Repetitive control for nonlinear systems

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Outline

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Introduction to Output Regulation

- Consider the following plant

$$\begin{cases} \dot{z} &= f(z, w, u) \\ e &= h(z, w) \\ y &= r(z, w) \end{cases}$$

- **Robust output regulation:** design

$$\begin{cases} \dot{\eta} &= \varphi(\eta, e, y) \\ u &= \beta(\eta, e, y) \end{cases}$$

such that robustly with respect to model uncertainties f, h:

i) bounded trajectories
ii) asymptotic regulation $\lim_{t\to\infty} e(t) = 0$

- **Standing assumption:** we know a model generator for w

$$\dot{w} = s(w)$$

which is "neutrally/critically stable"

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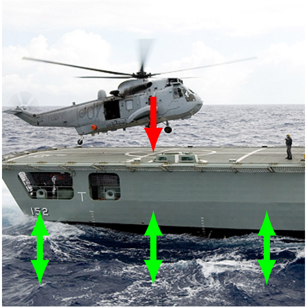
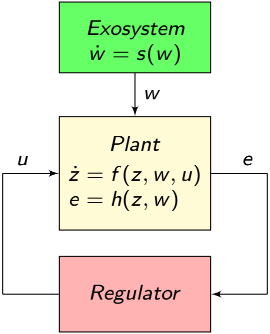
An Illustration

- Many applications can be put in this context:
  - Tracking
  - Disturbance rejection

Example: helicopter landing on a boat





Regulation goals:
- $e \to 0$
- robustly

The conceptual formulation

Output regulation: stabilization to a **non-trivial unknown manifold**

regulator equations: $$\left\{ \begin{array}{rcl} \frac{\partial \pi(w)}{\partial w} s(w) & = & f(\pi(w), w, c(w)) \\ 0 & = & h(\pi(w), w) \end{array} \right.$$

- $\pi(w)$: the steady-state manifold on which $e = 0$
- $c(w)$: the "friend", i.e. the steady-state input which makes π invariant

Peculiarity: characterization of the class of all possible exogenous inputs (disturbances/ references) as the set of all possible solutions of a fixed known differential equation

The exosystem-generated disturbances/references is a trade-off between:

- worst case disturbance (H_∞ control): too pessimistic
- exact knowledge of w (inversion-based control): too optimistic

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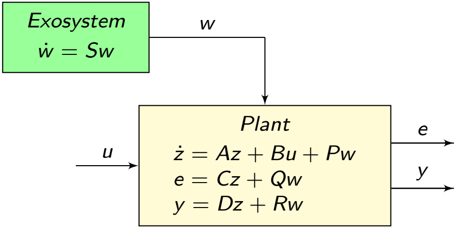
The solution in the linear case



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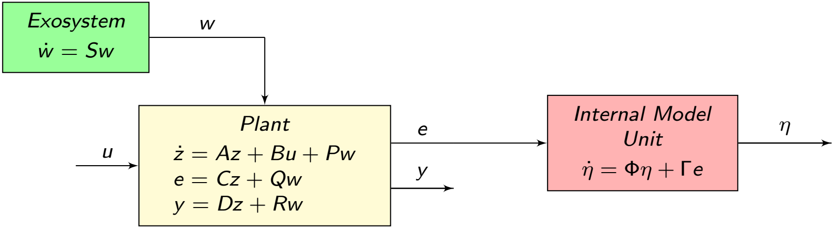

