Convergence of a Finite-Volume scheme for a heat equation with multiplicative noise

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joint work with Caroline Bauzet†, Kerstin Schmitz‡ and Aleksandra Zimmermann‡

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The continuous problem

$$\begin{cases} du - \Delta u \, dt = g(u) \, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
  - $W(0) = 0$ almost surely;
  - $W(t)$ is almost surely continuous in t;
  - $\forall 0 \leq s \leq t$, $W(t) - W(s) \sim \mathcal{N}(0, t-s)$;
  - $\forall 0 \leq t^1 \leq t^2 \leq \cdots \leq t^k$, $W(t^1)$, $W(t^2) - W(t^1)$, $\cdots$, $W(t^k) - W(t^{k-1})$ are independent.
- $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is a $\mathcal{F}_0$-measurable random variable.

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Definition

A variational solution to the heat equation with multiplicative Lipschitz noise is a $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process

$$u \in L^2(\Omega; \mathcal{C}([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

such that, for all $t \in [0,T]$, in $L^2(\Lambda)$, $\mathbb{P}$-a.s. in Ω,

$$u(t) - u_0 - \int_0^t \Delta u(s)\, ds = \int_0^t g(u(s))\, dW_s.$$

$\rightsquigarrow$ From classical results existence and uniqueness of a variational solution is well-known. **Pardoux(1975), Krylov-Rozovskii(1981), Liu-Röckner(2015), ...**

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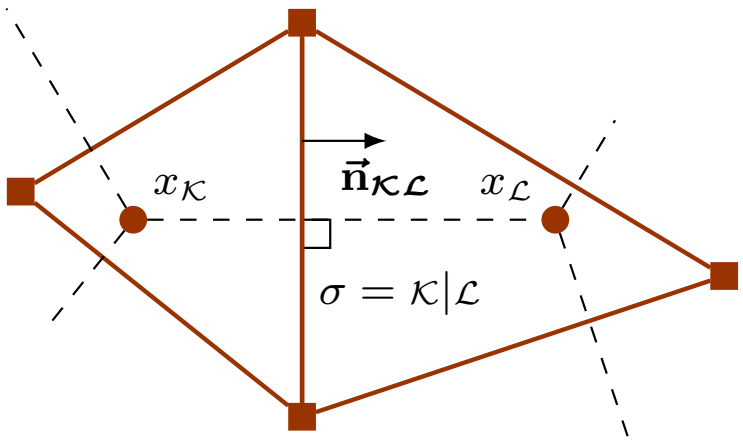
Goal: Convergence of the TPFA scheme towards the variational solution.

Consider the following Laplace problem

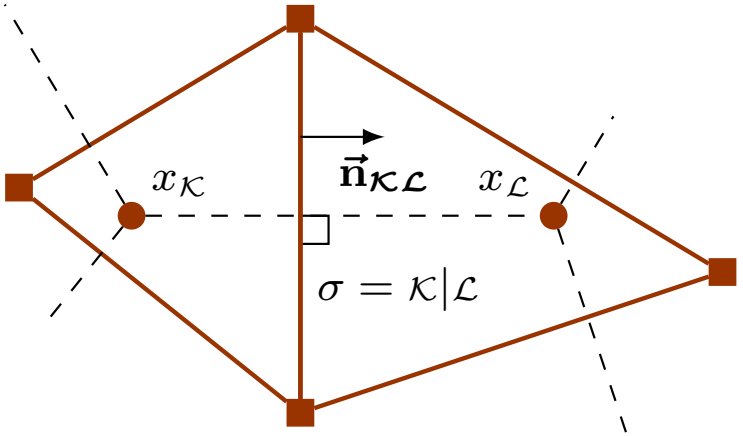
$$\begin{cases} -\Delta u = f, \text{ in } \Lambda \\ \quad \text{B.C}, \quad \text{on } \partial\Lambda. \end{cases}$$

and an admissible orthogonal mesh $\mathcal{T}$.



Flux balance equation on the control volume $\mathcal{K}$:

$$m_{\mathcal{K}} f_{\mathcal{K}} \overset{\text{def}}{=} \int_{\mathcal{K}} f = \int_{\mathcal{K}} -\Delta u = \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}} \underbrace{-\int_{\sigma} \nabla u \cdot \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}}_{\overset{\text{def}}{=} \overline{F}_{\mathcal{K},\sigma}(u)}.$$

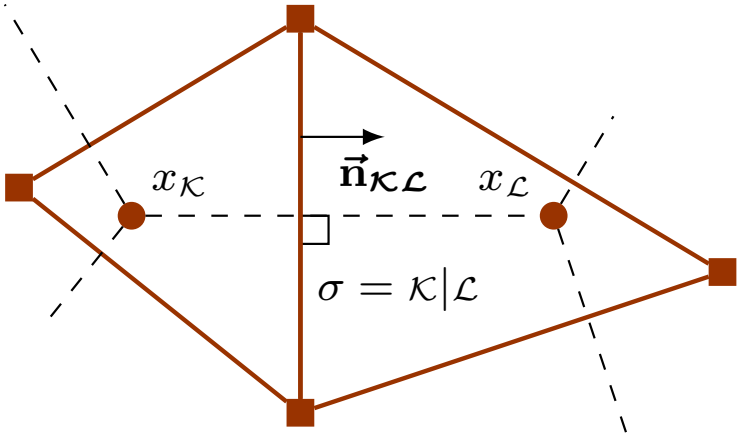


Cell-centered unknowns

$$m_\mathcal{K} f_\mathcal{K} = \sum_{\sigma \in \mathcal{E}_\mathcal{K}} \overline{F}_{\mathcal{K},\sigma}(u).$$

We are looking for $u_\mathcal{K} \sim u(x_\mathcal{K})$.

Notation : $u_\mathcal{T} = (u_\mathcal{K})_{\mathcal{K}\in\mathcal{T}} \in \mathbb{R}^\mathcal{T}$.



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Case of an interior edge

$\sigma \in \mathcal{E}_{int}$, $\sigma = \mathcal{K}|\mathcal{L}$.

$$u(x_\mathcal{L}) - u(x_\mathcal{K}) = (\nabla u(x)) \cdot (x_\mathcal{L} - x_\mathcal{K}) + \mathcal{O}\left(\text{size}(\mathcal{T})^2\right)$$

$$(x_\mathcal{L} - x_\mathcal{K}) = d_{\mathcal{K},\mathcal{L}} \vec{\mathbf{n}}_{\mathcal{KL}} \Rightarrow (\nabla u(x)) \cdot \vec{\mathbf{n}}_{\mathcal{KL}} = \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K},\mathcal{L}}} + \mathcal{O}\left(\text{size}(\mathcal{T})\right)$$

$$\Rightarrow \overline{F}_{\mathcal{K},\sigma}(u) = -\int_\sigma \nabla u \cdot \vec{\mathbf{n}}_{\sigma\mathcal{K}} = -m_\sigma \frac{u(x_\mathcal{L}) - u(x_\mathcal{K})}{d_{\mathcal{K},\mathcal{L}}} + \mathcal{O}\left(\text{size}(\mathcal{T})^2\right).$$

Thus, we define

$$F_{\mathcal{K},\sigma}(u_\mathcal{T}) \overset{\text{def}}{=} -m_\sigma \frac{u_\mathcal{L} - u_\mathcal{K}}{d_{\mathcal{K},\mathcal{L}}}.$$

Time discretization:

$N \in \mathbb{N}^* \Rightarrow$ Time step: $\Delta t = \dfrac{T}{N}$ and $\forall n \in [\![0, N]\!]$, $t^n = n\Delta t$.

Unknowns: We are looking for $u_{\mathcal{K}}^n \sim u(t^n, x_{\mathcal{K}})$

$$\textbf{Notation: } u_{\mathcal{T}}^n = \left((u_{\mathcal{K}}^n)_{\mathcal{K}\in\mathcal{T}}\right).$$

PDE's formulation:

$$\partial_t \left(u - \int_0^{\cdot} g(u) \mathrm{d}W \right) - \Delta u = 0$$

Finite-Volume scheme

Let $u_{\mathcal{T}}^n \in \mathbb{R}^{\mathcal{T}}$ be a $\mathcal{F}_{t^n}$-measurable random vector. We look for a $\mathcal{F}_{t^{n+1}}$-measurable random vector $u_{\mathcal{T}}^{n+1} \in \mathbb{R}^{\mathcal{T}}$ such that for almost every $\omega \in \Omega$,

$$\frac{m_{\mathcal{K}}}{\Delta t}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) = \frac{m_{\mathcal{K}}}{\Delta t} g(u_{\mathcal{K}}^n)\Delta_{n+1}W, \quad \forall \mathcal{K} \in \mathcal{T},$$

where

$$\Delta_{n+1}W := W(t^{n+1}) - W(t^n) \text{ for } n \in \{0, \dots, N-1\}.$$

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$\rightsquigarrow$ Existence of a discrete solution.

