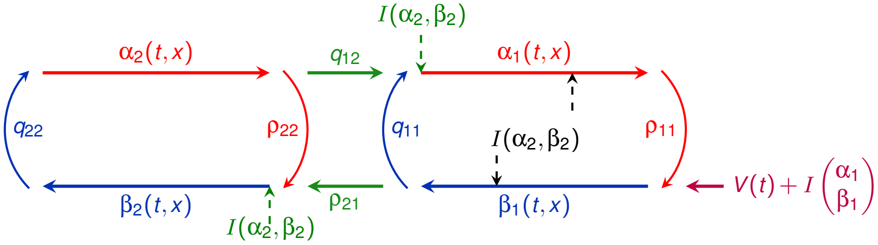
$$I(\alpha, \beta) = \int_0^1 L_1(\xi)\alpha(t,\xi) + L_2(\xi)\beta(t,\xi).$$

Objective

Using successive backstepping transformations we want to move the in-domain couplings at the actuated boundary.

- Use an affine integral transformation on the first syst.

$$\bar{\beta}_1(t,x) = \beta_1(t,x) - \int_0^x R(x,\xi)\beta_1(t,x)dx - \int_0^1 F^{\alpha}(x,\xi)\alpha_2(t,\xi)d\xi + F^{\beta}(x,\xi)\beta_2(t,\xi)d\xi,$$



Successive backstepping transformations

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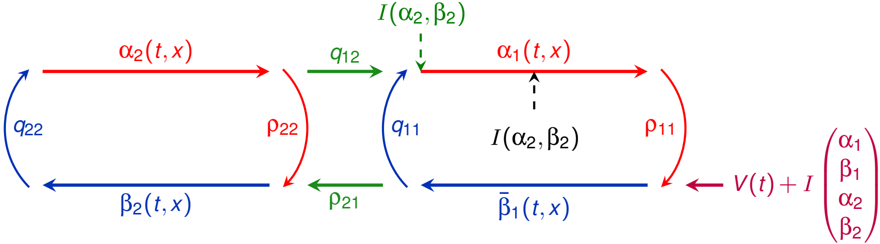

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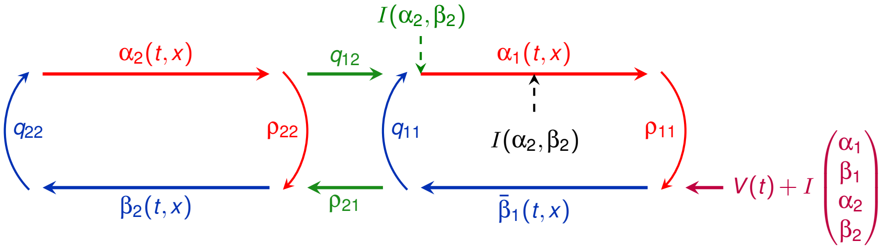

- Clear actuation path from $V(t)$ to subsystem (2).

Successive backstepping transformations

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Using successive backstepping transformations we want to move the in-domain couplings at the actuated boundary.

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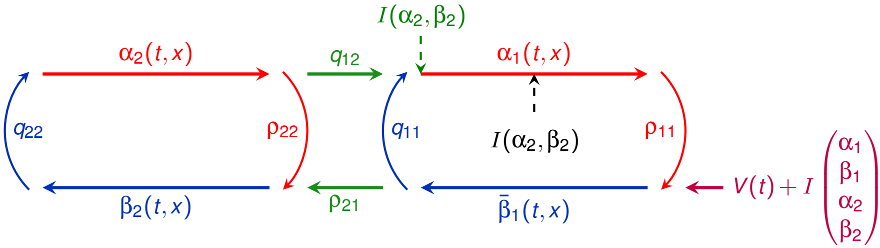
$$\bar{\beta}_1(t,x) = \beta_1(t,x) - \int_0^x R(x,\xi)\beta_1(t,x)dx - \int_0^1 F^{\alpha}(x,\xi)\alpha_2(t,\xi)d\xi + F^{\beta}(x,\xi)\beta_2(t,\xi)d\xi,$$



- Clear actuation path from $V(t)$ to subsystem (2).
- Stabilizing control law: $V(t) = -I(\alpha_1,\alpha_2,\beta_1,\beta_2)$.

Extensions and limitations of the approach

Extension to multiple subsystems

- Possible but technical: requires additional conditions on the boundary couplings.
- The transformations have to be modified when a new system is added to the chain

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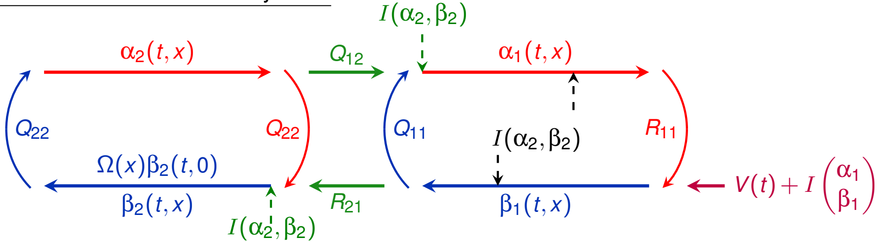
Extension to non-scalar subsystems



- System 2 is not autonomously exp. stable.
- The affine transformation does not work anymore.

