Exponential convergence towards consensus for non-symmetric linear first-order systems in finite and infinite dimensions

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First-order linear consensus model in finite dimension

$$\dot{y}_i(t) = \sum_{j=1}^{N} \sigma_{ij} \left(y_j(t) - y_i(t) \right) \qquad 1 \leqslant i \leqslant N \qquad \text{(Hegselmann-Krause)}$$

- $y_i(t)$: state of the agent i (position, opinion, etc).
- $\sigma_{ij} \geqslant 0$: interaction frequency of the agent i with the agent j.

We say we have consensus when $y_i(t) = y_j(t) = \bar{y}$ for all i, j.

The system is symmetric if $\sigma_{ij} = \sigma_{ji}$ for all i, j, and non-symmetric otherwise.

HK: basic model for collective (social) dynamics. Many other models, like Cucker-Smale second-order models.

Albi, Ayi, Bellomo, Bertozzi, Biccari, Borzi, Boudin, Caines, Caponigro, Carrillo, Couzin, Cristiani, Cucker, Degond, Desvillettes, Dimarco, Dong, Fornasier, Frasca, Ha, Haskovec, Illner, Kalise, Kim, Lee, Leonard, Levy, Liu, Lorenzi, Malhamé, Markowich, Mordecki, Motsch, Pareschi, Parrish, Perthame, Piccoli, Pouradier Duteil, Rosado, Rossi, Salvarani, Shen, Smale, Tadmor, Toscani, Tosin, Vecil, Vicsek, Wolfram, Zanella, Zuazua........

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Setting

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} \qquad A = \begin{pmatrix} -\sum_{k\neq 1} \sigma_{1k} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & -\sum_{k\neq 2} \sigma_{2k} & \cdots & \sigma_{2N} \\ \vdots & & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & -\sum_{k\neq N} \sigma_{Nk} \end{pmatrix}$$

the system is

$$\boxed{\dot{y}(t) = Ay(t)}$$

A: arbitrary $N \times N$ real matrix whose off-diagonal coefficients are $\geqslant 0$ and such that the sum of coefficients of any of its rows is zero.

First-order linear consensus model in finite dimension

In what follows we assume that $\forall i \quad \exists j \neq i \mid \sigma_{ij} > 0$: every agent has at least one interaction.

$$A = \begin{pmatrix} -\sum_{k\neq 1} \sigma_{1k} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & -\sum_{k\neq 2} \sigma_{2k} & \cdots & \sigma_{2N} \\ \vdots & & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & -\sum_{k\neq N} \sigma_{Nk} \end{pmatrix} \qquad e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Remarks:

- $Ae = 0$
- All eigenvalues of A (but 0) have a negative real part (Gershgorin circle theorem). Hence $\ker A = \mathbb{R}e$.

First-order linear consensus model in finite dimension

$$\boxed{\dot{y}(t) = Ay(t)}$$

In the symmetric case: $A = A^\top$

$$\frac{d}{dt}\sum_{i=1}^{N} y_i(t) = \langle \dot{y}(t), e\rangle = \langle Ay(t), e\rangle = \langle y(t), Ae\rangle = 0 \quad \Rightarrow \quad \bar{y}^e = \langle y(t), e\rangle e = \mathrm{Cst}$$

and (average)

$$\frac{1}{2}\frac{d}{dt}|y(t) - \bar{y}^e|^2 = \langle A(y(t) - \bar{y}^e), y(t) - \bar{y}^e\rangle < 0 \qquad \text{(variance)}$$

hence $y(t) \to \bar{y}^e$ exponentially: exponential convergence to consensus.

In the non-symmetric case, this simple argument cannot work because $\langle Az, z\rangle$ may be positive for some z.

$\to$ No existing "L^2-theory". See "L^∞-theory" by Jabin Motsch Tadmor (JDE 2014).

