Exponentially stable uncertain systems in infinite dimension: converse Lyapunov characterization and some applications

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Linear switched systems in Banach spaces

Consider $\dot{x} = A_\sigma x$ in a Banach space X. The linear operators $A_{\bar{\sigma}}$, $\bar{\sigma} \in \Xi$, might be unbounded and have different domains (example: network of linear hyperbolic conservation laws with switching topology).
A simple definition of solution for such a switched system is:

Definition

Assume A_ξ generates a strongly continuous semigroup $T_{\bar{\sigma}}(\cdot)$, $\forall \bar{\sigma} \in \Xi$.
For $\sigma : [0, \infty) \to \Xi$ piecewise constant left-continuous with switching times $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and for $t \in [t_k, t_k + 1)$, let

$$x(t; \sigma, x_0) = T_{\sigma(t_k)}(t - t_k) \circ T_{\sigma(t_{k-1})}(t_k - t_{k-1}) \circ \cdots \circ T_{\sigma(t_0)}(t_1) x_0.$$

When solutions are well defined for $\sigma : [0, \infty) \to \Xi$ measurable, one may use more general propagators

A more abstract nonlinear framework: forward complete dynamical system

Definition (A. Mironchenko & F. Wirth, 2019)

The triplet $\Sigma = (X, \mathcal{S}, \phi)$, with $\mathcal{S}$ a shift-invariant and closed by concatenation class of signals $\sigma : [0, \infty) \to \Xi$, and $\phi : \mathbb{R}_+ \times X \times \mathcal{S} \to X$ (the transition map), is said to be a forward complete dynamical system if:

i) $\forall\ (x_0, \sigma) \in X \times \mathcal{S}$: $\phi(0, x_0, \sigma) = x_0$

ii) $\forall\ (x_0, \sigma) \in X \times \mathcal{S}$, $\forall\ t \geq 0$, $\forall\ \tilde{\sigma} \in \mathcal{S}$: $\tilde{\sigma} = \sigma$ over $[0, t]$ then $\phi(t, x_0, \tilde{\sigma}) = \phi(t, x_0, \sigma)$;

iii) $\forall\ (x_0, \sigma) \in X \times \mathcal{S}$: the map $t \mapsto \phi(t, x_0, \sigma)$ is continuous

iv) $\forall\ t, \tau \geq 0$, $\forall\ (x_0, \sigma) \in X \times \mathcal{S}$: $\phi(\tau, \phi(t, x_0, \sigma), \sigma(t + \cdot)) = \phi(t + \tau, x_0, \sigma)$

Linear case: $\mathcal{S} = \mathrm{PC}([0, +\infty), \Xi)$ and $t \mapsto \phi(t, \cdot, \xi)$ strongly continuous linear semigroup for every $\xi \in \Xi$

The switching paradigm

The switching signal $\sigma(\cdot)$ is known only to belong to some known class $\mathcal{S}$ and cannot be chosen.

$t \mapsto \sigma(t)$ may be used to model:

- logical (= discrete valued) uncontrolled variables affecting the dynamics
- disturbances or time-varying uncertainties
- in both the above cases, the time-evolution may be difficult to model or not known: one may prefer to embed the dynamics of a time-dependent equation corresponding to $t \mapsto A_{\sigma(t)}$ into a parametric family of non-autonomous systems (and $\mathcal{S}$ is autonomous)
- relaxations of dynamical systems $\dot{x}(t) = A_{\sigma(x(t))}x(t)$ with σ possibly discontinuous $\rightarrow$ remove Zeno complications, take into account sector constraints
- not discussed today: the class $\mathcal{S}$ may encode constraints such as dwell-time, persistent excitation, periodicity, ...

