Groupe de travail GDR
Homotopy limits and colimits

September 26, 2023

The main goal of this workshop is to study the notion of homotopy limit and homotopy colimit. The directions given in the program for each talk may serve as a guide, but they do not claim to be exhaustive. Management is, of course, left to the speaker, depending also on his or her judgment about the structure of the panel. The schedule will consist of four talks listed in detail below and a brief introduction, in which some basic definitions and examples will be recalled, such as the notions of homotopy category of a category with weak equivalences, homotopy pullbacks and pushouts, as well as some applications in homological algebra and topology.

Talk 0: Introduction to the workshop

- **Hour:** 10:00-10:45
- **Speaker:** Nicola Carissimi

Talk 1: Model categories

- **Hour:** 11:00 - 12:00
- **Speaker:** Paul Laubie
- **Reference:** [DS15]
- **Program:**
  - Definition of the notion of model category and examples of Top and $\mathsf{Ch}(R)$ ([DS15, §3])
  - Definition of the homotopy relation ([DS15, §4.22]).
  - Construction of the homotopy category and its localization property ([DS15, §5.6, Prop 5.8 and Thm 6.2]).
  - Definition of (total) derived functors ([DS15, §9 and Def 9.5]).
  - Construction of the homotopy pushout and pullback ([DS15, §10.4 and §10.8]).

Talk 2: Simplicial model categories

- **Hour:** 13:30 - 14:30
- **Speaker:** Grégoire Marc
- **References:** [DS15], [Hir17] and [BK72]
- **Program:**
  - Definition of simplicial sets and its underlying model category sSet ([BK72, §VIII.2.1 and §VIII.3.1-§VIII.3.4]).
  - Quillen adjunctions ([DS15, Theorem 9.7]).
  - The Quillen equivalence $|-| : \mathsf{sSet} \longleftrightarrow \mathsf{Top} : Sin$ ([BK72, §VIII.3.6])

- Simplicial model category ([Hir17, Defs 9.1.2 and 9.1.6]).
- Examples of Top and sSet ([Hir17, Defs 9.1.13 and 9.1.15]).
- (Perhaps Tot and geometric realization in sSet, [Dug15, §3.1 and §5.3])

Talk 3: Homotopy limits and colimits in a simplicial model category

- **Hour:** 14:45 - 15:45
- **Speaker:** Quentin Lapie
- **References:** [Dug15] and [Hir17]
- **Program:**
  - Definition of holim and hocolim ([Hir17, §18.1], and perhaps [Dug15, §4.5 and §4.13])
  - Examples: homotopy pullback, homotopy pushout and the classifying space as a hocolim computation ([Dug15, Ex 4.11] and [Hir17, Prop 18.1.6])
  - The morphism hocolim $\longrightarrow$ colim ([Dug15, §4.10])
  - Homotopy invariance theorem ([Hir17, Thm 18.5.1])
  - Mapping space adjunction formula ([Hir17, Thm 18.1.10]).

Talk 4: Holim and hocolim as derived functors

- **Hour:** 16:00 - 17:00
- **Speaker:** Francesca Pratali
- **References:** [Dug15] and [Rie09]
- **Program:**
  - Definitions and existence of projective and injective model structures on $\mathcal{M}^{\mathcal{I}}$ ([Dug15, §13])
  - Existence of $\mathbb{L}\mathsf{colim}$ and $\mathbb{R}\mathsf{lim}$ with the appropriate model structure ([Dug15, §13] and perhaps [Rie09, §4.3])
  - Comparison of $\mathbb{L}\mathsf{colim}$ and $\mathbb{R}\mathsf{lim}$ with $\mathsf{hocolim}$ and holim when working on an appropriate simplicial model category ([Dug15, §9.10] and perhaps [Rie09, §5])
  - Adjunction $\mathsf{holim} \dashv \Delta \dashv \mathsf{hocolim}$ ([Rie09, §4.2])
  - Application: hocolim of a tower of cofibrations ([Dug15, Prop 14.10]).

References

[BK72] Bousfield and Kan. Homotopy limits, completions and localizations. https://people.math.rochester.edu/faculty/doug/otherpapers/bk-yellow.pdf, 1972.

[DS15] Dwyer and Spalinski. Homotopy theories and model categories. https://math.jhu.edu/~eriehl/616-s16/DwyerSpalinski.pdf, 2015.

[Dug15] Daniel Dugger. A primer on homotopy colimits. https://pages.uoregon.edu/ddugger/hocolim.pdf, 2015.

[Hir17] Philip S. Hirschhorn. Model categories and their localizations. https://people.math.rochester.edu/faculty/doug/otherpapers/hirschhornloc.pdf, 2017.

[Rie09] Emily Riehl. Homotopy (limits and) colimits). https://emilyriehl.github.io/files/hocolimits.pdf, 2009.