Need of uniqueness to obtain measurability

Piecewise constant approximation

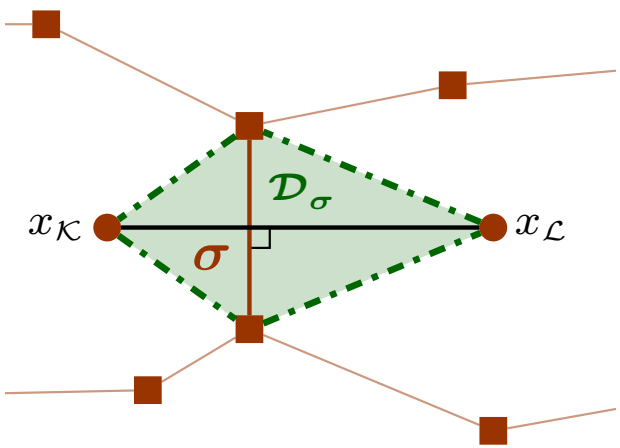
With each set of unknowns $v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, we associate the **piecewise constant** function

$$v_{\mathcal{T}}(x) = \sum_{K \in \mathcal{T}} v_K \mathbf{1}_K(x).$$

Discrete L^2-norm and H^1-seminorm

$$\|v_{\mathcal{T}}\|^2_{L^2(\Lambda)} = \sum_{K \in \mathcal{T}} m_K |v_K|^2 \quad \text{and} \quad |v_{\mathcal{T}}|^2_{1,\mathcal{T}} = \sum_{\sigma = K|L \in \mathcal{E}_{int}} \frac{m_\sigma}{d_{K,L}} (v_K - v_L)^2.$$

Diamond cells and Discrete gradient



$\forall v_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$, $\forall \sigma \in \mathcal{E}_{int}$, we set

$$\nabla^{\mathcal{D}} v_{\mathcal{T}} \overset{\text{def}}{=} d \frac{v_L - v_K}{d_{K,L}} \vec{\mathbf{n}}_{KL},$$

$$\nabla^{\mathcal{T}} v_{\mathcal{T}} \overset{\text{def}}{=} \sum_{\sigma = K|L \in \mathcal{E}_{int}} \mathbf{1}_{\mathcal{D}} \nabla^{\mathcal{D}} v_{\mathcal{T}} \in (L^2(\Omega))^2.$$

Link with the discrete H^1 semi-norm

$$|v_{\mathcal{T}}|^2_{1,\mathcal{T}} = \frac{1}{d} \left\| \nabla^{\mathcal{T}} v_{\mathcal{T}} \right\|^2_{L^2(\Lambda)}.$$

Let $(\mathcal{T}_m)_m$ and $(N_m)_m$ be s.t. $\text{size}(\mathcal{T}_m) \xrightarrow[m\to+\infty]{} 0$ and $\Delta t_m = \dfrac{T}{N_m} \xrightarrow[m\to+\infty]{} 0$.

(Bauzet-Nabet-Schmitz-Zimmermann, '22)

Theorem (Convergence)

Let $u_0 \in L^2(\Omega, H^1(\Lambda))$ $\mathcal{F}_0$-measurable, then

$$u^r_{\mathcal{T}_m,N_m}, u^l_{\mathcal{T}_m,N_m} \xrightarrow[m\to+\infty]{} u \ \text{ in } L^p(\Omega; L^2(0,T; L^2(\Lambda))) \ \text{ with } 1 \le p < 2,$$

where u is the unique variational solution of the heat equation with multiplicative Lipschitz noise.

$$u^r_{\mathcal{T},N}(t,x) := u^{n+1}_{\mathcal{K}}, \ t \in [t^n, t^{n+1}), \ x \in \kappa \text{ (**not** adapted)};$$

$$u^l_{\mathcal{T},N}(t,x) := u^{n}_{\mathcal{K}}, \ t \in [t_n, t^{n+1}), \ x \in \kappa \text{ (adapted)}.$$

Proposition (Bounds on the discrete solutions)

For any $n \in \{1, \dots, N\}$,

$$\mathbb{E}\left[\|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2\right] + 2\Delta t \sum_{i=0}^{n-1} \mathbb{E}\left[\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2\right] \leq C.$$

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Idea of proof:

$$m_{\mathcal{K}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n) + \Delta t \sum_{\sigma \in \mathcal{E}_{\mathcal{K}}^{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}) = m_{\mathcal{K}} g(u_{\mathcal{K}}^n) \underbrace{\Delta_{n+1} W}_{=\sqrt{\Delta t}\xi^{n+1}}, \quad \forall \mathcal{K} \in \mathcal{T}.$$

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$$\sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \frac{1}{2}\left(|u_{\mathcal{K}}^{n+1}|^2 - |u_{\mathcal{K}}^n|^2 + |u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n|^2\right) + \Delta t \sum_{\sigma=\mathcal{K}|\mathcal{L}\in\mathcal{E}_{int}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}|u_{\mathcal{K}}^{n+1} - u_{\mathcal{L}}^{n+1}|^2$$
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$$\frac{1}{2}\left(\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\right) + \Delta t \left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2 = \sum_{\mathcal{K} \in \mathcal{T}} m_{\mathcal{K}} g(u_{\mathcal{K}}^n) \Delta_{n+1} W u_{\mathcal{K}}^{n+1}$$

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$$\leq \mathbb{E}\left[|g(u_{\mathcal{K}}^n)\Delta_{n+1} W|^2\right] + \frac{1}{4}\mathbb{E}\left[\left|u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n\right|^2\right]$$

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$$\frac{1}{2}\mathbb{E}\Big[\left\|u_{\mathcal{T}}^{n+1}\right\|_{L^2(\Lambda)}^2 - \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2 + \left\|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\right\|_{L^2(\Lambda)}^2\Big] + \Delta t \mathbb{E}\Big[\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2\Big]$$
$$= \sum_{\mathcal{K}\in\mathcal{T}} m_{\mathcal{K}} \mathbb{E}\Big[g(u_{\mathcal{K}}^n)\Delta_{n+1} W u_{\mathcal{K}}^{n+1}\Big]$$

Key point: $\mathbb{E}\left[g(u_{\mathcal{K}}^n)\Delta_{n+1} W u_{\mathcal{K}}^n\right] = \mathbb{E}\left[g(u_{\mathcal{K}}^n)\left(W(t^{n+1}) - W(t^n)\right)u_{\mathcal{K}}^n\right] = 0$

$$\begin{aligned}
\mathbb{E}\left[g(u_{\mathcal{K}}^n)\Delta_{n+1} W u_{\mathcal{K}}^{n+1}\right] &= \mathbb{E}\left[g(u_{\mathcal{K}}^n)\Delta_{n+1} W \left(u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n\right)\right] \\
&\leq \mathbb{E}\left[|g(u_{\mathcal{K}}^n)\Delta_{n+1} W|^2\right] + \frac{1}{4}\mathbb{E}\left[\left|u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n\right|^2\right] \\
&\leq \Delta t \mathbb{E}\left[|g(u_{\mathcal{K}}^n)|^2\right] + \frac{1}{4}\mathbb{E}\left[\left|u_{\mathcal{K}}^{n+1} - u_{\mathcal{K}}^n\right|^2\right]
\end{aligned}$$

Key point: Itô isometry

Proposition (Bounds on the discrete solutions)

For any $n \in \{1, \dots, N\}$,

$$\mathbb{E}\left[\|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2\right] + \mathbb{E}\left[\sum_{i=0}^{n-1} \|u_{\mathcal{T}}^{n+1} - u_{\mathcal{T}}^n\|_{L^2(\Lambda)}^2\right] + 2\Delta t \sum_{i=0}^{n-1} \mathbb{E}\left[\left|u_{\mathcal{T}}^{n+1}\right|_{1,\mathcal{T}}^2\right] \leq C.$$

Consequences: Weak convergence

There exists $u \in L^2(\Omega; L^2(0,T; H^1(\Lambda))$ such that (up to a subsequence):

$$u_{\mathcal{T},N}^r, u_{\mathcal{T},N}^l \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} u \text{ weakly in } L^2(\Omega; L^2(0,T; L^2(\Lambda)),$$

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BUT weak convergence not sufficient for nonlinear term:

$$\exists g_u \in L^2(\Omega; L^2(0,T; L^2(\Lambda)) \text{ s.t. } g(u_{\mathcal{T},N}^l) \xrightarrow[\text{size}(\mathcal{T}),\Delta t \to 0]{} g_u$$

$$g_u = g(u)??$$

$\Rightarrow$ Need of stochastic compactness.