New objective

Develop a new modular approach to stabilize chains of non-scalar subsystems

Non-scalar interconnected systems

$$\partial_t u_i + \Lambda_i^+ \partial_x u_i = \Sigma_i^{++}(x) u_i + \Sigma_i^{++}(x) v_i,$$
$$\partial_t v_i - \Lambda_i^- \partial_x v_i = \Sigma_i^{-+}(x) u_i + \Sigma_i^{--}(x) v_i,$$

$$u_2(t,0) = Q_{22} v_2(t,0),\ v_2(t,1) = R_{22} u_2(t,1) + R_{21} u_1(t,0),$$
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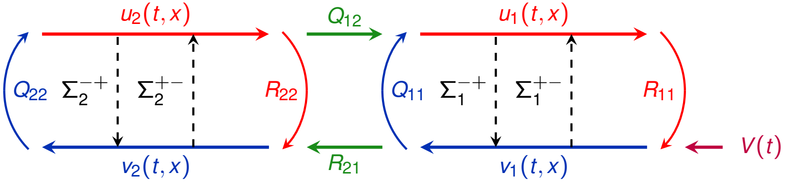


Assumption 1 : controllability

The matrix R_{21} is full row-rank (existence of a right inverse).

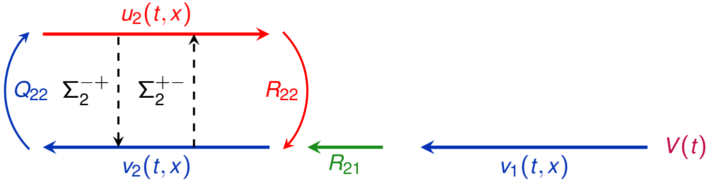
Conservative assumption but only specific results exist for underactuated systems

Assumption 2 : delay-robustness

The open-loop system without in-domain couplings is exp. stable.

A delayed-control effect

Let us focus on the second subsystem and assume $\Sigma_1^{-+} = 0$



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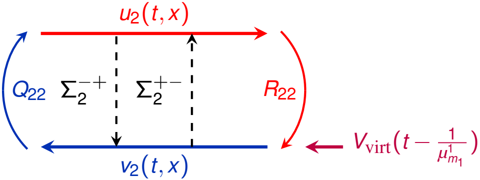

The actuation acts on the distal subsystem with a constant delay.

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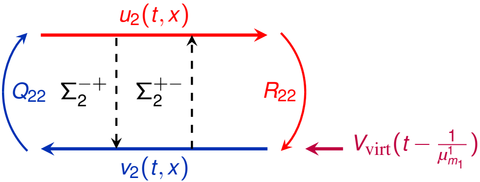


The actuation acts on the distal subsystem with a constant delay.

We already know how to stabilize such a system!

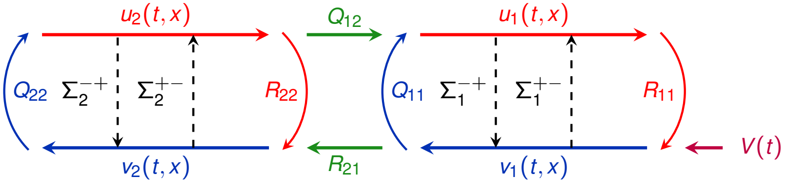
Stabilizing controller

We choose the **virtual control law** as

$$V_{\mathrm{virt}}(t) = \int_0^{\delta} f(\nu) z(t-\nu) + g(\nu) V_{\mathrm{virt}}(t-\nu) d\nu,$$

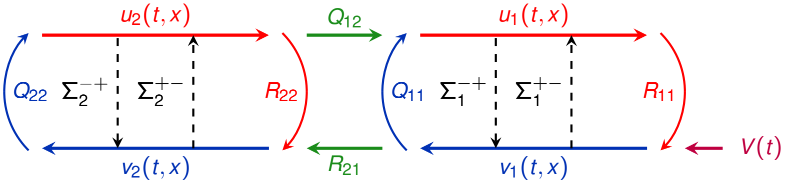
where z is defined from (u_2, v_2) using backstepping transformations and where f and g are the solutions of appropriate Volterra equations.

Tracking of the virtual control input



We now want $R_{21}v_1(t,0)$ to **track** the signal $V_{virt}(t-\frac{1}{\mu^1_{m_1}})$.