In infinite dimension

$\Omega \subset \mathbb{R}^d$ open bounded, $\quad 0 \leqslant \sigma \in L^\infty(\Omega^2)$, $\quad y : \Omega \times \mathbb{R}_+ \to \mathbb{R}$

$$\frac{\partial y}{\partial t}(t,x) = \int_\Omega \sigma(x,x')(y(t,x') - y(t,x))\,dx'$$

i.e.,

$$\boxed{\dot{y}(t) = Ay(t)}$$

with

$$(Az)(x) = \int_\Omega \sigma(x,x')(z(x') - z(x))\,dx'$$

$$A = K - M_S \qquad (Kz)(x) = \int_\Omega \sigma(x,x')z(x')\,dx', \quad M_S = S\,\mathrm{Id}, \quad S = Ke$$

- $K : L^2(\Omega) \to L^2(\Omega)$ (Hilbert-Schmidt) compact operator
- $M_S : L^2(\Omega) \to L^2(\Omega)$ multiplication operator
- $e(x) = 1 \quad \forall x \in \Omega \qquad \to$ note that $Ae = 0$.

Objective: understand the asymptotic behavior of $y(t)$.

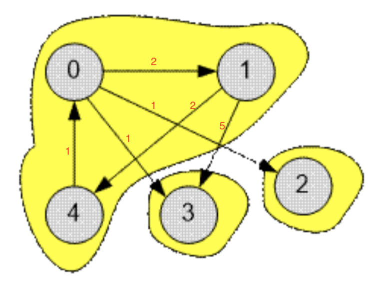
Strong connectivity of the graph

In finite dimension: To $\sigma = (\sigma_{ij})$ is associated the directed graph G of $1, 2, \ldots, N$, which has an edge from i to j when $\sigma_{ij} > 0$.

When an entry of A is zero, there is no direct interaction between the corresponding agents and when an entry of A is positive, they are are directly connected.

G is *strongly connected* if, for any $i \neq j$, there exists a path, joining i to j in G, of distinct indices satisfying

$$i_0 = i, \quad i_r = j, \qquad \sigma_{i_k i_{k+1}} > 0, \quad 0 \leqslant k \leqslant r - 1.$$



$$\sigma = \begin{pmatrix} -4 & 2 & 1 & 1 & 0 \\ 0 & -7 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

not strongly connected
(3 strongly connected components)

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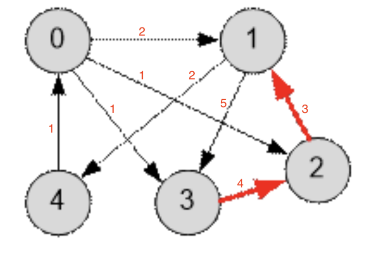

$$\sigma = \begin{pmatrix} -4 & 2 & 1 & 1 & 0 \\ 0 & -7 & 0 & 5 & 2 \\ 0 & 3 & -3 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

strongly connected

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$$i_0 = i, \quad i_r = j, \qquad \sigma_{i_k i_{k+1}} > 0, \quad 0 \leqslant k \leqslant r-1.$$

In infinite dimension: The vertices of the directed graph G associated to $\sigma \in L^\infty(\Omega^2)$ are the Lebesgue points x of σ in Ω, i.e., such that $x' \mapsto \sigma(x, x')$ is defined a.e. in Ω.

Given any two vertices $x_1 \neq x_2$, we say that (x_1, x_2) is an arc if $x_2 \in \operatorname{ess\,supp} \sigma(x_1, \cdot)$.

The directed graph G is strongly connected if:

1. For any vertices $x \neq x'$, there exists a path joining x to x' in G, i.e., two-by-two distinct Lebesgue points x_0, ..., x_r, for some $r \in \mathbb{N}^*$ such that

$$x_0 = x, \quad x_r = x', \qquad x_{k+1} \in \operatorname{ess\,supp} \sigma(x_k, \cdot), \quad 0 \leqslant k \leqslant r-1.$$

2. $\delta = \operatorname{ess\,inf} S > 0$: means that (almost) every agent can interact with a significant continuum of agents in Ω. (actually, relaxable assumption)

Main result

Theorem (Boudin Salvarani Trélat, SIMA 2022)

Assume that the graph associated to σ is strongly connected.

- $\exists ! v \in \ker A^*$ s.t. $v > 0$ and $\langle v, e\rangle = 1$, and the weighted mean $\bar{\mathbf{y}}^{\mathbf{v}} = \langle \mathbf{y}(\mathbf{t}), \mathbf{v}\rangle \mathbf{e}$ of any solution y is (time) constant.
- $\exists \rho > 0 \mid \quad \forall y$ solution $\quad \forall \varepsilon \in (0, \rho) \quad \exists M_\varepsilon > 0$ s.t.

$$\boxed{\| y(t) - \bar{y}^v \| \leqslant M_\varepsilon \| y(0) - \bar{y}^v \| e^{(-\rho+\varepsilon)t} \qquad \forall t \geqslant 0}$$

- In finite dimension, $\rho = |\mathrm{Re}\, \lambda_2|$

see also Olfati-Saber Murray (TAC 2004), Weber Theisen Motsch (JSP 2019).