Uniform exponential stability

Uniform exponential stability (UES) of $\Sigma = (X, \mathcal{S}, \phi)$: $\exists C, \nu > 0$ such that $\|\phi(t, x_0, \sigma)\| \leq Ce^{-\nu t}\|x_0\|$, $\forall t \geq 0$, $\forall \sigma \in \mathcal{S}$

The (exponential/asymptotic/...) stability of each semigroup $T_{\bar{\sigma}}(\cdot)$ does not imply the stability of the switched system ("too many cooks spoil the broth")

Main goal:

- direct Lyapunov theorems with as few conditions on the Lyapunov function as possible
- converse Lyapunov theorems (stability $\implies$ existence of a Lyapunov function) (idea going back to Filippov's Selection Lemma for differential inclusions)
- use Lyapunov functions to deduce robustness and interconnect switching dynamics

Part I: Linear case

Converse Lyapunov theorem in Banach spaces [Hante, S., 2011]

Σ linear. The following three conditions are equivalent:

(A) UES

(B) There exists $V : X \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on X,

$$c\|x\|_X^2 \leq V(x) \leq C\|x\|_X^2, \quad x \in X$$

$$\limsup_{t \to 0^+} \frac{V(T_{\bar{\sigma}}(t)x) - V(x)}{t} \leq -\|x\|_X^2, \quad \bar{\sigma} \in \Xi,\ x \in X.$$

(C) There exist $M \geq 1$ and $\omega > 0$ such that, $\forall \sigma \in \mathcal{S}$, $\forall x_0$,

$$\|x(t; \sigma, x_0)\|_X \leq M e^{\omega t} \|x_0\|_X, \quad t \geq 0,$$

and there exists $V : X \to [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm in X,

$$V(x) \leq C\|x\|_X^2, \quad x \in X$$

$$\liminf_{t \to 0^+} \frac{V(T_{\bar{\sigma}}(t)x) - V(x)}{t} \leq -\|x\|_X^2, \quad \bar{\sigma} \in \Xi,\ x \in X.$$

Remarks and open problems

- V squared norm is equivalent to: positive definite, homogeneous of degree 2, continuous, and convex
- The result allows to exploit Lyapunov functions also when the stability has been deduced by non-Lyapunov methods (motivating example: S. Amin, F. M. Hante, and A. M. Bayen, 2008 - switched linear hyperbolic conservation laws with reflecting boundaries)
- (C) $\Longrightarrow$ (B) can be used to pass from a constructive non-coercive Lyapunov function to a coercive one (even in the unswitched case $\Xi = \{\bar{\sigma}\}$, without contradicting the existence of exponentially stable dynamics without coercive quadratic Lyapunov functions [Chernoff 1976])

Idea of the proof

(A) $\Longrightarrow$ (C) obtained taking

$$V(x) = \sup_{\sigma\in\mathcal{S}} \int_0^\infty \|T_\sigma(t)x\|^2 dt$$

(alternative choice: $V(x) = \int_0^\infty \sup_{\sigma\in\mathcal{S}} \|T_\sigma(t)x\|^2 dt$)

- well-posedness is a direct consequence of UES
- positive-definiteness and 2-homogeneity by construction
- decay along trajectories:

$$V(T_{\bar\sigma}(t)x) - V(x) \leq V(T_{\bar\sigma}(t)x) - \sup_{\sigma\in\mathcal{S},\sigma|_{[0,t]}\equiv\bar\sigma} \int_0^\infty \|T_\sigma(t)x\|^2 dt$$
$$= -\int_0^t \|T_{\bar\sigma}(\tau)x\|^2 d\tau$$

(underlying property: $\mathcal{S}$ closed under concatenation)

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$$V(T_{\bar\sigma}(t)x) - V(x) \le V(T_{\bar\sigma}(t)x) - \sup_{\sigma \in \mathcal{S}, \sigma|_{[0,t]} \equiv \bar\sigma} \int_0^\infty \|T_\sigma(t)x\|^2 dt$$

$$= -\int_0^t \|T_{\bar\sigma}(\tau)x\|^2 d\tau$$

(underlying property: $\mathcal{S}$ closed under concatenation)

(A) $\implies$ (B): do as above replacing Ξ by $\Xi \cup \{-\nu I_X\}$ for some $\nu > 0$ (the corresponding Lyapunov function is not quadratic even in the unswitched case $\Xi = \{\bar\sigma\}$)

(C)$\Longrightarrow$ (A): a Datko-type lemma

Adaptation of [Triggiani, 1994]

Assume that

(a) there exist $M \geq 1$ and $w > 0$ such that $\forall \sigma \in \mathcal{S}$, $\forall x \in X$,

$$\|T_\sigma(t)x\|_X \leq Me^{\omega t}\|x\|_X, \quad t \geq 0,$$

(b) there exist $c \geq 0$ and $p \in [1, +\infty)$ such that

$$\int_0^{+\infty} \|T_\sigma(t)x\|_X^p \leq c\|x\|_X^p,$$

for every $x \in X$ and every $\sigma \in \mathcal{S}$.