Tracking of the virtual control input

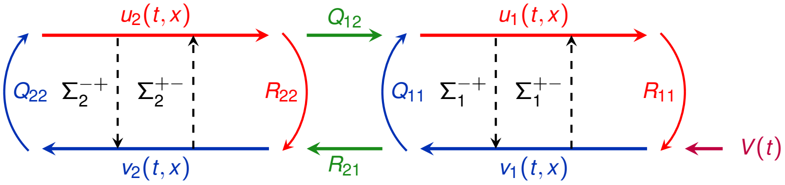


We now want $R_{21}v_1(t,0)$ to **track** the signal $V_{virt}(t-\frac{1}{\mu^1_{m_1}})$.

- Consider the backstepping transformation

$$\beta_1(t,x) = v_1(t,x) + \int_0^x K_1(x,y)u_1(t,y) + L_1(x,y)v_1(t,y)dy$$
$$+ \int_0^1 K_2(x,y)u_2(t,y) + L_2(x,y)v_2(t,y)dy$$

Classical backstepping transformation with an affine part.



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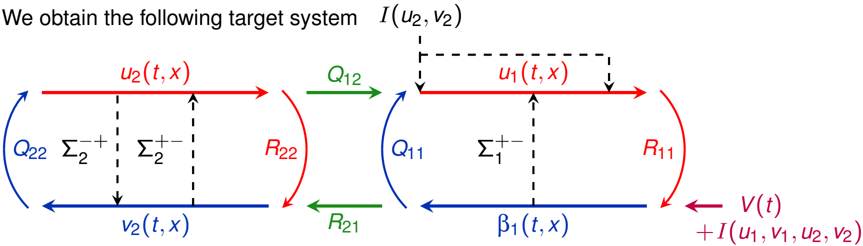
$$+ \int_0^1 K_2(x,y)u_2(t,y) + L_2(x,y)v_2(t,y)dy$$

  Classical backstepping transformation with an affine part.
- The kernels K_2 and L_2 verify $K_2(0,y) = L_2(0,y) = 0 \Rightarrow \beta_1(t,0) = v_1(t,0)$.

- We obtain the following target system



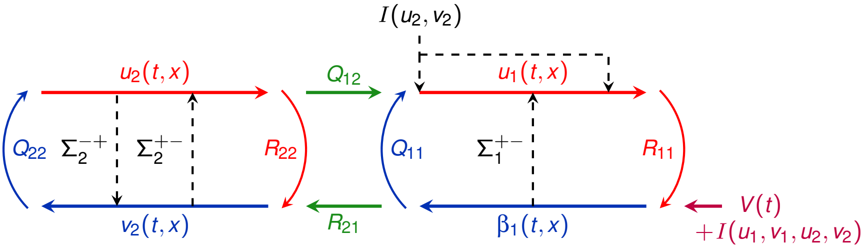
$$\partial_t \beta_1(t,x) - \Lambda_1^- \partial_x \beta_1(t,x) = \Omega(x)\beta_1(t,0) = \begin{pmatrix} 0 & & & \\ 0 & 0 & \star & \\ \vdots & & \ddots & \\ 0 & \dots & & 0 \end{pmatrix} \beta_1(t,0)$$

Tracking control law see [Hu and al.]

Let $V_i(t) = -(R_{11}u^1(t,1) + I(\cdot))_i + \zeta_i\left(t + \frac{1}{\mu_i^1}\right) - \sum_{j=i+1}^{m_1} \int_0^{\frac{1}{\mu_i^1}} \Omega_{i,j}(\mu_i^1 \nu)\zeta_j\left(t + \frac{1}{\mu_i^1} - \nu\right) d\nu$

where ζ is an arbitrary known function. Then, for any $t \geq \sum_{j=1}^{m_p} \frac{1}{\mu_j^1}$, $\beta_1(t,0) \equiv \zeta(t)$.

- We obtain the following target system



Stabilizing control law

The control law

$$V_i(t) = -(R_{11}u^1(t,1) + I(\cdot))_i + \zeta_i\left(t + \frac{1}{\mu_i^1}\right) - \sum_{j=i+1}^{m_1}\int_0^{\frac{1}{\mu_i^1}} \Omega_{i,j}(\mu_i^1\nu)\zeta_j\left(t + \frac{1}{\mu_i^1} - \nu\right)d\nu$$

with $\zeta(t) = R_{21}^T(R_{21}R_{21}^T)^{-1}V_{\mathrm{virt}}(t - \frac{1}{\mu_1^1})$, **exponentially stabilizes** the interconnected system.

Summary of the approach, extensions and limitations

- The proposed control strategy combines several ingredients
  - The backstepping approach,
  - State-predictors (virtual controller),
  - Tracking component.
- Possible to design a state-observer.
- Low-pass filter the control law to guarantee robustness.