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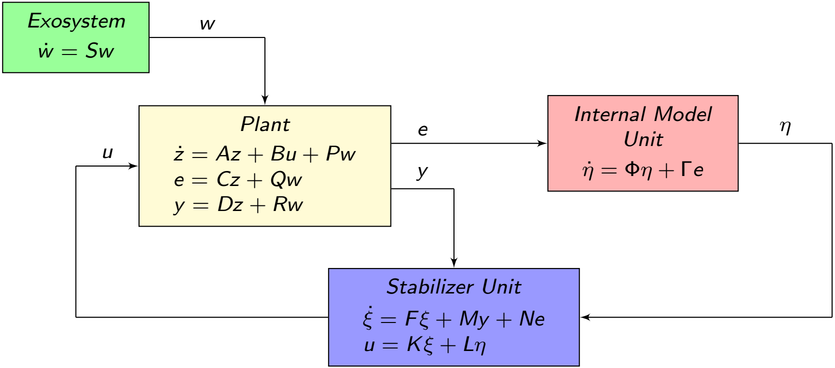

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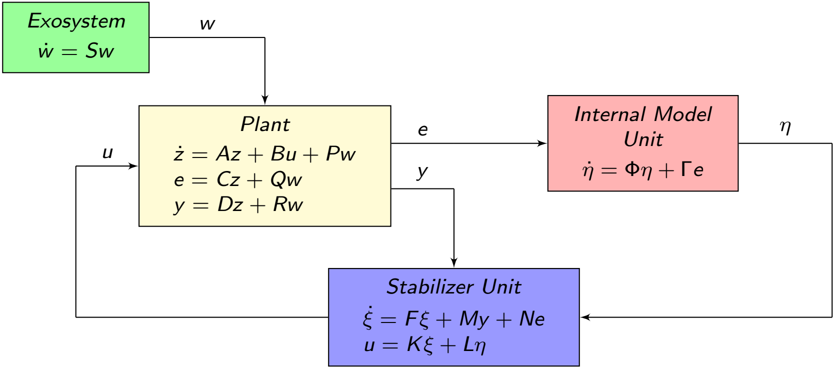

1) Design an Internal Model Unit containing a copy of the exosystem: $\sigma(\Phi) = \sigma(S)$
2) Design a Stabilizer Unit stabilizing the extended plant
3) *Magically*, steady-state solutions satisfy $e = 0$ even with model parameter uncertainties

The internal model principle for linear systems

The internal model principle (Francis and Wonham 1976)

The robust output regulation problem is solved IFF the regulator "incorporates a copy of the dynamic structure of the disturbance and reference signals".

Examples:

- Integral action $\dfrac{1}{s}$ to track/reject constant signals $w(t) = w_0$
- Linear oscillator $\dfrac{1}{s^2 + \omega^2}$ to track/reject sinusoidal signals $w(t) = a\sin(\omega t + \varphi)$

Remarks:

- This principle was proved for finite-dimensional linear systems and extended afterwards to infinite-dimensional linear operators

  [Paunonen, Pohjolainen, SIAM 2010]

- Robustness is referred to parametric uncertainties
- For (finite-dimensional) nonlinear systems, the theory is still incomplete and things are more complicated!

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Interpretation of the internal-model principle

- The eigenvalues of the internal-model unit becomes zeros in the transfer function: zero-blocking effect

$$e(s) = H(s)w(s), \qquad H(s) = \frac{s(s^2+\omega^2)(\cdots)}{s^d + a_1 s^{d-1} + \cdots + s a_{d-1} + a_d},$$

- Since the closed-loop system is stable with $w = 0$, the closed-loop system has bounded trajectories with $w \neq 0$: analysis of steady-state trajectories of

$$\dot{\eta} = e, \qquad \dot{\eta} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \eta + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e$$

implies $e = 0$ based on resonance arguments

A simple nonlinear example

Consider a system

$$\dot{z} = -z^3 - z + u$$
$$e = z - r(t)$$

and suppose that the reference is given by

$$r(t) = \sin(\omega t)$$

Then the steady-state pair $\pi(t), c(t)$ is computed as

$$\pi(t) = \sin(\omega t)$$
$$c(t) = \cos(\omega t) + \tfrac{7}{4}\sin(\omega t) - \tfrac{1}{4}\sin(3\omega t)$$

The cubic term z^3 adds a high-order harmonic in $c(t)$!!!