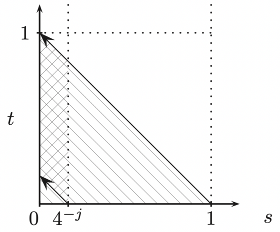
Then there exist $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0,\ \sigma \in \mathcal{S}$$

The uniform exponential boundedness cannot be removed

$\Xi = \mathbb{N}$, $X = L^p(0,1)$, $p \in [1,\infty)$,

$$(T_j(t)x)(s) = \begin{cases} 2^{\frac{1}{p}} x(s+t), & \text{if } s \in [0, 1-t] \cap [4^{-j} - t, 4^{-j}), \\ x(s+t), & \text{if } s \in [0, 1-t] \setminus [4^{-j} - t, 4^{-j}), \\ 0, & \text{if } s \in (1-t, 1] \cap [0,1] \end{cases} \qquad j \in \Xi$$



One can check that $V(x) = \sup_{\sigma \in \mathcal{S}} \int_0^\infty \|T_\sigma(t)x\|_X^2 \, dt \le \frac{3}{2} \|x\|_X^2$
and $\liminf_{t \to 0+} \frac{V(T_j(t)x) - V(x)}{t} \le -\|x\|^2$
However, the system is not UES (but $T_\sigma(t)x = 0 \ \forall t \ge 1 \ \forall x \in X$)

The Hilbert case

Proposition

X separable Hilbert space, Σ linear and UES. Then there exists $\mathcal{B} \subset \mathcal{L}(X)$ compact for the weak operator topology, made of self-adjoint operators, such that the function $V : X \to \mathbb{R}$ given by

$$V(x) = \max_{B \in \mathcal{B}} \langle x, Bx \rangle$$

is a Lyapunov function that is directionally differentiable in the sense of Fréchet with

$$V'(x, \psi) = \max_{\hat{B} \in S(x)} 2 \left\langle \psi, \hat{B}x \right\rangle, \quad S(x) = \mathrm{argmax}_{B \in \mathcal{B}} \langle x, Bx \rangle$$

- Idea: B is of the form $Bx = \int_0^\infty T_\sigma^*(t) T_\sigma(t) x dt$, $\sigma \in \mathcal{S}$ (plus limit points)
- when $\dim(X) < \infty$ one can further smoothen V. Open problem: do the same in the Hilbert case

Retarded systems

Consider

$$\dot{x}(t) = \sum_{i=1}^{p} A_{\sigma(t),i}\, x\Big(t - \tau_i\big(\sigma(t)\big)\Big), \quad x(t) \in \mathbb{R}^d$$

Retarded systems

Consider

$$\dot{x}(t) = \sum_{i=1}^{p} A_{\sigma(t),i} x\Big(t - \tau_i\big(\sigma(t)\big)\Big), \quad x(t) \in \mathbb{R}^d$$

or, more generally,

$$\dot{x} = \Gamma(t) x_t$$

where $x_t : [-r, 0] \to \mathbb{R}^d$ is the history function $x_t(\theta) = x(t + \theta)$, and $\Gamma(\cdot)$ is piecewise constant or measurable in a set Ξ of bounded operators

$$L : C([-r, 0], \mathbb{R}^d) \to \mathbb{R}^d.$$

Strongly continuous semigroup for every $L \in \Xi$ $\rightarrow$ e.g. [Hale, Lunel, 1993]

Converse Lyapunov theorem for retarded systems [Haidar, Mason, S., 2015]