Summary of the approach, extensions and limitations

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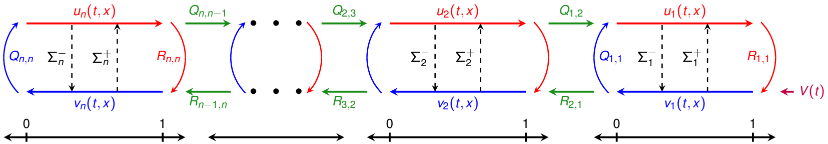
Extension to multiple subsystems

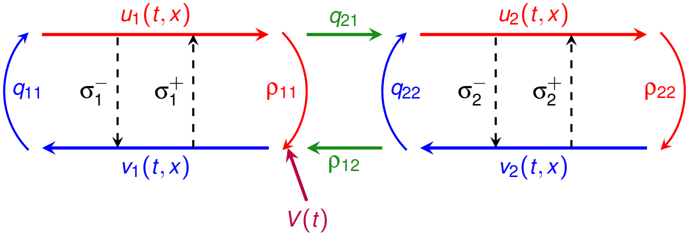


- Possible but technical: the backstepping transformation requires an additional component to avoid causality issues.
- ***Recursive dynamics interconnection framework***: the control law is designed recursively (starting with the last subsystem).

$$\partial_t u_i(t,x) + \lambda_i \partial_x u_i(t,x) = \sigma_i^+(x) v_i(t,x),$$
$$\partial_t v_i(t,x) - \mu_i \partial_x v_i(t,x) = \sigma_i^-(x) u_i(t,x),$$

with the boundary conditions

$$u_1(t,0) = q_{11} v_1(t,0), \quad v_2(t,1) = \rho_{22} u_2(t,1),$$
$$v_1(t,1) = V(t) + \rho_{11} u_1(t,1) + \rho_{12} v_2(t,0), \quad u_2(t,0) = q_{22} v_2(t,0) + q_{21} u_1(t,1).$$



Stabilization at the junction of two scalar interconnected systems

$$\partial_t u_i(t,x) + \lambda_i \partial_x u_i(t,x) = \sigma_i^+(x) v_i(t,x),$$
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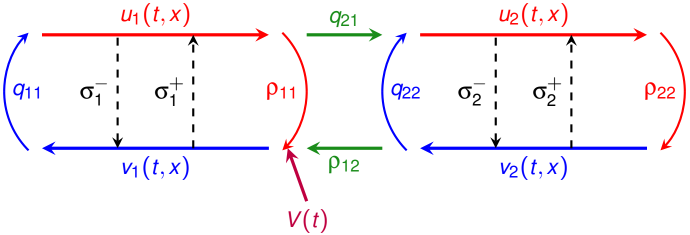

Delay robustness assumption

The open-loop system without in-domain couplings is exp. stable.

This implies $|\rho_{11} q_{11}| < 1$ and $|\rho_{22} q_{22}| < 1$.

Stabilization at the junction of two scalar interconnected systems

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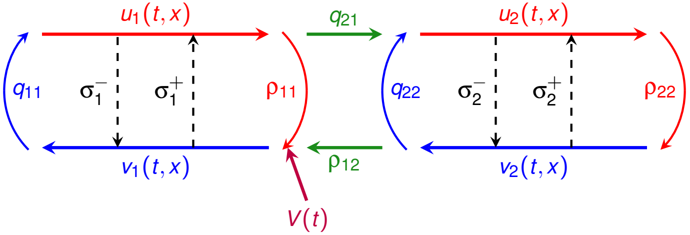

Action from the subsystem "1" on the subsystem "2".

The boundary coupling coefficient q_{21} satisfies $q_{21} \neq 0$.

If $q_{21} = 0$, it is impossible to act on subsystem "2" using the control input on subsystem "1".

$$\partial_t u_i(t,x) + \lambda_i \partial_x u_i(t,x) = \sigma_i^+(x) v_i(t,x),$$
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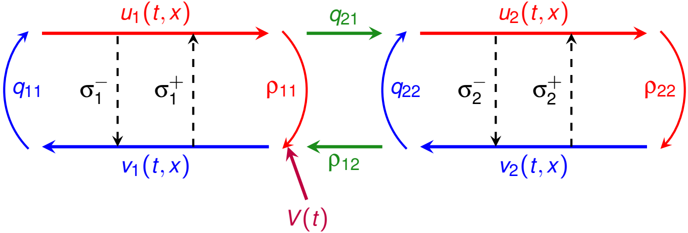

Condition on the boundary couplings

The coupling coefficients q_{11} and ρ_{22} satisfy $q_{11} \neq 0$, and $\rho_{22} \neq 0$.

Conservative assumption. If $q_{11} = 0$, the control input can act on subsystem “2” through distributed terms only.

Controllability condition

- The interconnected system may not be controllable.