Conceptual difficulties

- Nonlinearities introduces high-order deformations that may be not present in the exosystem
- Nonlinearities may be introduced by model uncertainties and/or by the Stabilizer Unit

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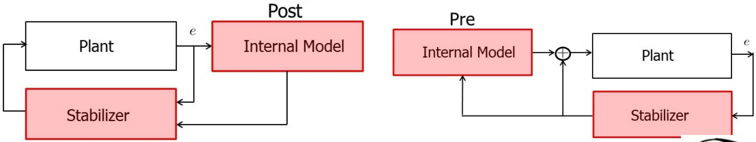
✗ Chicken-egg dilemma: first design Stabilizer Unit and then Internal Model Unit or vice-versa?



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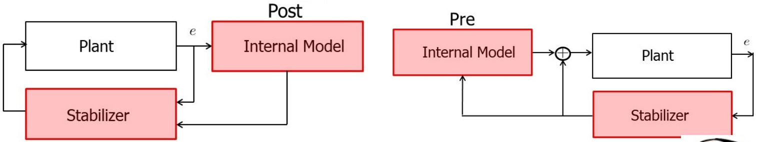

✗ Robustness to model uncertainties in which sense?

Some Milestones

Linear output regulation and internal model principle

- Francis, Wonham (1976), *The internal model principle of control theory*, Automatica
- Davison (1976), *The robust control of a servomechanism problem for linear time-invariant multivariable systems*, IEEE TAC
- Paunonen, Pohjolainen (2010), *Internal model theory for distributed parameter systems*, SICON

Necessary conditions, regulator equations and design of asymptotic regulators

- Byrnes, Isidori (2003), *Limit Sets, Zero Dynamics, and Internal Models in the Problem of Nonlinear Output Regulation*, IEEE TAC
- Byrnes, Isidori (2004), *Nonlinear internal models for output regulation*, IEEE TAC
- Marconi, Praly, Isidori (2007), *Output stabilization via nonlinear Luenberger observers*, SICON

Outline

The context of periodic signals

- Let's focus on signals w which are T-periodic, with T known
- Our regulation problem becomes:

$$\begin{cases} \dot{z} & = & f(z, w, u) \\ e & = & h(z, w) \end{cases}$$

Goal:

Find a dynamical regulator such that, for all T-periodic signals w, closed-loop trajectories are bounded and $\lim_{t \to \infty} e(t) = 0$.

Main idea:

The regulator must be able to generate any possible T-periodic signal at steady-state

The context of periodic signals

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Repetitive control: some background

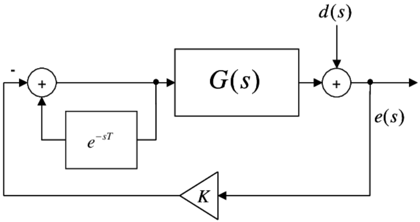
- Introduced by Hara, Yamamoto, Omata, Nakano in the 1988 for linear systems

- It consists of introducing the transfer function

$$R(s) = \frac{\exp(-Ts)}{1 - \exp(-Ts)}$$

  in the closed-loop system

- $R(s)$ is a universal generator of T-periodic signals



Repetitive Control: some comments

- ✗ Repetitive control schemes have been developed mostly for linear systems: the proof is based on transfer function analysis (Nyquist..)
- ✗ It is shown to work only for systems with zero relative degree between input and regulated output
- How to analyse the interconnection with a nonlinear system ?

Some bibliography:

- Hara, Yamamoto, Omata, Nakano (1988), *Repetitive control system: A new type servo system for periodic exogenous signals*, IEEE TAC
- Weiss, Häfele (1999), *Repetitive control of MIMO systems using H_∞ design*, Automatica
- Verrelli, Tomei (2023), *Adaptive learning control for nonlinear systems: A single learning estimation scheme is enough*, Automatica.