$\Xi \subset \mathcal{L}(C([-r,0],\mathbb{R}^d),\mathbb{R}^d)$ bounded. The following statements are equivalent:

(i) The system is UES in $C([-r,0],\mathbb{R}^d)$

(ii) The system is UES in $H^1([-r,0],\mathbb{R}^d)$

(iii) There exists $V : C([-r,0],\mathbb{R}^d) \to [0,\infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $C([-r,0],\mathbb{R}^d)$,

$$\underline{c}\|\psi\|_C^2 \leq V(\psi) \leq \bar{c}\|\psi\|_C^2$$

$$\limsup_{t\to 0^+} \frac{V(T_{\bar{\sigma}}(t)\psi) - V(\psi)}{t} \leq -\|\psi\|_C^2, \quad \bar{\sigma} \in \Xi, \psi \in C([-r,0],\mathbb{R}^d)$$

(iv) There exists a directionally Fréchet differentiable function $V : H^1([-r,0],\mathbb{R}^d) \to [0,\infty)$ such that $\sqrt{V(\cdot)}$ is a norm on $H^1([-r,0],\mathbb{R}^d)$,

$$\underline{c}\|\psi\|_{H^1}^2 \leq V(\psi) \leq \bar{c}\|\psi\|_{H^1}^2$$

$$\limsup_{t\to 0^+} \frac{V(T_{\bar{\sigma}}(t)\psi) - V(\psi)}{t} \leq -\|\psi\|_{H^1}^2, \quad \bar{\sigma} \in \Xi, \psi \in H^1$$

Converse Lyapunov theorem for retarded systems [Haidar, Mason, S., 2015]

(v) There exists a continuous function

$$V : C([-r,0],\mathbb{R}^d) \to [0,\infty)$$

such that

$$V(\psi) \le c\|\psi\|_C^2$$

$$\liminf_{t\to 0^+} \frac{V(T_{\bar\sigma}(t)\psi) - V(\psi)}{t} \le -|\psi(0)|^2, \quad \bar\sigma \in \Xi, \psi \in C([-r,0],\mathbb{R}^d)$$

(vi) There exists a continuous function

$$V : H^1([-r,0],\mathbb{R}^d) \to [0,\infty)$$

such that

$$V(\psi) \le c\|\psi\|_{H^1}^2$$

for some constant $c > 0$ and

$$\liminf_{t\to 0^+} \frac{V(T_{\bar\sigma}(t)\psi) - V(\psi)}{t} \le -|\psi(0)|^2, \quad \bar\sigma \in \Xi, \psi \in H^1$$

Remarks

- uniform exponential boundedness not assumed but always true
- for retarded equations measurable signals can be considered, giving rise to equivalent UES conditions
- bound on the derivative depending only on $\psi(0)$ $\rightarrow$ one recovers $\|\psi\|_C$ by integration and $\|\psi\|_{H^1}$ by boundedness of the operators in Ξ
- exponential stability in L^2 norm: following [Curtain, Zwart, 1995] the solutions are well-defined in $X = L^2([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$. In this case, however, the uniform exponential boundedness of the solutions is not guaranteed

Robustness

$\mathcal{C} = C([-r,0],\mathbb{R}^d)$

Lemma

$\Xi, \Theta \subset \mathcal{L}(\mathcal{C},\mathbb{R}^d)$ bounded. Let $V : \mathcal{C} \to \mathbb{R}$ be the square of a norm such that $V(\psi) \le \overline{c}\|\psi\|_{\mathcal{C}}^2$ for some $\overline{c} > 0$ and for every $\psi \in \mathcal{C}$. Then there exists $K > 0$ such that

$$\overline{D}_{L+\Lambda}V(\psi) \le \overline{D}_L V(\psi) + K\|\psi\|_{\mathcal{C}}^2 \quad \forall L \in \Xi,\ \forall \Lambda \in \Theta,\ \forall \psi \in \mathcal{C}$$