- In infinite dimension, $\rho = s(A_2)$ (spectral bound) where $A_2 : \mathrm{Im}\, A \to \mathrm{Im}\, A$ is the isomorphism defined by $A_2 z = Az$ for every $z \in \mathrm{Im}\, A$.

- In the absence of strong connectivity: exponential convergence to clusters defined by strongly connected components of σ.

Main steps of the proof

Step 1: properties of A and A^*, definition of the weight

In finite dimension, for any $z \in \mathbb{R}^N$,

$$(A^* z)_i = \sum_j \sigma_{ji} z_j - \Big(\sum_j \sigma_{ij}\Big) z_i \qquad 1 \leqslant i \leqslant N,$$

In infinite dimension, for any $z \in L^2(\Omega)$,

$$\begin{aligned} A^* z(x) &= \int_\Omega \sigma(x', x) z(x')\, dx' - \left(\int_\Omega \sigma(x, x')\, dx'\right) z(x) \\ &= \int_\Omega \sigma(x', x) z(x')\, dx' - S(x) z(x) \qquad \text{for a.e. } x \in \Omega. \end{aligned}$$

Main steps of the proof

Proposition

1. $\ker A = \ker A^2$ is a one-dimensional subspace of X spanned by e.
2. $\ker A^* = \ker(A^*)^2$ is a one-dimensional subspace of X.
3. 0 is a simple eigenvalue of both A and A^*.

This is proved thanks to the strong connectivity assumption.

Consequence: there exists a unique $v \in \ker A^*$ such that $\langle v, e \rangle = 1$ (normalization).

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Actually:

$$\boxed{v > 0}$$

Proof by an homotopy argument:

$$[0,1] \ni \lambda \longmapsto \sigma_\lambda = \lambda\sigma + (1-\lambda)\|\sigma\|_\infty$$

Start from the symmetric case, $\lambda = 0$, $v_0 = e$, and prove, using analyticity and strong connectivity, that $v_\lambda > 0$ along the path.

Main steps of the proof

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Actually:

$$\boxed{v > 0}$$

$\rightarrow$ **v is a weight**

$\Rightarrow$ we define a *weighted* Hilbert structure on $X = \mathbb{R}^N$ or $L^2(\Omega)$:

$$\langle y, z \rangle_v = \sum_{i=1}^{N} v_i y_i z_i \qquad \langle y, z \rangle_v = \int_\Omega y(x) z(x)\, v(x)\, dx$$

(note that $v(x)\, dx$ is an absolutely continuous probability measure)

Weighted mean:

$$\bar{y}^v = \langle y, v \rangle e = \langle y, e \rangle_v \, e$$

Main steps of the proof

Actually:

Lemma

- $(\ker A)^{\perp_v} = \mathrm{Im}\, A$
- $\ker A^{*_v} = \ker A = \mathrm{Span}\, e, \quad \mathrm{Im}\, A^{*_v} = \mathrm{Im}\, A$
- $X = \ker A \overset{\perp_v}{\oplus} \mathrm{Im}\, A = \ker A^{*} \overset{\perp}{\oplus} \mathrm{Im}\, A$
- $\mathrm{Im}\, A = \mathrm{Im}\, A^2$

(however A is *not* v-selfadjoint)

<u>Consequence:</u> $\pi : X \to \mathrm{Im}\, A \quad v$-orthogonal projection

- $\forall y \in X \qquad y = \bar{y}^v + \pi y$
- $\begin{array}{rcl} A_2 : \mathrm{Im}\, A & \longrightarrow & \mathrm{Im}\, A \\ y & \longmapsto & Ay \end{array}$ isomorphism.

Main steps of the proof

Changing the basis to $X = \ker A \overset{\perp_V}{\oplus} \mathrm{Im}\, A$:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix} \qquad \Rightarrow \qquad y(t) - \bar{y}^V = e^{tA}(y(0) - \bar{y}^V) = e^{tA}\pi y(0) = e^{tA_2}\pi y(0)$$

<u>In finite dimension:</u> A_2 Hurwitz $\Rightarrow$ exponential convergence at the sharp rate $|\mathrm{Re}\,\lambda_2|$.