Repetitive Control as a transport equation

A delay

$$v(t) = e(t - T)$$

can be equivalently expressed with a transport equation of the form:

$$\begin{aligned} \partial_t \eta(t,x) &= -\frac{1}{T}\partial_x \eta(t,x) && \forall\,(t,x) \in \mathbb{R}_+ \times [0,1], \\ \eta(t,0) &= e(t) && \forall\, t \in \mathbb{R}_+, \\ \eta(0,x) &= 0 && \forall\, x \in [0,1], \\ v(t) &= \eta(t,1) && \forall\, t \in \mathbb{R}_+. \end{aligned}$$

Indeed, the general solution to such a hyperbolic PDE is given by

$$\eta(t + (x - x')T, x) = \eta(t, x')$$

from which we obtain

$$v(t+T) = \eta(t+T, 1) = \eta(t, 0) = e(t)$$

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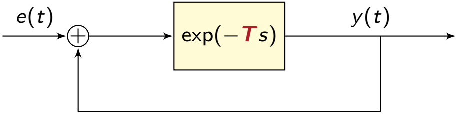
Repetitive Control: equivalent representations

- Transfer function RC



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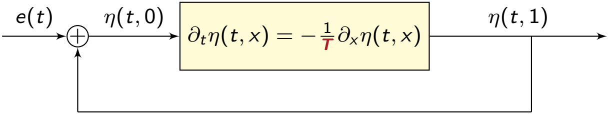


The context of periodic signals - rephrased

- Our regulation problem becomes:

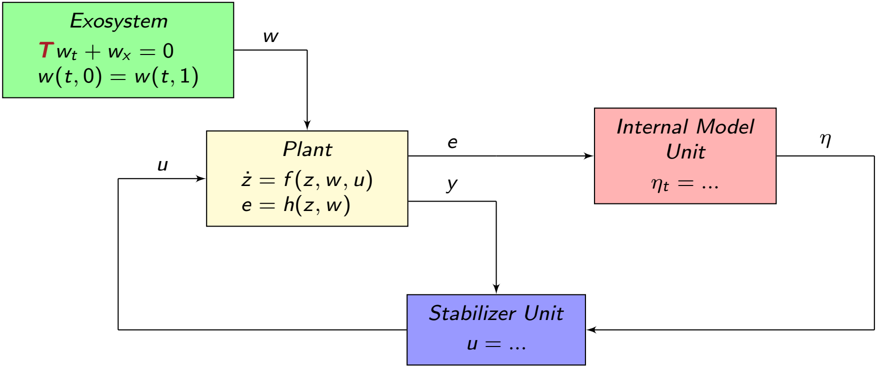
$$\begin{cases} \dot{z} &= f(z, w, u) \\ e &= h(z, w) \end{cases}$$

where w is generated as

$$\begin{aligned} &\partial_t w(t,x) + \frac{1}{T}\partial_x w(t,x) = 0, \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ &w(t,0) = w(t,1), \\ &w(0,x) = w_0(x). \end{aligned}$$

- Internal model property: we need to incorporate such a PDE in the control scheme

The overall-scheme



Open problem:

- Design of the internal model unit
- Design of the stabilizer

Design of the internal-model unit

- We need an interconnection term
- Boundary interconnection:

$$\begin{aligned} \partial_t \eta(t,x) &= -\tfrac{1}{T}\partial_x \eta(t,x) && \forall\,(t,x) \in \mathbb{R}_+ \times [0,1], \\ \eta(t,0) &= \eta(t,1) + \gamma e(t) && \forall\, t \in \mathbb{R}_+, \\ \eta(0,x) &= 0 && \forall\, x \in [0,1], \end{aligned}$$

- Distributed interconnection:

$$\begin{aligned} \partial_t \eta(t,x) &= -\tfrac{1}{T}\partial_x \eta(t,x) + \gamma(x) e(t) && \forall\,(t,x) \in \mathbb{R}_+ \times [0,1], \\ \eta(t,0) &= \eta(t,1) && \forall\, t \in \mathbb{R}_+, \\ \eta(0,x) &= 0 && \forall\, x \in [0,1], \end{aligned}$$

- Both designs are possible

Design of the stabilizer unit ???