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- Operator formulation: $\frac{d}{dt}w = A\big(w\big) + BU$, where $B^{\star}(\begin{pmatrix} u_1 & v_1 & u_2 & v_2 \end{pmatrix}^{\top}) = \mu_1 v_1(1)$, and

$$A : D(A) \subset L^2([0,1],\mathbb{R}^4) \to L^2([0,1],\mathbb{R}^4)$$
$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} -\lambda_1 \partial_x u_1 + \sigma_1^{+}(\cdot) v_1 \\ \mu_1 \partial_x v_1 + \sigma_1^{-}(\cdot) u_1 \\ -\lambda_2 \partial_x u_2 + \sigma_2^{-}(\cdot) v_2 \\ \mu_2 \partial_x v_2 + \sigma_2^{-}(\cdot) u_2 \end{pmatrix},$$

with $D(A) = \{(u_1, v_1, u_2, v_2) \in H^1([0,1],\mathbb{R}^4) |\ u_1(0) = q_{11} v_1(0),\ v_2(1) = \rho_{22} u_2(1),$
$v_1(1) = \rho_{11} u_1(1) + \rho_{12} u_2(1),\ u_2(0) = q_{22} v_2(0) + q_{21} u_1(1)\}$.

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Controllability condition (Coron, Fattorini)

The operators $A^\star$ and $B^\star$ verify

$$\forall s \in \mathbb{C}, \ker(s - A^\star) \cap \ker(B^\star) = \{0\}.$$

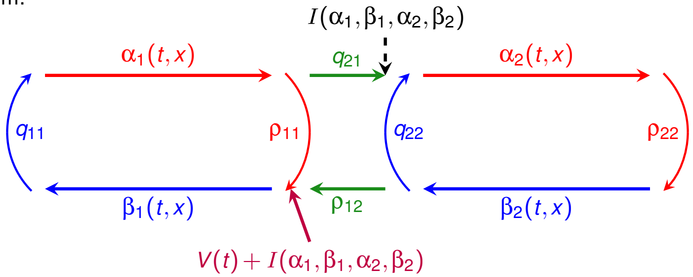
We apply classical backstepping transformations on each subsystem

$$\begin{cases} u_1(t,x) = \alpha_1(t,x) - \int_0^x L_1^{11}(x,y)\alpha_1(t,y) + L_1^{12}(x,y)\beta_1(t,y)\mathrm{d}y, \\ v_1(t,x) = \beta_1(t,x) - \int_0^x L_1^{21}(x,y)\alpha_1(t,y) + L_1^{22}(x,y)\beta_1(t,y)\mathrm{d}y, \end{cases}$$

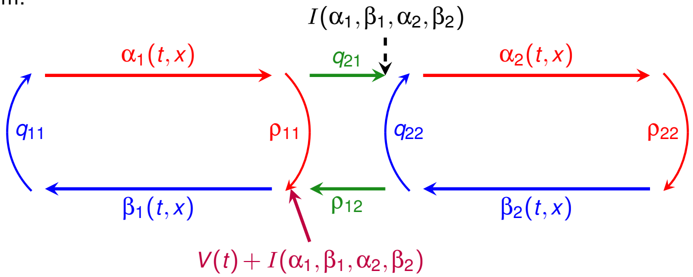
$$\begin{cases} u_2(t,x) = \alpha_2(t,x) - \int_x^1 L_2^{11}(x,y)\alpha_2(t,y) + L_2^{12}(x,y)\beta_2(t,y)\mathrm{d}y, \\ v_2(t,x) = \beta_2(t,x) - \int_x^1 L_2^{21}(x,y)\alpha_2(t,y) + L_2^{22}(x,y)\beta_2(t,y)\mathrm{d}y, \end{cases}$$

Objective: move the in-domain couplings at the boundaries.

Target system:



Target system:



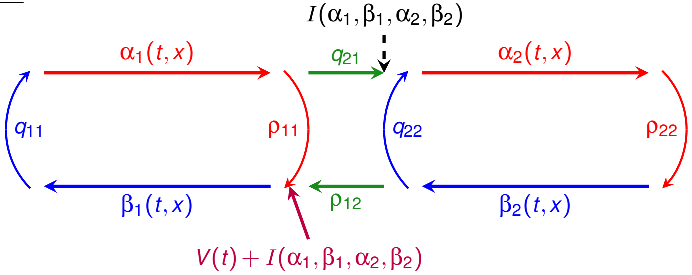
Time-delay representation

Denote $z_1(t) = \beta_1(t,1)$ and $z_2(t) = \alpha_2(t,0)$. We have for all $t \geq \max\{\tau_i = \frac{1}{\lambda_i} + \frac{1}{\mu_i}\}$

$$\begin{aligned} z_1(t) =& \rho_{11} q_{11} z_1(t-\tau_1) + \rho_{12}\rho_{22} z_2(t-\tau_2) + V(t) \\ &+ \int_0^{\tau_1} H_{11}(\nu) z_1(t-\nu) d\nu + \int_0^{\tau_2} H_{12}(\nu) z_2(t-\nu) d\nu, \\ z_2(t) =& q_{21} q_{11} z_1(t-\tau_1) + q_{22}\rho_{22} z_2(t-\tau_2) \\ &+ \int_0^{\tau_1} H_{21}(\nu) z_1(t-\nu) d\nu + \int_0^{\tau_2} H_{22}(\nu) z_2(t-\nu) d\nu. \end{aligned}$$

Backstepping transformation and time-delay representation

Target system:



Time-delay representation

Denote $z_1(t)=\beta_1(t,1)$ and $z_2(t)=\alpha_2(t,0)$. We have for all $t \geq \max\{\tau_i = \frac{1}{\lambda_i}+\frac{1}{\mu_i}\}$

$$
\begin{aligned}
z_1(t) &= \bar{V}(t),\\
z_2(t) &= a\bar{V}(t-\tau_1)+bz_2(t-\tau_2)+\int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu+\int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu,
\end{aligned}
$$

with $a \neq 0$ and $|b|<1$.