Theorem (Skorokhod's representation theorem)

Let $(X_m)_m$ be a sequence of random variables on a measurable space s.t. $X_m \xrightarrow{\mathcal{L}} X$. Then, there exists a probability space $(\Omega', \mathcal{A}', \mathbb{P}')$ and random variables $Y, (Y_m)_m$ s.t.:

$$\mathcal{L}(X_m) = \mathcal{L}(Y_m),\ \forall m, \quad \mathcal{L}(X) = \mathcal{L}(Y), \text{ and } \quad Y_m \to Y \quad \mathbb{P}'\text{-a.s. in } \Omega'.$$

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- Prokhorov's theorem: If $(\mathbb{P} \circ (u^l_{\mathcal{T},N})^{-1})_m$ on $L^2(0,T;L^2(\Lambda))$ is tight, then it is relatively compact.

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- $K_\varepsilon = B_{\mathcal{W}}(0,R)$ and Markov inequality

$$[\mathbb{P} \circ (u^l_{\mathcal{T},N})^{-1}](B_{\mathcal{W}}(0,R)) \geq 1 - \frac{1}{R^2}\mathbb{E}\left[\|u^l_{\mathcal{T},N}\|^2_{\mathcal{W}}\right].$$

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- For any $\alpha \in (0, 1/2)$,

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Skorokhod's theorem:

On a new probability space $(\Omega', \mathcal{A}', \mathbb{P}')$:

- there exist random variables v_0, $(v_m)_m$, u_∞ with $\mathcal{L}(v_0) = \mathcal{L}(u_0)$, $\mathcal{L}(v_m) = \mathcal{L}(u^l_{\mathcal{T}_m, N_m})$ for all $m \in \mathbb{N}$, $\mathcal{L}(u_\infty) = \mu_\infty$ and

$$v_m \xrightarrow[m \to +\infty]{} u_\infty \text{ in } L^2(0,T; L^2(\Lambda)), \ \mathbb{P}'\text{-a.s. in } \Omega';$$

- there exists a stochastic process W_∞ and a sequence of Brownian motions $(W_m)_m$ such that

$$W_m \xrightarrow[m \to +\infty]{} W_\infty \text{ in } \mathcal{C}([0,T]), \ \mathbb{P}'\text{-a.s. in } \Omega'.$$

Consequences:

- For $m \in \mathbb{N}^*$, v_m is a step function i.e.

$$\exists v^l_{\mathcal{T}_m, N_m} \in \mathbb{R}^{\mathcal{T}_m \times N_m} \text{ s.t. } v_m = v^l_{\mathcal{T}_m, N_m} \ \mathbb{P}'\text{-a.s. in } \Omega'$$
$$\Rightarrow \quad v_m(t,x) = v^n_\mathcal{K}, \quad \forall t \in [t^n, t^{n+1}), \forall x \in \mathcal{K}.$$

- For $m \in \mathbb{N}^*$, any $n \in \{0, \dots, N_m - 1\}$ and any $\mathcal{K} \in \mathcal{T}_m$, the random vector $v^{n+1}_{\mathcal{T}_m}$ is solution to

$$m_\mathcal{K}(v^{n+1}_\mathcal{K} - v^n_\mathcal{K}) + \Delta t \sum_{\sigma \in \mathcal{E}^{int}_\mathcal{K}} \frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v^{n+1}_\mathcal{K} - v^{n+1}_\mathcal{L}) - m_\mathcal{K} g(v^n_\mathcal{K}) \Delta_{n+1} W_m = 0$$

$\mathbb{P}'$-a.s. in Ω', where $\Delta_{n+1} W_m := W_m(t^{n+1}) - W_m(t^n)$.

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- For any $m \in \mathbb{N}$, there exists a filtration $(\mathfrak{F}_t^m)_{t\geq 0}$ such that $(v_m)_m$ is adapted to $(\mathfrak{F}_t^m)_{t\geq 0}$ and $W_m = (W_m(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^m)_{t\geq 0}$.

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- There exists a filtration $(\mathfrak{F}_t^\infty)_{t\geq 0}$ such that u_∞ has a predictable $d\mathbb{P}'\otimes dt$-representative and $W_\infty=(W_\infty(t))_{t\geq 0}$ is a Brownian motion with respect to $(\mathfrak{F}_t^\infty)_{t\geq 0}$.

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  - $v^l_{\mathcal{T}_m,N_m} \xrightarrow[m\to+\infty]{} u_\infty$ in $L^2(0,T;L^2(\Lambda))$, $\mathbb{P}'$-a.s. in Ω'

    $\Rightarrow g(v^l_{\mathcal{T}_m,N_m}) \xrightarrow[m\to+\infty]{} g(u_\infty)$ in probability in $L^2(0,T;L^2(\Lambda))$.

(Debussche,Glatt-Holtz,Temam, '11)

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$$\sum_{\mathcal{K}\in\mathcal{T}}\sum_{n=0}^{N-1} m_{\mathcal{K}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{K}}^{n})+\Delta t\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}^{int}}\frac{m_\sigma}{d_{\mathcal{K},\mathcal{L}}}(v_{\mathcal{K}}^{n+1}-v_{\mathcal{L}}^{n+1})=\int_\Lambda\int_0^T g(v^l_{\mathcal{T},N})\,dW_m(t)\,dx$$

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Up to a subsequence **(Debussche,Glatt-Holtz,Temam, '11)**

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Proposition (Bauzet, Nabet, Schmitz, Zimmermann, '22)

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W^\infty, u_\infty, v_0)$ is a martingale solution for the heat equation with multiplicative Lipschitz noise.

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2. **(Gyöngy-Krylov, '96)**
   - From any arbitrary subsequences $(u_\nu^l)_\nu, (u_\rho^l)_\rho$ of $(u_m^l)_m$, we construct two martingale solutions u_1^∞, u_2^∞ with respect to the same stochastic basis and with the same initial value.
   - Pathwise uniqueness implies
   $$1 = \mathbb{P}'(\{u_\infty^1 = u_\infty^2\}) = \mathbb{P}' \circ (u_\infty^1, u_\infty^2)^{-1}(\{(x,y) \in L^2(0,T; L^2(\Lambda)^2 \mid x = y\})$$
   $$\Rightarrow \; u^l_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in probability in } L^2(0,T; L^2(\Lambda)).$$

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Proposition (Bauzet, Nabet, Schmitz, Zimmermann, '22)

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W^\infty, u_\infty, v_0)$ is a martingale solution for the heat equation with multiplicative Lipschitz noise.

1. Let u_1, u_2 be two solutions w.r.t. the $\mathcal{F}_0$-initial values $u_0^1, u_0^2 \in L^2(\Omega; L^2(\Lambda))$ on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\geq 0})$, then
$$\mathbb{E}\left[\|u_1(t) - u_2(t)\|_{L^2(\Lambda)}^2\right] \leq C\mathbb{E}\left[\|u_0^1 - u_0^2\|_{L^2(\Lambda)}^2\right], \quad \forall t \in [0,T].$$

2. **(Gyöngy-Krylov, '96)**
   - From any arbitrary subsequences $(u_\nu^l)_\nu, (u_\rho^l)_\rho$ of $(u_m^l)_m$, we construct two martingale solutions u_1^∞, u_2^∞ with respect to the same stochastic basis and with the same initial value.
   - Pathwise uniqueness implies
   $$1 = \mathbb{P}'(\{u_\infty^1 = u_\infty^2\}) = \mathbb{P}' \circ (u_\infty^1, u_\infty^2)^{-1}(\{(x,y) \in L^2(0,T;L^2(\Lambda)^2 \mid x = y\})$$
   $$\Rightarrow \; u^l_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in probability in } L^2(0,T;L^2(\Lambda)).$$
   - Up to a subsequence,
   $$u^l_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \quad \mathbb{P}\text{-a.s. in } L^2(0,T;L^2(\Lambda)).$$
   - Uniform bounds in $L^2(\Omega; L^2(0,T;L^2(\Lambda))) \oplus$ Vitali's theorem:
   $$u^l_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in } L^p(\Omega; L^2(0,T;L^2(\Lambda))) \text{ for } 1 \leq p < 2.$$

Proposition (Bauzet, Nabet, Schmitz, Zimmermann, '22)

$(\Omega', \mathbb{P}', \mathcal{A}', (\mathfrak{F}_t^\infty)_{t\geq 0}, W^\infty, u_\infty, v_0)$ is a martingale solution for the heat equation with multiplicative Lipschitz noise.