- The resulting extended system reads

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- There exists a feedback $u = \alpha(z, e, \eta)$ such that the closed-loop system is..??

Which property do we need?

Transient and asymptotic behaviours

- When $w = 0$ for all $t \geq 0$ we would like the closed-loop system origin to be asymptotically stable

- When $w \neq 0$ and $w(t + \boldsymbol{T}) = w(t)$ we would like the closed-loop trajectories to converge to a $\boldsymbol{T}$ periodic solution

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- Resonance arguments:

$$\begin{aligned} \partial_t \eta(t,x) &= -\tfrac{1}{\boldsymbol{T}} \partial_x \eta(t,x) & \qquad \forall\, (t,x) \in \mathbb{R}_+ \times [0,1], \\ \eta(t,0) &= \eta(t,1) + e(t) & \qquad \forall\, t \in \mathbb{R}_+, \end{aligned}$$

  if both η and e are $\boldsymbol{T}$-periodic and bounded, then $e(t) = 0$ for all t

- Equivalently, the corresponding solution $\bar{\eta}$ at any instants $t, t + \boldsymbol{T}, \dots, t + N\boldsymbol{T}$ is given by

$$\bar{\eta}(k+1) = \bar{\eta}(k) + \bar{e}(k)$$

  with $\bar{e}(k) = \bar{e}_0$ for all k, which implies $\bar{e}_0 = 0$.

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- In finite dimensional literature the aformentioned property is also referred to as entrainment to periodic inputs

Entrainment to periodic inputs property: a sufficient condition

Incremental input-to-state stability (δISS)

A system

$$\dot{z} = f(w, z)$$

is δISS with respect to w if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any two initial conditions z_a, z_b and pair of inputs w_a, w_b then the corresponding solutions satisfy

$$|Z(t, x_a) - Z(t, x_b)| \leq \beta(t, |z_a - z_b|) + \sup_{s\in[0,t]} \gamma(|w_a(s) - w_b(s)|_\infty)$$

δISS $\implies$ Entrainment to periodic inputs

Suppose the system

$$\dot{z} = f(w, z)$$

is δISS w.r.t. w. Then, it has the Entrainment to periodic inputs property, namely, for any T-periodic w, then z has an asymptotically stable T-periodic trajectory.

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Design of the stabilizer unit

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- How to design a feedback to stabilize the previous system to obtain the desired property?
- Few design techniques for nonlinear ODE - PDE interconnections..

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Minimum-phase systems

- We focus on systems in the following form

$$\dot{\zeta} = f(w, \zeta, e)$$
$$\dot{e} = q(w, \zeta, e) + u$$

with $z = (\zeta, e) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and satisfying the following:

ASS 1: Minimum phase: the system $\dot{\zeta} = f(w, \zeta, e)$ is δISS with respect to w and e

ASS 2: Lipschitzness: the function q is globally Lipschitz

- **Step 1:** Under the previous assumptions, the feedback law

$$u = -\boldsymbol{\sigma} e + v, \qquad \boldsymbol{\sigma} > 0$$

makes the (ζ, e) dynamics δISS with respect to w and v

Feedback design

- We have an extended system

$$\dot{\zeta} = f(w,\zeta,e) \qquad \partial_t \eta(t,x) = -\tfrac{1}{T}\partial_x \eta(t,x)$$
$$\dot{e} = q(w,\zeta,e) - \sigma e + v \qquad \eta(t,0) = e(t) + \eta(t,1)$$