The exp. stability of z_1 and z_2 will imply the exp. stability of the original system.

An IDE with distributed actuation

$$z_2(t) = a\bar{V}(t-\tau_1) + bz_2(t-\tau_2) + \int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu + \int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu.$$

- The difficulties to stabilize the IDE are related to the simultaneous presence of a distributed-delay term for the actuation and the state.

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- The difficulties to stabilize the IDE are related to the simultaneous presence of a distributed-delay term for the actuation and the state.
- Laplace transform: $F_0(s)z_2(s) = F_1(s)\bar{V}(s)$, where the holomorphic function F_0 and F_1 are defined by

$$F_0(s) = 1 - be^{-\tau_1 s} - \int_0^{\tau_2} H_{21}(\nu)e^{-\nu s}d\nu, \quad F_1(s) = ae^{-\tau_1 s} + \int_0^{\tau_1} H_{22}(\nu)e^{-\nu s}d\nu.$$

Controllability condition [Mounier]

The functions F_0 and F_1 cannot simultaneously vanish, for all $s \in \mathbb{C}$, $rank[F_0(s), F_1(s)] = 1$.

Equivalent to the previous controllability condition.

An IDE with distributed actuation

$$z_2(t) = a\bar{V}(t-\tau_1) + bz_2(t-\tau_2) + \int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu + \int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu.$$

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Equivalent to the previous controllability condition.

- From now, we assume that $\tau_1 = (N+1)\tau_2 \rightarrow$ non restrictive as it is always possible to artificially delay the control law $\bar{V}(t)$.

Design of a state-feedback controller

$$z_2(t) = a\bar{V}(t-\tau_1) + bz_2(t-\tau_2) + \int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu + \int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu.$$

We consider the following candidate control law

$$\bar{V}(t) = \int_0^{\tau_2} f(\nu)z_2(t-\nu)d\nu + \int_0^{\tau_1} g(\nu)\bar{V}(t-\nu)d\nu,$$

with f and g **piecewise continuous matrix-valued functions**.

Objective

Find f and g such that the control law $\bar{V}$ stabilizes the system.

$$z_2(t) = a\bar{V}(t-\tau_1) + bz_2(t-\tau_2) + \int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu + \int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu.$$

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with f and g **piecewise continuous matrix-valued functions**.

Objective

Find f and g such that the control law $\bar{V}$ stabilizes the system.

We can show that

$$z_2(t) = bz_2(t-\tau_2) + \int_0^{\tau_2} I_1(\nu)z_2(t-\nu)d\nu + \int_{\tau_2}^{\tau_1} I_2(\nu)z_2(t-\nu)d\nu + \int_{\tau_1}^{\tau_1+\tau_2} I_3(\nu)z_2(t-\nu)d\nu$$

where

$$\begin{aligned}
I_1(\nu) =& g(\nu) + H_{22}(\nu) - \int_0^{\nu} f(\eta)H_{21}(\nu-\eta)d\eta - \int_0^{\nu} g(\eta)H_{22}(\nu-\eta)d\eta,\\
I_2(\nu) =& g(\nu) - bg(\nu-\tau_2) - \int_0^{\tau_2} f(\eta)H_{21}(\nu-\eta)d\eta - \int_{\nu-\tau_2}^{\nu} g(\eta)H_{22}(\nu-\eta)d\eta,\\
I_3(\nu) =& af(\nu-\tau_1) - bg(\nu-\tau_2) - \int_{\nu-\tau_1}^{\tau_2} f(\eta)H_{21}(\nu-\eta)d\eta - \int_{\nu-\tau_2}^{\tau_1} g(\eta)H_{22}(\nu-\eta)d\eta,
\end{aligned}$$

$$z_2(t) = a\bar{V}(t-\tau_1) + bz_2(t-\tau_2) + \int_0^{\tau_1} H_{21}(\nu)\bar{V}(t-\nu)d\nu + \int_0^{\tau_2} H_{22}(\nu)z_2(t-\nu)d\nu.$$

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$$\bar{V}(t) = \int_0^{\tau_2} f(\nu)z_2(t-\nu)d\nu + \int_0^{\tau_1} g(\nu)\bar{V}(t-\nu)d\nu,$$

with f and g **piecewise continuous matrix-valued functions**.

Objective

Find f and g such that the control law $\bar{V}$ stabilizes the system.