1. Let u_1, u_2 be two solutions w.r.t. the $\mathcal{F}_0$-initial values $u_0^1, u_0^2 \in L^2(\Omega; L^2(\Lambda))$ on $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W(t))_{t\geq 0})$, then

$$\mathbb{E}\left[\|u_1(t) - u_2(t)\|_{L^2(\Lambda)}^2\right] \leq C\mathbb{E}\left[\|u_0^1 - u_0^2\|_{L^2(\Lambda)}^2\right], \quad \forall t \in [0,T].$$

2. $$u^l_{\mathcal{T}_m, N_m} \xrightarrow[m\to+\infty]{} u \text{ in } L^p(\Omega; L^2(0,T; L^2(\Lambda))) \text{ for } 1 \leq p < 2.$$

3. u is the unique variational solution of the heat equation with multiplicative Lipschitz noise.

$$\begin{cases} du - \Delta u\, dt = Lu\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda. \end{cases}$$

Initial data

$$u_0(x,y) = 2\cos(\pi x)\cos(\pi y) \Rightarrow \Delta u_0 = -2\pi^2 u_0 = -\lambda_2 u_0$$

Black Scholes model

$u(t,(x,y)) = \mathcal{S}(t)u_0(x,y)$ solution iff $\mathcal{S}(t)$ satisfies

$$\begin{cases} d\mathcal{S}(t) = -\lambda_2 \mathcal{S}(t)\, dt + L\mathcal{S}(t)\, dW(t); \\ \mathcal{S}(0) = 1. \end{cases}$$

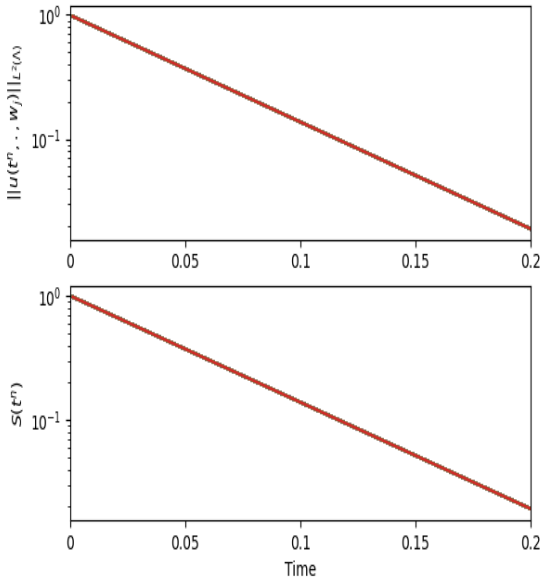
$$\mathcal{S}(t) = \mathcal{S}(0) e^{(-\lambda_2 - \frac{L^2}{2})t} e^{LW(t)} \text{ and } \|u(t,(x,y))\|_{L^2(\Lambda)} = \mathcal{S}(t)$$

Comparison between $\mathcal{S}(t^n)$ and $\|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}$

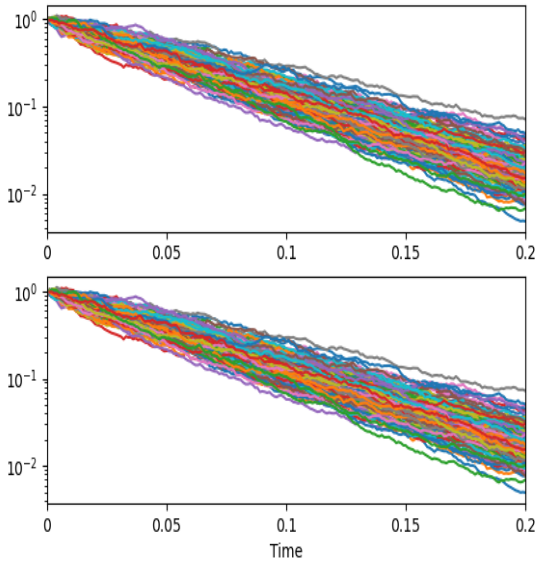
- Time discretisation:

$$T = 0.2, \quad N = 200, \quad \mathrm{d}t = 10^{-3}$$

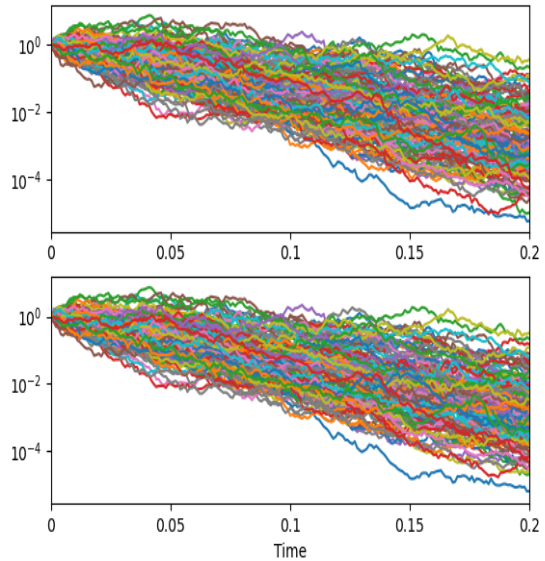
- Space discretization: Mesh size $\sim 1.66.10^{-2}$