<u>In infinite dimension:</u> study of the spectrum of A and A_2:

- $\mathfrak{S}(A) \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z \leqslant 0\}$ and $\mathfrak{S}(A_2) = \mathfrak{S}(A) \setminus \{0\} \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\}$: by strong connectivity.
  $\Rightarrow$ A is dissipative and A_2 is strictly dissipative.
- Spectrum of A = discrete spectrum and essential spectrum
- $A = K - M_S$ with K compact thus $\mathfrak{S}(K)$ countable and $\mathfrak{S}(M_S) = \mathrm{ess\,ran}(S)$
- Finally: $\mathrm{s}(A_2) = \sup\{\mathrm{Re}\, z \mid z \in \mathfrak{S}(A_2)\} < 0$ (spectral bound).
- Spectral mapping theorem $\Rightarrow$ spectral bound = spectral growth of e^{tA_2}.

$\Rightarrow$ exponential convergence to consensus, at sharp rate $|s(A_2)|$.

Further comments: discrete-time setting

Discrete time

$$y_i^{n+1} = \sum_{j=1}^{N} \gamma_{ij} \Delta t \, y_j^n \quad \Leftrightarrow \quad \frac{y_i^{n+1} - y_i^n}{\Delta t} = \sum_{j \neq i} \gamma_{ij} (y_j^n - y_i^n) \qquad 1 \leqslant i \leqslant N, \quad n \in \mathbb{N}$$

If the graph associated with σ is strongly connected then

$$\exists \rho_* \in (0,1) \quad \exists M_* > 0 \quad | \quad \|y^n - \bar{y}^v\| \leqslant M_* \, \|y^0 - \bar{y}^v\| \, \rho_*^n \qquad \forall n \in \mathbb{N}.$$

Further comments: kinetic limit

Kinetic limit

Passing to the kinetic limit when $N \to +\infty$ in

$$\dot{x}_i(t) = 0, \qquad \dot{\xi}_i(t) = \frac{1}{N}\sum_j \sigma(x_i, x_j)(\xi_j(t) - \xi_i(t)) \qquad \text{with} \quad \sigma(x_i, x_j) = \sigma_{ij}$$

gives a probability measure $\mu(t) = f(t, x, \xi)\, dx\, d\xi$ on $\Omega \times \mathbb{R}^d$ solution of

$$\boxed{\partial_t \mu + \mathrm{div}_\xi(X[\mu]\mu) = 0}$$

with

$$X[\mu](x, \xi) = \int_{\Omega \times \mathbb{R}^d} \sigma(x, x')(\xi_* - \xi)\, \frac{1}{F(x')}\, d\mu(x', \xi_*) \qquad \text{and} \quad F(x) = \int_{\mathbb{R}^d} f(t, x, \xi)\, d\xi$$

and we have

$$\boxed{y(t, x) = \frac{1}{F(x)} \int_{\mathbb{R}^d} \xi\, f(t, x, \xi)\, d\xi = \frac{\int_{\mathbb{R}^d} \xi\, f(t, x, \xi)\, d\xi}{\int_{\mathbb{R}^d} f(t, x, \xi)\, d\xi}}$$

(see ongoing works with T. Paul)

Further comments: weighted variance

Weighted expectation

$$\mathbb{E}_v(y) = \langle y, v\rangle = \langle y, e\rangle_v = \begin{cases} \displaystyle\sum_{i=1}^{N} v_i y_i & \text{if} \quad X = \mathbb{R}^N \\ \displaystyle\int_\Omega v(x)y(x)\,dx & \text{if} \quad X = L^2(\Omega) \end{cases}$$

Note that $\bar{y}^v = \mathbb{E}_v(y)e$.