- **Step 2:** we add a second stabilizing term for the η-dynamics designed as

$$v(t) = \mu \int_0^1 \big[\eta(t,x) - M(x)e(t)\big] M(x)\, dx$$

with $M : [0,1] \to \mathbb{R}$ defined as solution to the following two-boundary value problem

$$\begin{cases} M'(x) = \sigma\, TM(x)\,, \\ M(0) = M(1) + 1\,, \end{cases} \qquad M(x) = \frac{\exp(\sigma T x)}{1 - \exp(\sigma T)}.$$

Feedback design

- The feedback law

$$v(t) = \mu \int_0^1 \left[\eta(t,x) - M(x)e(t)\right] M(x)\, dx$$

  is inspired by the forwarding-approach developed in the 90's for cascade of nonlinear systems

- The e-dynamics is fast and we aim at stabilizing the η-dynamics to a manifold that depends on e

- We build on a Lyapunov functional of the form

$$\int_0^1 (\eta(t,x) - M(x)e(t))^2 dx$$

  in order to stabilize η on the manifold $M(x)e(t)$..

- If e goes to zero we can hope that $\eta(t,x)$ goes to zero as well!

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Robust Asymptotic Regulation with Repetitive Control

$$\dot{\zeta} = f(w, \zeta, e) \qquad \partial_t \eta(t,x) = -\tfrac{1}{T}\partial_x \eta(t,x) \quad x \in [0,1]$$

$$\dot{e} = q(w, \zeta, e) + u \qquad \eta(t,1) = \eta(t,0) + e(t)$$

$$u = -\sigma e + \mu \int_0^1 (\eta(t,x) - M(t,x)e) M(x) dx$$

Theorem (Repetitive Control)

Select $\sigma > 0$ large enough and $\mu > 0$. Then, for any initial conditions in $\mathbb{R}^n \times \mathbb{R} \times L^2(0,1)$ and $w \in C^2([0,T], \mathbb{R}^{n_w})$:

- *Stability Requirement: $(\zeta(t), e(t), \eta(t))$ bounded in $\mathbb{R}^n \times \mathbb{R} \times L^2(0,1)$*
- *Steady-State: T periodic $(\bar{\zeta}, \bar{e}, \bar{\eta})$ asymptotically stable*
- *Asymptotic Behavior: $\bar{e} = 0$*
- *Robustness Requirement: for any (f, q) "close" to the nominal one*

Sketch of the proof

- **Step 1:** show the existence of the internal-model property:
  $\forall$ T-periodic function $q(t, \bar{z}(t), 0)$ $\exists$ an initial condition $\bar{\eta}(x)$ such that

$$\begin{aligned}
\partial_t \bar{\eta}(t,x) &= -\tfrac{1}{T}\partial_x \bar{\eta}(t,x) & \forall\,(t,x) \in \mathbb{R}_+ \times [0,1],\\
\bar{\eta}(t,0) &= \bar{\eta}(t,1) & \forall\, t \in \mathbb{R}_+,\\
\bar{\eta}(0,x) &= \bar{\eta}(x) & \forall\, x \in [0,1],\\
q(t,\bar{z}(t),0) &= -\mu \int_0^1 \bar{\eta}(t,x) M(x) dx
\end{aligned}$$

- **Step 2:** Show the stability of $(\bar{z}(t), 0, \bar{\eta}(t,x))$ via the Lyapunov functional

$$V(z,e,\eta) = W(z,\bar{z}(t)) + e^2 + \mu \int_0^1 (\eta(t,x) - \bar{\eta}(t,x) - M(x)e(t))^2 dx$$

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Outline

Some comments

- ✗ Infinite-dimensional controllers are not implementable
- ✗ Using a delay in the feedback loop may generate instability in the presence of input-delays (e.g. physical actuator with some delay)
- ✗ The "realization" of the controller can achieve only practical regulation

Finite-dimensional continous-time realization

Given a system of the form

$$\dot{\zeta} = f(w, \zeta, e)$$
$$\dot{e} = q(w, \zeta, e) + u$$

we look for a controller of the form

$$\dot{\eta}_n = \Phi_n \eta_n + G_n e$$
$$u = -\sigma e + \mu M_n^\top (\eta_n - M_n e)$$

with $\eta_n \in \mathbb{R}^n$ in which the matrices (Φ_n, G_n, M_n) to be chosen such that

$$\lim_{t\to\infty} |e(t)| \leq \varepsilon$$

How to chose the realization?