Hence small perturbations of Σ UES are still UES.
Example: $\dot{x}(t) = -x(t-\tau(t))$, $\tau \in [0,r]$, known to be UES for $r < 3/2$ [Myshkis, 1951]. Hence

$$\dot{x}(t) = -x(t-\tau(t)) + \int_{-\bar{r}}^{0} a(s)x(t+s)ds$$

with $a \in L^1([-\bar{r},0],\mathbb{R})$ and $\bar{r} \ge r$ is still UES for $\|a\|_{L^1}$ small enough

Interconnection

$X_1 = C([-r,0],\mathbb{R}^{n_1})$, $X_2 = C([-r,0],\mathbb{R}^{n_2})$, $X = X_1 \times X_2$
Let Ξ_i be a bounded subset of $\mathcal{L}\left(X,\mathbb{R}^{n_i}\right)$, $i = 1,2$, and consider

$$\Sigma_1 : \dot{x}(t) = \Gamma_1(t)(x_t, 0),\ \Gamma_1(t) \in \Xi_1,$$
$$\Sigma_2 : \dot{y}(t) = \Gamma_2(t)(0, y_t),\ \Gamma_2(t) \in \Xi_2$$

and the interconnected system $\Sigma : \dot{z}(t) = \Gamma(t)z_t,\ \Gamma(t) \in \Xi_1 \times \Xi_2$

Theorem

Assume that Σ_1 and Σ_2 are UES. Let $V_i : X_i \to [0,\infty)$, $i = 1,2$, be coercive Lyapunov functionals for Σ_1 and Σ_2. Let $\bar{c}_1, \bar{c}_2$ be the upper-bound constants for V_1 and V_2, respectively. Let

$$\mu = \sup\left\{ \frac{\|(L_1(0,\psi_2), L_2(\psi_1,0))\|}{\|\psi\|_X} \mid (L_1,L_2) \in \Xi_1 \times \Xi_2,\ 0 \neq \psi \in X \right\}$$

If $2\max(\bar{c}_1,\bar{c}_2)\mu < 1$ then the interconnected system Σ is UES

Part II: Nonlinear case

Datko-type theorem

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$*. Let* $t_1 > 0$ *and* $G_0 \geq 1$ *be such that*

$$\|\phi(t,x,\sigma)\| \leq G_0 \|x\|, \ \forall\, t \in [0, t_1], \ \forall\, x \in X, \ \forall\, \sigma \in \mathcal{S}$$

The following statements are equivalent

i) Σ *is UES*

ii) *there exist* $p, k > 0$ *such that*

$$\int_0^{+\infty} \|\phi(t,x,\sigma)\|^p dt \leq k^p \|x\|^p, \forall\, x \in X, \ \forall\, \sigma \in \mathcal{S}$$

Nonlinear version of Datko (unswitched): [Ichikawa, 1984]

Semi-global Datko-type theorem

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$*. Let* $t_1 > 0$ *and let* G_0 *be a class* $\mathcal{K}_\infty$ *function such that*

$$\|\phi(t, x, \sigma)\| \leq G_0(\|x\|), \ \forall \, t \in [0, t_1], \ \forall \, x \in X, \ \forall \, \sigma \in \mathcal{S}$$

The following statements are equivalent

i) Σ *is uniformly semiglobally exponentially stable (USGES)*

ii) there exist $p > 0$ *and* $k : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\int_0^{+\infty} \|\phi(t, x, \sigma)\|^p dt \leq k(\|x\|)\|x\|^p, \forall \, x \in X, \ \forall \, \sigma \in \mathcal{S}$$

Exponential stability characterizations: global non-coercive

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. *Let* $t_1 > 0$, $G_0 \geq 1$ *such that*

$$\|\phi(t, x, \sigma)\| \leq G_0 \|x\|, \quad \forall\, t \in [0, t_1],\ \forall\, x \in X,\ \forall\, \sigma \in \mathcal{S}$$

and $V : X \to \mathbb{R}_+$, $p, c > 0$ *such that*

$$V(x) \leq c\|x\|^p, \quad \forall\, x \in X$$
$$\underline{D}_\sigma V(x) \leq -\|x\|^p, \quad \forall\, x \in X, \forall\, \sigma \in \mathcal{S}$$

and the map $t \mapsto V(\phi(t, x, \sigma))$ *is continuous from the left, then* Σ *is* UES