We can show that

$$z_2(t) = bz_2(t-\tau_2) + \int_0^{\tau_2} l_1(\nu)z_2(t-\nu)d\nu + \int_{\tau_2}^{\tau_1} l_2(\nu)z_2(t-\nu)d\nu + \int_{\tau_1}^{\tau_1+\tau_2} l_3(\nu)z_2(t-\nu)d\nu.$$

If $l_1 = 0$, $l_2 = 0$, and $l_3 = 0$, then z_2 will exponentially converge to zero (since $|b| < 1$).

Objective

Find f and g such that $I_1 = 0$, $I_2 = 0$ and $I_3 = 0$.

- Introduce g_k defined on $[0, \tau_2]$ s.t. for all $\nu \in [k\tau_2, (k+1)\tau_2]$, $g_k(\nu) = g(\nu + k\tau_2)$.

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- Introduce g_k defined on $[0, \tau_2]$ s.t. for all $\nu \in [k\tau_2, (k+1)\tau_2]$, $g_k(\nu) = g(\nu + k\tau_2)$.
- The system $I_1(\nu) = 0, I_2(\nu) = 0, I_3(\nu) = 0$ is equivalent to

$$af(\nu) - bg_N(\nu) - \int_\nu^{\tau_2} g_N(\eta)H_{22}(\nu + \tau_2 - \eta)d\eta - \int_\nu^{\tau_2} f(\eta)H_{21}(\nu + \tau_1 - \eta)d\eta = 0,$$

$$g_k(\nu) - bg_{k-1}(\nu) - \int_\nu^{\tau_2} g_{k-1}(\eta)H_{22}(\nu - \eta + \tau_2)d\eta - \int_0^\nu g_k(\eta)H_{22}(\nu - \eta)d\eta$$

$$- \int_0^{\tau_2} f(\eta)H_{21}(\nu + k\tau_2 - \eta)d\eta = 0,$$

$$g_0(\nu) - \int_0^\nu g_0(\eta)H_{22}(\nu - \eta)d\eta - \int_0^\nu f(\eta)H_{21}(\nu - \eta)d\eta = -H_{22}(\nu),$$

which can be rewritten as

$$\mathcal{T}_0(f, g_N, \ldots, g_0) = (-H_{22}, 0, \ldots, 0), \rightarrow$$ **Fredholm integral equation ($a \neq 0$)**

Fredholm equation and invertibility of a Fredholm operator

Consider the **Fredholm integral operator** $\mathcal{T} : L^2([a,b],\mathbb{R}^n) \to L^2([a,b],\mathbb{R}^n)$ defined by

$$\mathcal{T}\left(z(\cdot)\right) = Mz(\cdot) - \int_a^b K(\cdot,y)z(y)dy,$$

where M is an invertible matrix and K is bounded piecewise continuous.

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Invertibility of the operator $\mathcal{T}$ [Coron]

Consider two linear operators $\mathcal{A}, \mathcal{B}$, such that $D(\mathcal{A}) = D(\mathcal{B}) \subset L^2([a,b],\mathbb{R}^n)$. Assume that

1. $\ker(\mathcal{T}) \subset D(\mathcal{A})$,
2. $\ker(\mathcal{T}) \subset \ker(\mathcal{B})$,
3. $\forall z \in \ker(\mathcal{T}),\ \mathcal{T}\mathcal{A}z = 0$,
4. $\forall s \in \mathbb{C}$, $\ker(s\mathrm{Id} - \mathcal{A}) \cap \ker(\mathcal{B}) = \{0\}$.

Then, the operator $\mathcal{T}$ is invertible.

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Then, the operator $\mathcal{T}$ is invertible.

Proof: Since the integral part of $\mathcal{T}$ is a compact operator, the Fredholm alternative implies that $\dim\ker(\mathcal{T}) < \infty$. The different conditions imply that $\ker(\mathcal{T}) = \{0\}$ and $\mathcal{T}$ is injective. Using the Fredholm alternative, we obtain that $\mathcal{T}$ is invertible.

Design of the state feedback controller

We want to show that the operator $\mathcal{T}_0$ is invertible.

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- Introduce the operators $A_{\mathcal{T}}$ defined on $D(A_{\mathcal{T}}) \subset L^2([0,\tau_2],\mathbb{R})^{N+2}$ by

$$A_{\mathcal{T}} : D(A_{\mathcal{T}}) \to L^2([0,\tau_2],\mathbb{R})^{N+2}$$
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where $D(A_{\mathcal{T}}) = \{(\phi, \psi_N, \ldots, \psi_0) \in (H^1([0,\tau_2],\mathbb{R}))^{N+2},\ \phi(\tau_2) = b\phi(0),\ \psi_N(\tau_2) = a\phi(0), \psi_k(\tau_2) = \psi_{k+1}(0),\ 0 \le k < N\}$.

Design of the state feedback controller

We want to show that the operator $\mathcal{T}_0$ is invertible.