- System

$$\dot{\eta}_n = \Phi_n \eta_n + G_n e$$
$$u = -\sigma e + \mu M_n^\top (\eta_n - M_n e)$$

is an approximation of

$$\frac{\partial}{\partial t}\eta(t,x) = \frac{1}{T}\frac{\partial}{\partial x}\eta(t,x)$$
$$\eta(t,1) = \eta(t,0) + e(t)$$
$$u(t) = -\sigma e(t) + \mu \int_0^1 M(x)(\eta(t,x) - M(x)e(t))dx \ ,$$

- Note that the objective is not to approximate as better as possible the PDE but to minimize ε for closed-loop solutions

$$\lim_{t\to\infty} |e(t)| \leq \varepsilon$$

An approximation based on Fourier series

We select the following approximation based on Fourier series:

$$\Phi_{2L+1} = \mathrm{blkdiag}(0, S, \ldots, LS), \qquad S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \omega = \frac{2\pi}{T}$$

$$G_{2L+1} = \mathrm{col}(1, G, \ldots, G) \qquad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

that is:

$$\begin{aligned} \dot\eta_0 &= 0 \\ \dot\eta_k &= kS\eta_k + Ge \quad k \in \{1, 2, \ldots, L\} \end{aligned}$$

Harmonic internal model property

Consider the internal model unit

$$\dot{\eta} = S\eta + Ge, \qquad S = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \quad \omega = \frac{2\pi}{T}, \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and suppose that both (η, e) have a steady-state T-periodic trajectory (η_{ss}, e_{ss}).

They satisfy:

$$\eta_{ss}(t+T) = \exp(ST)\eta(t) + \int_t^{t+T} \exp(S(T+t-s))Ge(s)ds$$

Using $\exp(ST) = I$ it yields:

$$0 = \int_0^T \cos(\omega t)e_{ss}(t) = \int_0^T \sin(\omega t)e_{ss}(t)$$

The Fourier coefficient of e_{ss} associated to the harmonic of S is zero!

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Towards Harmonic Regulation

- Suppose e_{ss} is C^1.. then it can be expressed as a Fourier series

$$e_{ss}(t) = c_0 + \sum_{k=1}^{\infty} c_k^s \sin(k\omega t) + c_k^c \cos(k\omega t)$$

and we now that $c_1^s = c_1^c = 0$

- If enough Fourier coefficients are zero, we can reduce the L^2 norm of e_{ss}..

$$\text{Parseval's theorem:} \quad \frac{1}{T}\int_0^T e_{ss}^2(t)dt = \frac{c_0^2}{2} + \sum_{k=1}^{\infty}\left((c_k^s)^2 + (c_k^c)^2\right)$$

- Harmonic Regulation Objective: we try to nullify the Fourier coefficients of the steady-state error output

Harmonic Regulation

$$\dot\zeta = f(w,\zeta,e) \qquad \dot\eta_0 = e$$

$$\dot e = q(w,\zeta,e) + u \qquad \dot\eta_k = kS\eta_k + Ge \quad k \in \{1,2,\ldots,L\}$$

$$-\sigma M_k = kSM_k + G \qquad u = -\sigma e + \mu \sum_{k=0}^{L} M_k^\top(\eta_k - M_k e)$$

Theorem (Uniform approximate regulation and asymptotic regulation)

Select σ large enough and $\mu > 0$. For any $L > 0$ all trajectories starting inside $Z \times E \times \Xi$ with any $w \in C^2([0,T],W)$ satisfy