Exponential stability characterizations: global coercive

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ *and* $p > 0$*. Assume that the transition map* ϕ *is uniformly continuous. If* Σ *is* UES *then there exists* $V : X \to \mathbb{R}_+$ *continuous and* $c, C > 0$ *such that*

$$c\|x\|^p \leq V(x) \leq C\|x\|^p, \quad \forall\, x \in X$$
$$\overline{D}_\sigma V(x) \leq -\|x\|^p, \quad \forall\, x \in X, \forall\, \sigma \in \mathcal{S}$$

Non-coercive Lyapunov theorem: comparison with related results

Definition (A. Mironchenko & F. Wirth, 2019)

(RFC) if for any $C > 0$ and any $\tau > 0$:

$$\sup_{\|x\| \le C, t \in [0,\tau], \sigma \in \mathcal{S}} \|\phi(t,x,\sigma)\| < \infty$$

(REP) for every $\varepsilon, h > 0$, there exists $\delta > 0$:

$$\|x\| \le \delta \implies \sup_{t \in [0,h], \sigma \in \mathcal{S}} \|\phi(t,x,\sigma)\| \le \varepsilon$$

Theorem (A. Mironchenko & F. Wirth, 2019)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ and assume that Σ satisfies the RFC and REP conditions. If Σ admits a non-coercive Lyapunov functional, then it is uniformly globally asymptotically stable.

Exponential stability characterizations: semi-global non-coercive

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$. *Let* $t_1 > 0$ *and let* G_0 *be a class* $\mathcal{K}_\infty$ *function such that*

$$\|\phi(t, x, \sigma)\| \le G_0(\|x\|),\ \forall\, t \in [0, t_1],\ \forall\, x \in X,\ \forall\, \sigma \in \mathcal{S}.$$

If for every $r > 0$ *there exists* $V_r : X \to \mathbb{R}_+$ *and* $p_r, c_r > 0$ *such that*

$$\begin{aligned} V_r(x) &\le c_r \|x\|^{p_r}, \quad \forall\, x \in B_X(0, r) \\ \underline{D}_\sigma V_r(x) &\le -\|x\|^{p_r}, \quad \forall\, x \in B_X(0, r), \forall\, \sigma \in \mathcal{S} \end{aligned}$$

$V_r(\phi(\cdot, x, \sigma))$ *is continuous from the left, and, moreover,* $\limsup_{r \to +\infty} {G_0}^{-1}(r) \min\left\{1, \left(\frac{t_1}{c_r}\right)^{\frac{1}{p_r}}\right\} = +\infty$, *then system* Σ *is* USGES

Exponential stability characterizations: semi-global coercive

Theorem (Haidar, Chitour, Mason and S., 2021)

Consider a forward complete dynamical system $\Sigma = (X, \mathcal{S}, \phi)$ *and assume that the transition map* ϕ *is uniformly continuous. If* Σ *is USGES then for every* $r > 0$ *there exist* $\underline{c}_r, \overline{c}_r > 0$ *and a continuous functional* $V_r : X \to \mathbb{R}_+$, *such that*

$$\underline{c}_r \|x\| \leq V_r(x) \leq \overline{c}_r \|x\|, \quad \forall\, x \in B_X(0, r),$$
$$\overline{D}_\sigma V_r(x) \leq -\|x\|, \quad \forall\, x \in B_X(0, r),\ \forall\, \sigma \in \mathcal{S}.$$

Moreover, if the transition map Φ *is uniformly Lipschitz continuous (respectively, uniformly Lipschitz continuous on bounded sets),* V_r *can be taken Lipschitz continuous (respectively, Lipschitz continuous on bounded sets).*

Example: retarded switching control systems

Consider the uncertain retarded functional differential equation

$$\Sigma : \begin{array}{rcll} \dot{x}(t) & = & f_{\sigma(t)}(x_t), & a.e.\ t \geq 0, \\ x(\theta) & = & x_0(\theta), & \theta \in [-\Delta, 0], \end{array}$$

- $x(t) \in \mathbb{R}^n$; n is a positive integer;
- $x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the history of $x(t)$ defined by

$$x_t(\theta) = x(t+\theta), \quad \forall\, \theta \in [-\Delta, 0];$$

- $x_0 \in \mathcal{C} := C([-\Delta, 0], \mathbb{R}^n)$ is the initial condition.