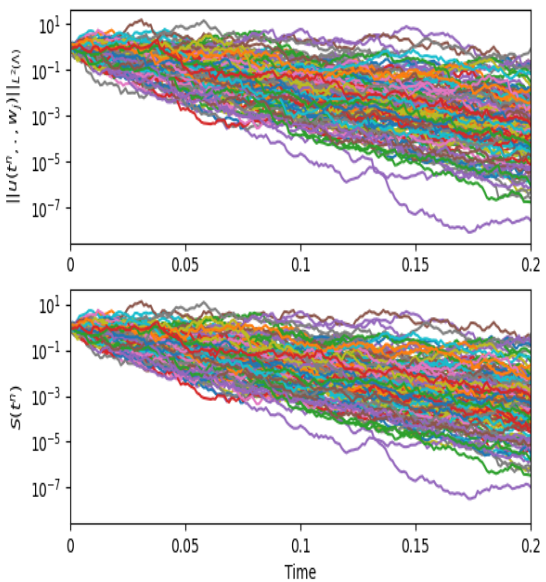
$L = 0$



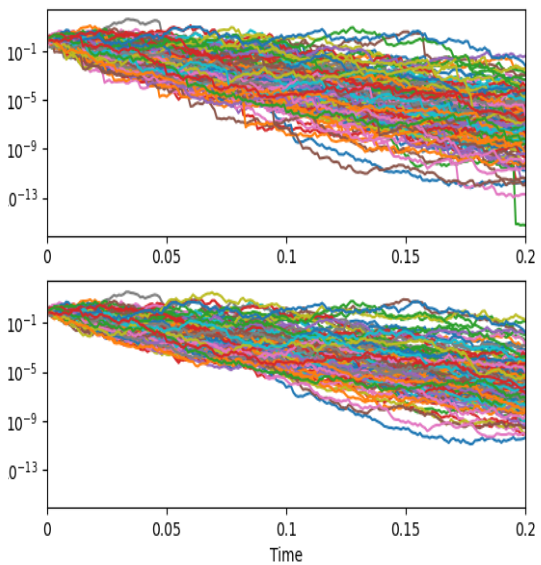
$L = 1$



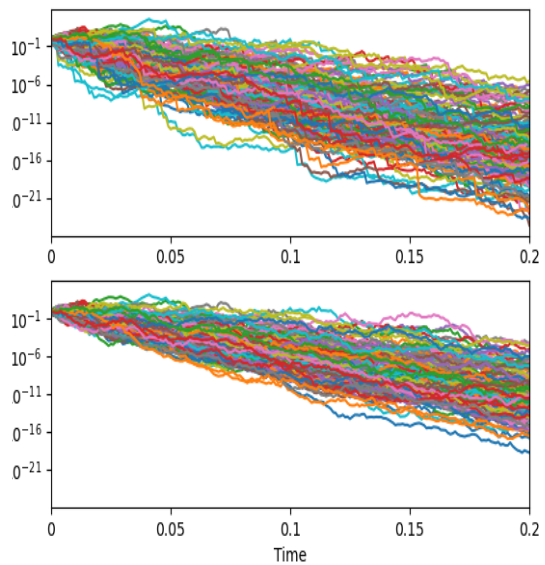
$L = 5$



$L = 2\pi$

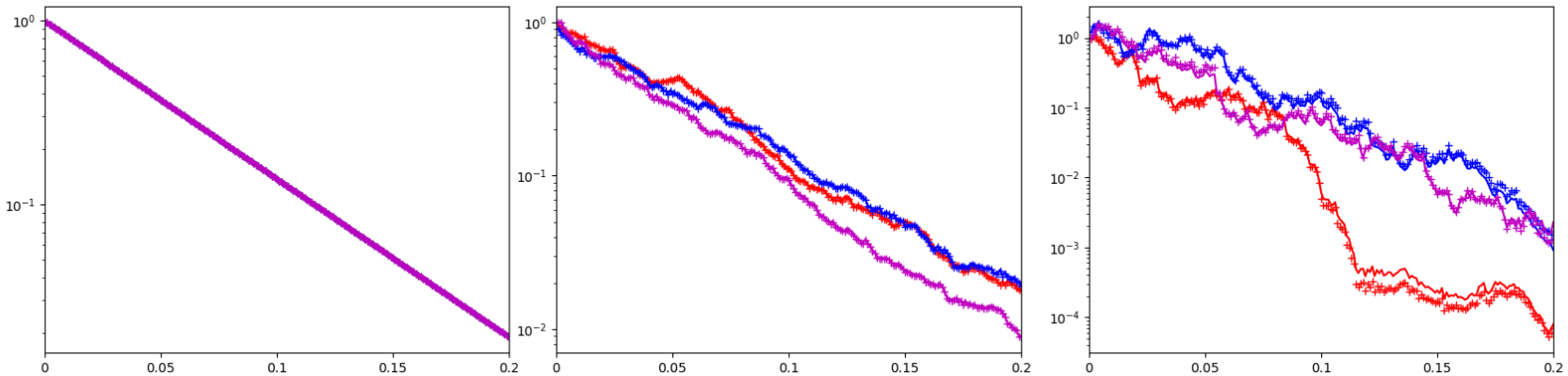


$L = 10$

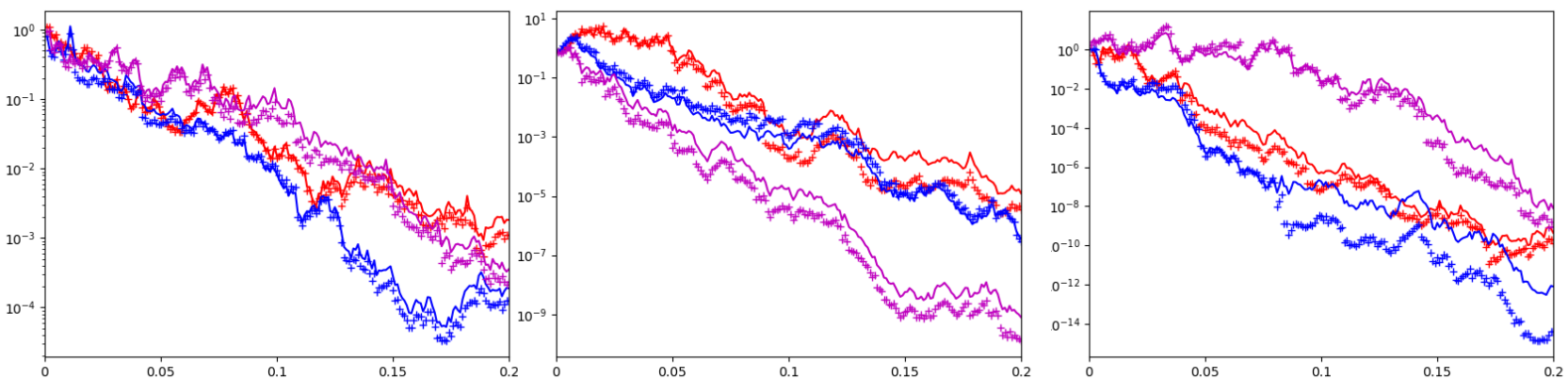


$L = 15$

Zoom on 3 trajectories



$L = 0$ $L = 1$ $L = 5$

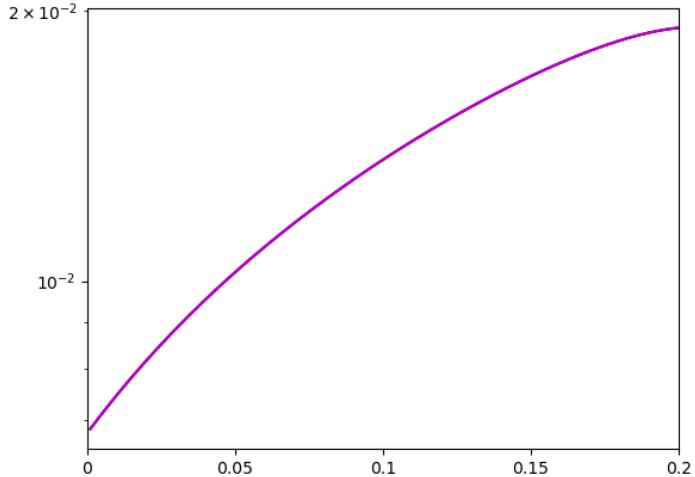


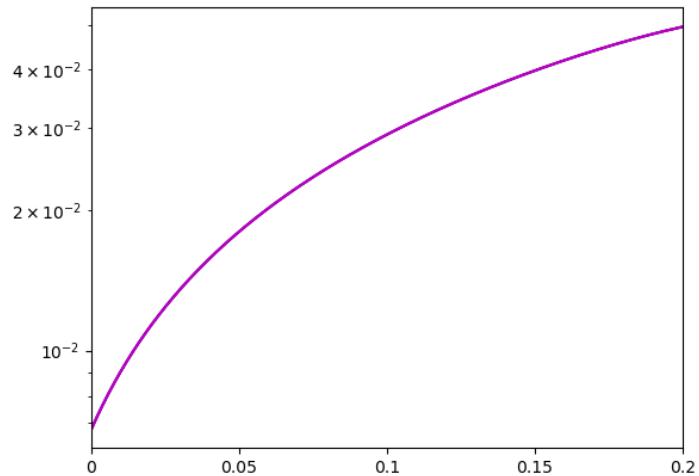
$L = 2\pi$ $L = 10$ $L = 15$

——— $\mathcal{S}(t^n)$ $+++$ $\|u_{\mathcal{T}}^n\|_{L^2(\Lambda)}$

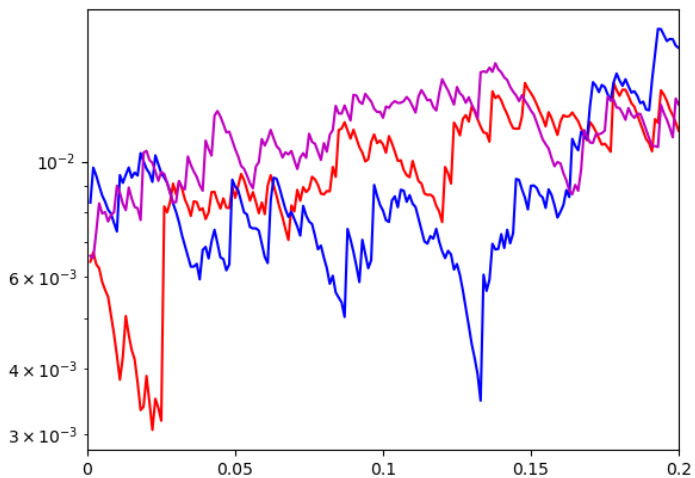
Influence of the time step

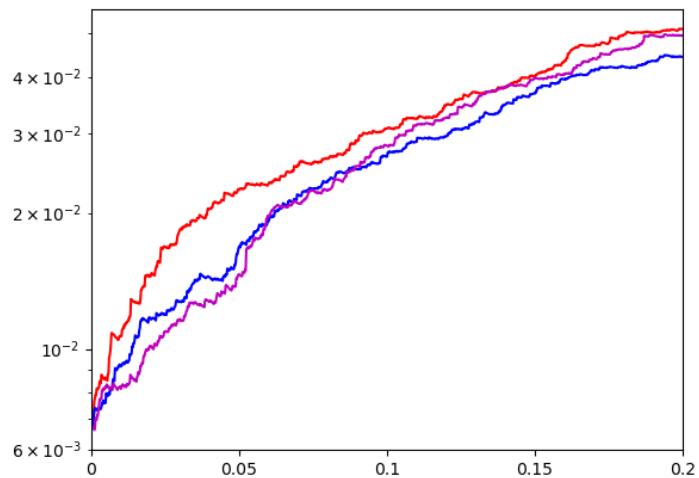
$$\frac{\left|\mathcal{S}(t^n) - \|u_{\mathcal{T}}^n\|\right|}{\mathcal{S}(t^n)}$$