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Existence of f and g

There exist unique piecewise continuous functions f and g such that $I_1(\nu) = 0$, $I_2(\nu) = 0$, and $I_3(\nu) = 0$.

State-feedback control law

Consider the functions I_1, I_2 and I_3 and let f and g be the unique piecewise continuous functions that lead to $I_1(\nu) = 0$, $I_2(\nu) = 0$, and $I_3(\nu) = 0$. Then, the closed-loop system with the control law

$$V(t) = -\rho_{11} q_{11} z_1(t-\tau_1) - \rho_{12}\rho_{22} z_2(t-\tau_2) + \bar{V}(t) - \int_0^{\tau_1} H_{11}(\nu) z_1(t-\nu) d\nu - \int_0^{\tau_2} H_{12}(\nu) z_2(t-\nu) d\nu,$$

where $\bar{V} = \int_0^{\tau_2} f(\nu) z_2(t-\nu) d\nu + \int_0^{\tau_1} g(\nu) \bar{U}(t-\nu) d\nu$ is exponentially stable. Moreover, the control law $V(t)$ exponentially converges to zero and can be low-pass filtered such that the resulting filtered control operator is strictly proper while stabilizing the plant

Control law and extensions

State-feedback control law

Consider the functions l_1, l_2 and l_3 and let f and g be the unique piecewise continuous functions that lead to $l_1(\nu) = 0$, $l_2(\nu) = 0$, and $l_3(\nu) = 0$. Then, the closed-loop system with the control law

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- An observer with in-between measurement can be obtained using an analogous approach $\rightarrow$ **Output-feedback controller**.

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General conclusions: Stabilization of networks with a chain structure

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- Chains with actuation at the extremity $\rightarrow$ **cascade structure**
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  - Possibility to add ODEs in the chain.
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General conclusions: Stabilization of networks with a chain structure

- Key idea: rewriting hyperbolic systems as IDEs using **backstepping**
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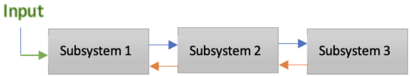
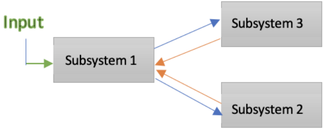
  - **Recursive dynamics interconnection framework:** combines backstepping, predictions, tracking.
  - Possibility to add ODEs in the chain.
  - Computational effort? Model reduction?
- Chains with actuation at one of the junction $\rightarrow$ **Not always controllable**
  - IDE with a distributed effect of the actuation.
  - Controller obtained by solving a **Fredholm equation**.
  - More than two subsystems? Non-scalar subsystems? Cycle?
  - Actuators at several nodes?

Perspectives

Extension to systems with a more complex graph structure





Perspectives

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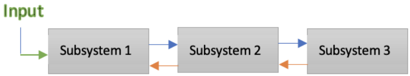

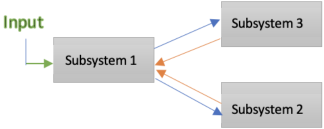

Controllability and control design

Does a given configuration of actuators makes the system controllable? How to design appropriate modular, scalable, and numerically implementable control laws?

Perspectives

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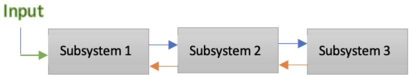

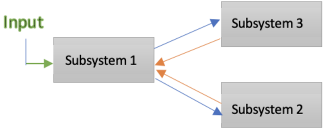

Controllability and control design

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Actuators placement

Considering a given number of actuators, what are the admissible locations that guarantee controllability?

Qualitative analysis to understand the links between the structure of the network (e.g., number of cycles, incidence matrix) and its **controllability/observability** properties.

Perspectives

- In-domain stabilization of hyperbolic systems

$$
\begin{aligned}
\partial_t u(t,x) + \lambda \partial_x u(t,x) &= \sigma^+(x) v(t,x) + h_u(x) V(t), \\
\partial_t v(t,x) - \mu \partial_x v(t,x) &= \sigma^-(x) v(t,x) + h_v(x) V(t),
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with the boundary conditions

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u(t,0) = q v(t,0), \quad v(t,1) = \rho u(t,1),
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- In-domain stabilization of hyperbolic systems

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with the boundary conditions

$$u(t,0) = q v(t,0), \quad v(t,1) = \rho u(t,1),$$

Can be rewritten as the following IDE

$$z(t) = \rho q z(t-\tau) + \int_0^\tau N_z(\nu) z(t-\nu) d\nu + \int_0^\tau N_V(\nu) V(t-\nu) d\nu,$$

- Control design for the general class of IDEs, links with the structural properties?

General class of IDEs

$$z(t) = \sum_{k=1}^{N} A_k z(t-\tau_k) + \int_0^{\tau_N} f(\nu) z(t-\nu) d\nu + \sum_{k=1}^{N} B_k V(t) + \int_0^{\tau_N} g(\nu) V(t-\nu) d\nu,$$