Weighted variance

$$\mathrm{Var}_v(y) = \mathbb{E}_v\left((y - \mathbb{E}_v(y))^2\right) = \mathbb{E}_v(y^2) - \mathbb{E}_v(y)^2 = \|\pi y\|_v^2$$

In finite dimension:

$$\mathrm{Var}_v(y) = \sum_{i=1}^{N} v_i(y_i - \langle y, e\rangle_v)^2 = \sum_{i=1}^{N} v_i y_i^2 - \langle y, e\rangle_v^2 = \frac{1}{2}\sum_{i,j=1}^{N} v_i v_j (y_i - y_j)^2$$

In infinite dimension:

$$\begin{aligned}\mathrm{Var}_v(y) = \int_\Omega v(x)(y(x) - \bar{y}^v)^2\,dx &= \int_\Omega v(x)y(x)^2\,dx - (\bar{y}^v)^2 \\ &= \frac{1}{2}\iint_{\Omega^2} v(x)v(x')(y(x) - y(x'))^2\,dx'\,dx\end{aligned}$$

Further comments: weighted variance

Setting $V_v(t) = \frac{1}{2}\mathrm{Var}_v(y(t))$ where $\dot{y}(t) = Ay(t)$, we have

$$\dot{V}_v = \langle y, Ay\rangle_v = -Q(y) = -Q_2(\pi y)$$

where

$$Q(y) = -\langle y, Ay\rangle_v = \begin{cases} \displaystyle\frac{1}{2}\sum_{i,j=1}^{N} v_i\,\sigma_{ij}(y_j - y_i)^2 & \text{in finite dimension} \\ \displaystyle\frac{1}{2}\iint_{\Omega^2} v(x)\sigma(x,x')(y(x)-y(x'))^2\,dx'\,dx & \text{in infinite dimension} \end{cases}$$

$$Q_2(z) = -\langle z, A_2 z\rangle_v$$

We recover convergence to consensus thanks to the LaSalle invariance principle.

$\rightarrow$ This is an "L^2 theory" in the non-symmetric case.

Open issues

- Use the v-weighted variance as a Lyapunov functional in control problems.
- Incorporate noise and/or nonlinearities in the system and establish robustness.
- Study the case of $\sigma(t)$ or $\sigma(|x_i - x_j|)$ and obtain the sharp asymptotic convergence rate.
- Study non-symmetric second-order models (generalized Cucker-Smale models).

Ongoing work: control of vote opinions

(with L. Boudin and F. Salvarani)

In finite dimension:

$$\dot{y}_i(t) = \underbrace{\sum_{j=1}^{n} \sigma_{i,j} \left(y_j(t) - y_i(t)\right)}_{Ay(t):\ \text{binary interactions}} + \underbrace{\int_0^t \beta(t-s)\left(y_i(s) - y_i(t)\right)\, ds}_{\text{memory term: self-thinking}} + \underbrace{\sum_{k=1}^{m} \alpha_k \left(u_k(t) - y_i(t)\right)}_{\text{influence of media}}$$

$u_k(t)$: opinion provided by media $\rightarrow$ control
α_k: influence amplitude

In infinite dimension:

$$\partial_t y(t,x) = (Ay(t))(x) + \int_0^t \beta(t-s)(y(s,x) - y(t,x))\, ds + \sum_{k=1}^{m} \alpha_k (u_k(t) - y(t,x))$$

Theorem

Assume that $\int_0^{+\infty} |\beta(s)|\, ds < \bar{\alpha} = \sum_{k=1}^m \alpha_k$. For every $\bar{y} \in \mathbb{R}$, any m-tuple of constant controls $(u_1, \dots, u_m)$ such that $\sum_{k=1}^m \alpha_k u_k = \bar{\alpha}\bar{y}$ steers exponentially all solutions to the point $\bar{y}e$.

Proof by using the Lyapunov function $V(t) = \mathrm{Var}_v(y(t)) + C \int_0^t \int_t^{+\infty} |\beta(u-s)|\, du\, \|y(s)\|^2\, ds$.

Ongoing work: connectivity optimization

(with N. Ayi, L. Boudin and N. Pouradier Duteil)

Question

How to choose at best σ, among all functions such that $0 \leqslant \sigma \leqslant \sigma_{\max}$ and $\iint_{\Omega^2} \sigma = 1$, so as to maximize the exponential decay rate?

In some sense we seek to "maximize the connectivity" of the graph.

Thanks to the weighted variance and to a probabilistic argument, we model the problem as

$$\boxed{\sup_{\substack{0 \leqslant \sigma \leqslant \sigma_{\max} \\ \iint \sigma = 1}} \inf_{x \in \Omega} \left(\int_\Omega \sigma(x, x')\, dx' \ v_\sigma(x) \right)}$$

This is a very nonlinear problem.

For the moment, we know that a σ such that $0 < \sigma < \sigma_{\max}$ cannot be a maximizer...