$L = 0$





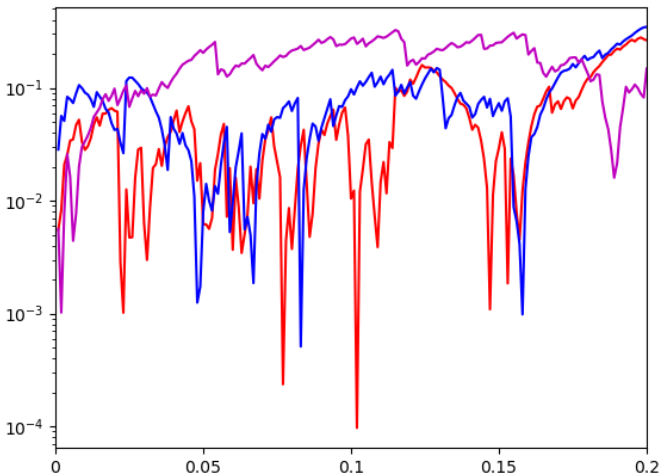
$L = 1$

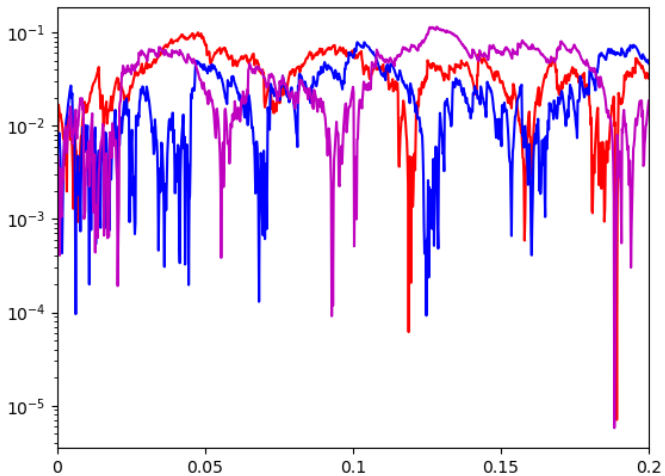
$\Delta t = 10^{-3}$

$\Delta t = 2.10^{-4}$

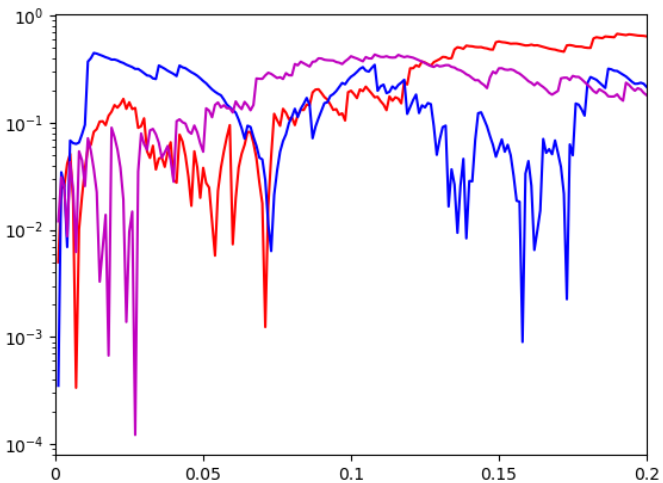
Influence of the time step

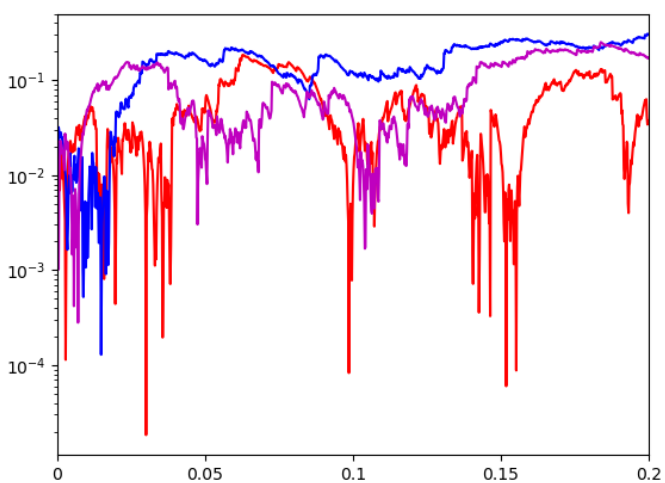
$$\frac{\left|\mathcal{S}(t^n) - \|u_{\mathcal{T}}^n\|\right|}{\mathcal{S}(t^n)}$$





$L = 5$





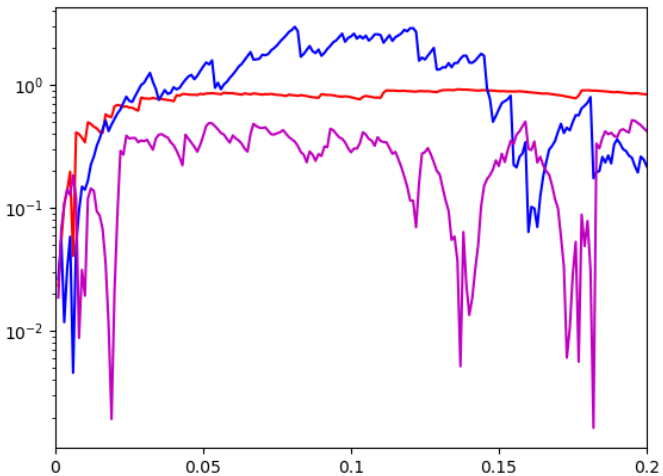
$L = 2\pi$

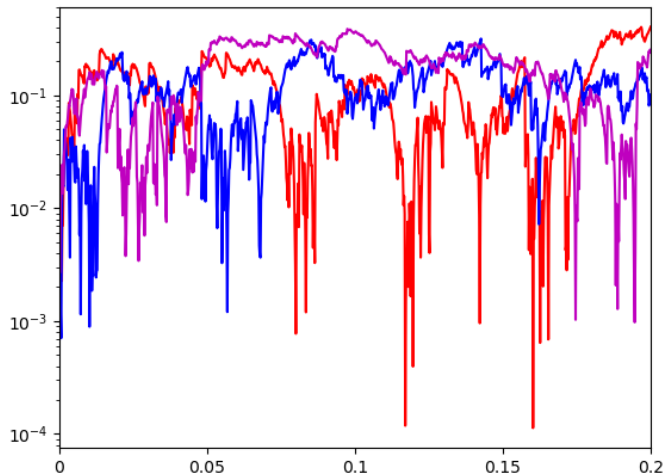
$\Delta t = 10^{-3}$

$\Delta t = 2.10^{-4}$

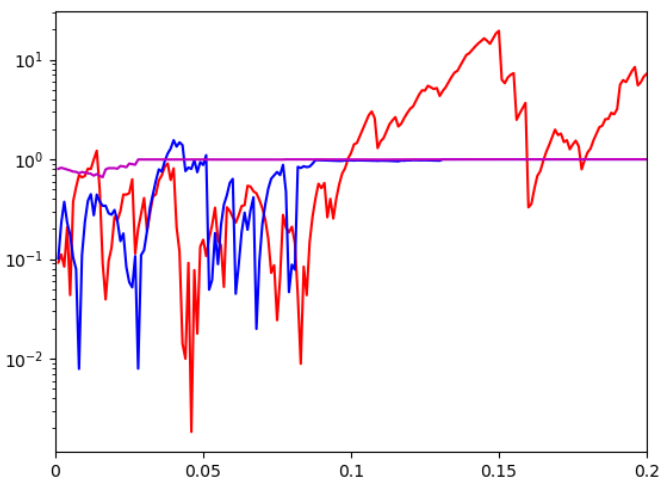
Influence of the time step

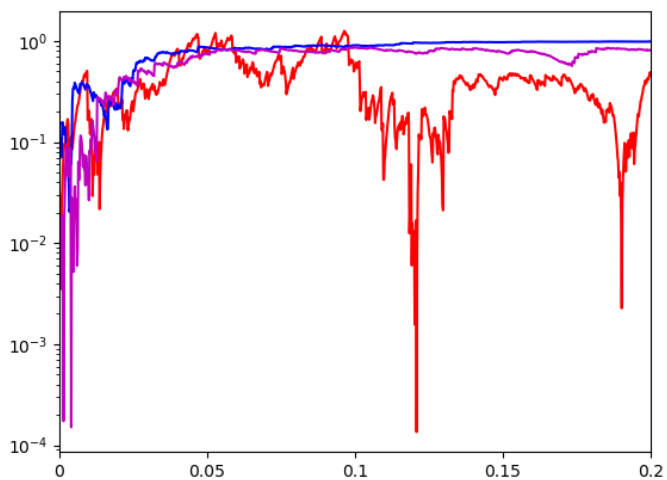
$$\frac{\left|\mathcal{S}(t^n) - \|u_{\mathcal{T}}^n\|\right|}{\mathcal{S}(t^n)}$$





$L = 10$





$L = 15$

$\Delta t = 10^{-3}$

$\Delta t = 2.10^{-4}$

$$\begin{cases} du - \Delta u\, dt = Lu\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda. \end{cases}$$

Black Scholes model

$u(t,(x,y)) = \mathcal{S}(t)u_0(x,y)$ with $\|u_0\|_{L^2(\Lambda)} = 1$

$$\mathcal{S}(t) = e^{(-\lambda_2 - \frac{L^2}{2})t} e^{LW(t)} \text{ and } \|u(t,(x,y))\|_{L^2(\Lambda)} = \mathcal{S}(t)$$