Example: retarded switching control systems

Theorem

Assume that f_q, $q \in \Xi$, are uniformly globally Lipschitz, i.e.,

$$\|f_q(\phi_1) - f_q(\phi_2)\| \leq l\|\phi_1 - \phi_2\|, \; \forall \; \phi_1, \phi_2 \in \mathcal{C}, \; \forall \; q \in \Xi,$$

for some $l > 0$. The following statements are equivalent:

i) *Σ is UES*

ii) *there exists a continuous functional $V : \mathcal{C} \to \mathbb{R}_+$ and $p, c > 0$ such that*

$$V(\phi) \leq c\|\phi\|^p, \quad \forall \; \phi \in \mathcal{C},$$

and

$$\overline{D}V(\phi) \leq -\|\phi\|^p, \quad \forall \; \phi \in \mathcal{C}$$

iii) *coercive version*

Sampling data control for semilinear switching systems

$$\Sigma: \quad \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), u(t))$$

with A generator of a linear C_0-group, $f_q : X \times U \to X$ uniformly Lipschitz continuous, $f_q(0,0) = 0$

Suppose that Σ in closed-loop with $u(t) = K(x(t))$ is UES, with $K : X \to U$ Lipschitz and $K(0) = 0$

Only discrete output measurements are available

$$y(t) = x(t_k), \quad \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \geq 0.$$

Theorem (Haidar, Chitour, Mason and S., 2021)

Let A, f, and K as above. There exists $\delta^\star > 0$ such that Σ in closed loop with the predictor-based sampled data controller

$$u(t) = K(T(t - t_k)x(t_k)), \quad \forall\, t \in [t_k, t_{k+1}),\ \forall\, k \geq 0,$$

is UES *provided that* $\sup_{k \geq 0}(t_{k+1} - t_k) \leq \delta^\star$

Example: switching damped wave equation

$\Omega \subset \mathbb{R}^n$ regular

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + \rho_{\sigma(t)}(u) = 0 & \text{in } \Omega \times \mathbb{R}_+ \\ \psi = 0 & \text{on } \partial\Omega \times \mathbb{R}_+ \\ \psi(0) = \psi_0 \psi'(0) = \psi_1 & \text{on } \Omega \end{cases} \tag{1}$$

where $\rho_q : \mathbb{R} \to \mathbb{R}$, for $q \in \Xi$, is a uniformly Lipschitz continuous function satisfying

$$\rho_q(0) = 0, \quad \alpha|v| \leq |\rho_q(v)| \leq \frac{|v|}{\alpha}, \quad \forall\ v \in \mathbb{R},\ \forall\ q \in \Xi$$

for some $\alpha > 0$.
UES of the system with feedback $u = \frac{\partial \psi}{\partial t}$ $\rightarrow$ [Martinez, 2000]
The previous theorem may be applied.

ISS of semilinear switching control systems

$$\Sigma : \quad \dot{x}(t) = Ax(t) + f_{\sigma(t)}(x(t), u(t))$$

A infinitesimal generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on X, $\sigma \in \mathcal{PC}$, $f_q : X \times U \to X$ uniformly Lipschitz continuous, $f_q(0,0) = 0$

Theorem

Assume that Σ is UES. Then for every $1 \leq p \leq +\infty$ and $\sigma \in \mathcal{PC}$, the input-to-state map $u \mapsto \phi_u(\cdot, 0, \sigma)$ is well defined as a map from $L^p(U)$ to $L^p(X)$ and has a finite L^p-gain independent of σ, i.e., there exists $c_p > 0$ such that

$$\|\phi_u(\cdot, 0, \sigma)\|_{L^p(X)} \leq c_p \|u\|_{L^p(U)}, \quad \forall\ u \in L^p(U),\ \forall\ \sigma \in \mathrm{PC}$$