- *Stability Requirement:* $(\zeta(t), e(t), \eta(t))$ *bounded for all* $t \geq 0$
- *Steady-State:* T *periodic* $(\zeta_L^\circ, e_L^\circ, \eta_L^\circ)$ *asymptotically stable*
- *Harmonic Regulation Requirement: the Fourier coefficients of* e_L° *at* $k\frac{2\pi}{T}$, $k = 0,1,\ldots,L$, *are zero*
- *Practical Regulation: for any* $b > 0$ *there exists* $L > 0$ *so that* $\int_0^T |e_L^\circ(t)|^2 dt \leq b$
- *Asymptotic Behavior:* $\lim_{L\to\infty} \limsup_{t\to\infty} |e_L^\circ(t)| = 0$
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Some comments

- First we fix σ to make the closed-loop dynamics δISS on some given compact set $Z \times E$
- All the values (domain of attraction, size of w, gain σ) are then uniform in L (e.g. they don't depend on L)
- We don't need to modify our pre-stabilizer when we add oscillators
- The L^2 norm of $e_L{}^\circ$ satisfies

$$\int_0^T |e_L^\circ(t)|^2 dt \leq \frac{\gamma}{L}$$

with $\gamma > 0$ independent of L

Outline

Numerical Simulation

$$\dot{z} = -z + z^2 - z^3 + e + \delta(t) \qquad \delta(t) = 2\exp(\sin(2t)) + 10\sin(t+0.1)$$

$$\dot{e} = \arctan(z) + e + u + \rho(t) \qquad \rho(t) = 0.5 + 5\sin(t-0.3)^3 + \cos(2t)$$

✓ Minimum-phase assumption verified: 0-GES when $e = 0$, $\delta(t) = 0$

✓ δ, ρ are $\boldsymbol{T} = 1$ periodic and smooth

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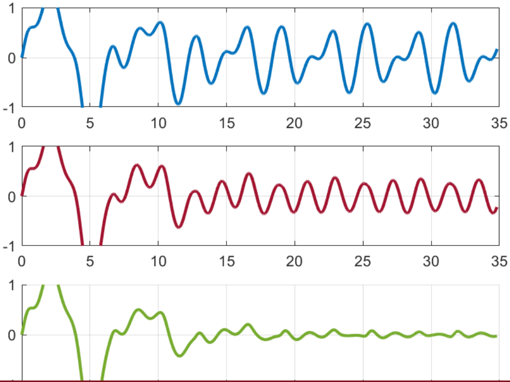
$$u = -\sigma e + \sigma \sum_{k=0}^{L} M_k^\top (\eta_k - M_k e), \qquad \sigma = 3, \qquad L < \infty$$



- $L = 1$ (number oscillators)
- $L = 2$ (number oscillators)
- $L = 3$ (number oscillators)

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L (number oscillators)	$\limsup_{t\to\infty} \lvert e(t)\rvert$
0	2.8613
1	0.7240
2	0.3420
3	0.0566
5	0.0210
10	0.0024
15	0.0004

Outline

Conclusions and Takeaway messages:

- We investigated the problem of robust asymptotic output regulation for periodic signals (references/disturbances)
- We proposed an infinite-dimensional regulator (repetitive-controller) solving the problem
- The realization of such a large-scale controller has to be done according to the properties that we want to achieve on the regulated output
- Using the Fourier basis provides better results than other basis (e.g. Tau-Legendre model/ Padé approximation or Chebyshev model)
- Comparison with low-pass filter based implementation or pure discrete-time realization?

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- Comparison with low-pass filter based implementation or pure discrete-time realization?

Future perspectives

- Other infinite-dimensional controllers for output regulation (different properties of the disturbances/references)
- Incremental stabilization techniques for interconnected (nonlinear) ODEs-PDEs
- Model reduction of PDE regulators with regulation objectives

Some References

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