- $e^{-\frac{L^2}{2}t + LW(t)}$ is a martingale:

$$\frac{1}{J}\sum_J \|u_{\mathcal{T}}^n\|_{L^2(\Lambda)} \sim \mathbb{E}\left[\|u(t^n,(x,y))\|_{L^2(\Lambda)}\right] = e^{-\lambda_2 t^n}$$

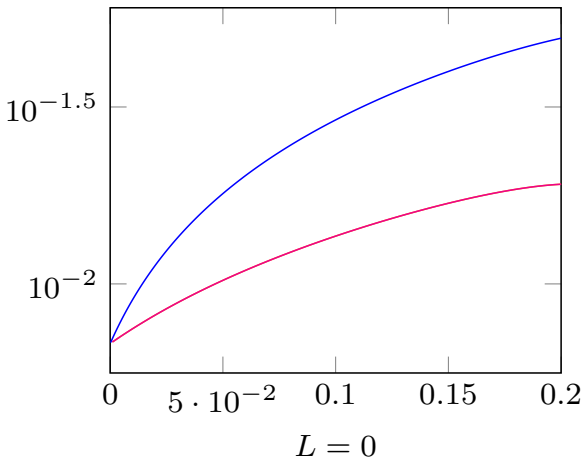
- $\|u(t,(x,y))\|^2_{L^2(\Lambda)} = e^{(-2\lambda_2 - L^2)t} e^{2LW(t)}$ and $e^{-2L^2 t + 2LW(t)}$ is a martingale:

$$\frac{1}{J}\sum_J \|u_{\mathcal{T}}^n\|^2_{L^2(\Lambda)} \sim \mathbb{E}\left[\|u(t^n,(x,y))\|^2_{L^2(\Lambda)}\right] = e^{(L^2 - 2\lambda_2)t^n}$$

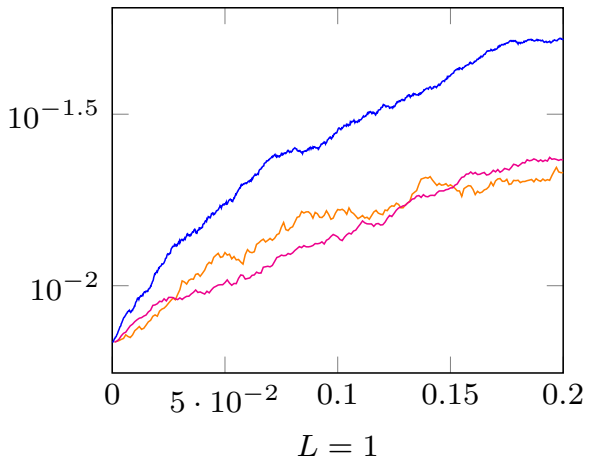
J: total number of trajectories

Comparison between the mean of the norm

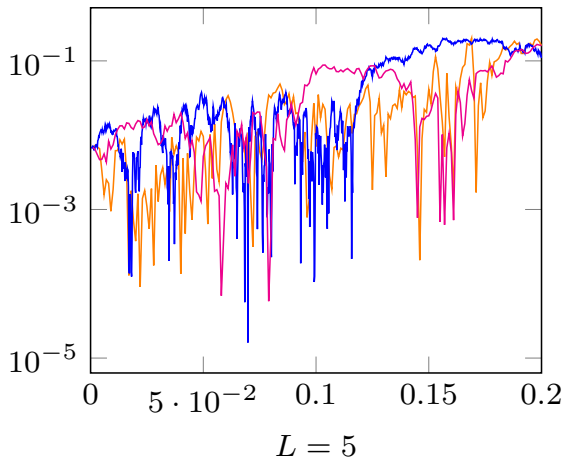
$$\frac{\left|\frac{1}{J}\sum_J \|u_T^n\|_{L^2(\Lambda)} - e^{-\lambda_2 t^n}\right|}{e^{-\lambda_2 t^n}}$$



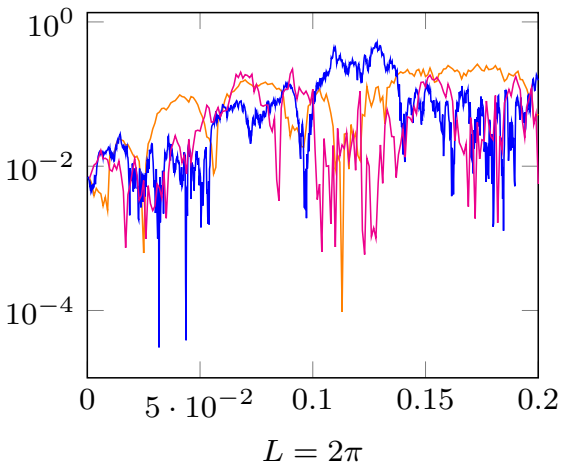
$L = 0$



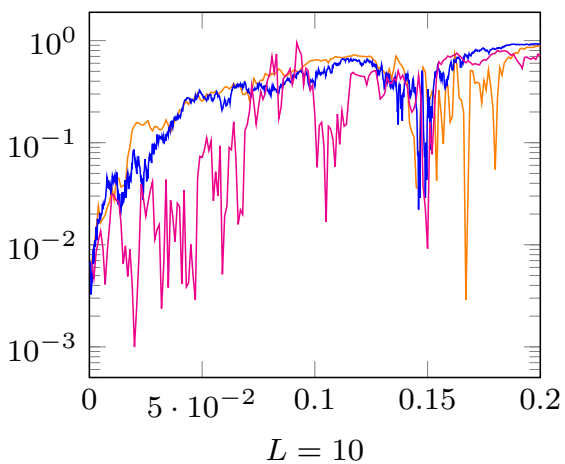
$L = 1$



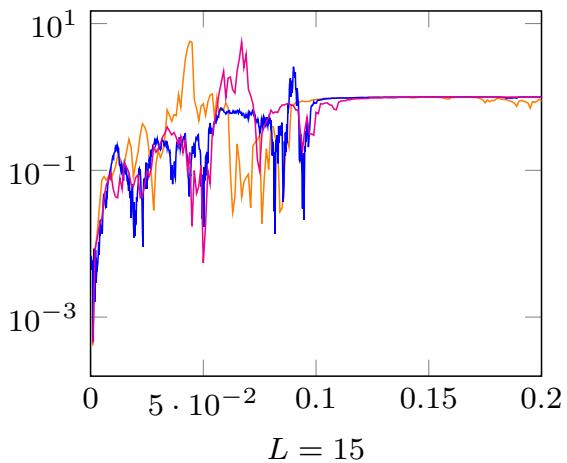
$L = 5$



$L = 2\pi$



$L = 10$

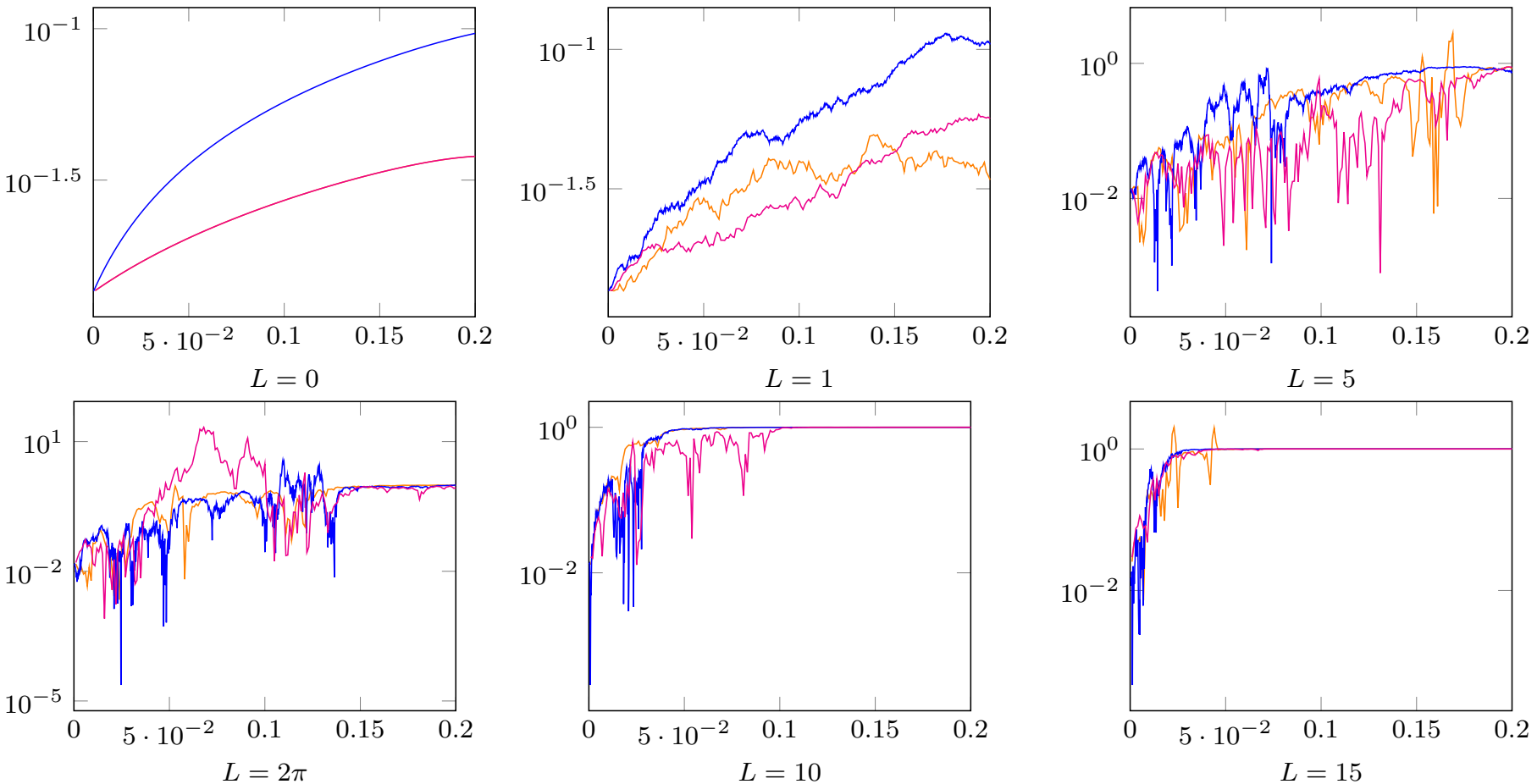


$L = 15$

$\Delta t = 2.10^{-4}, J = 2048$ $\Delta t = 10^{-3}, J = 2048$ $\Delta t = 10^{-3}, J = 5000$

Comparison between the mean of the square of the norm

$$\frac{\left|\frac{1}{J}\sum_J \|u_{\mathcal{T}}^n\|^2_{L^2(\Lambda)} - e^{(L^2-\lambda_2)t^n}\right|}{e^{(L^2-\lambda_2)t^n}}$$



$\Delta t = 2.10^{-4}, J = 2048$ $\Delta t = 10^{-3}, J = 2048$ $\Delta t = 10^{-3}, J = 5000$

$$\begin{cases} du - \Delta u\, dt = Lu\, dW(t), & \text{in } \Omega \times (0,T) \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \\ u(0,\cdot) = u_0, & \text{in } \Omega \times \Lambda. \end{cases}$$

Initial data

$$u_0(x,y) = 2\cos(\pi x)\cos(\pi y) \Rightarrow \Delta u_0 = -2\pi^2 u_0 = -\lambda_2 u_0$$

Solution to the deterministic heat equation

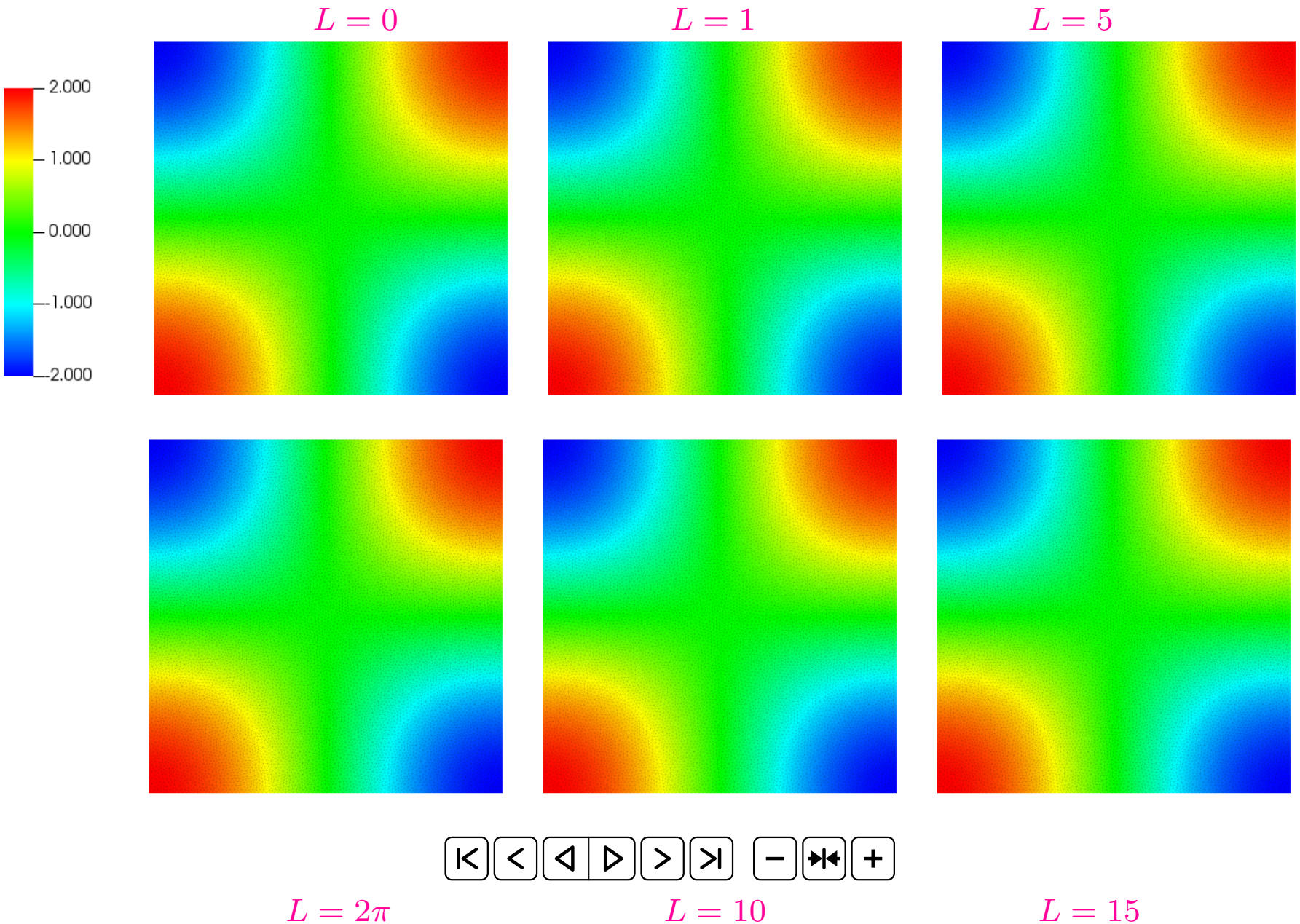
$$u(t,(x,y)) = 2e^{-2\pi^2 t}\cos(\pi x)\cos(\pi y)$$

Data

- Time discretisation:

$$T = 0.2, \quad N = 200, \quad \mathrm{d}t = 10^{-3}$$

- Space discretization: Mesh size $\sim 1.66.10^{-2}$
- Number of trajectories: $J = 2048$


$L = 0$
$L = 1$
$L = 5$
2.000
1.000
0.000
-1.000
-2.000
$L = 2\pi$
$L = 10$
$L = 15$

The continuous problem

$$\begin{cases} du - \Delta u\, dt + \operatorname{div}\big(\mathbf{v} f(u)\big)\, dt = g(u)\, dW(t) + \beta(u)\, dt, & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$-measurable.
- $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, nondecreasing;
- $g, \beta : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with $\beta(0) = 0$.
- $\mathbf{v} \in \mathscr{C}^1([0,T] \times \overline{\Lambda}; \mathbb{R}^d)$ s.t. $\operatorname{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on $[0,T] \times \partial\Lambda$.

Existence

(Bauzet-Nabet-Schmitz-Zimmermann, Proceeding FVCA X)

Convergence: $f = Id$

(Bauzet-Schmitz-Zimmermann, Preprint)

The continuous problem

$$\begin{cases} du - \Delta u\, dt + \operatorname{div}\big(\mathbf{v} f(u)\big)\, dt = g(u)\, dW(t) + \beta(u)\, dt, & \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) = u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{cases}$$

where:

- $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (W_t)_{t\geq 0})$ is a stochastic basis with a real-valued Brownian motion $(W_t)_{t\geq 0}$;
- $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$-measurable.
- $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, nondecreasing;
- $g, \beta : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with $\beta(0) = 0$.
- $\mathbf{v} \in \mathscr{C}^1([0,T] \times \overline{\Lambda}; \mathbb{R}^d)$ s.t. $\operatorname{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on $[0,T] \times \partial\Lambda$.

Existence

(Bauzet-Nabet-Schmitz-Zimmermann, Proceeding FVCA X)

Convergence: $f = Id$

(Bauzet-Schmitz-Zimmermann, Preprint)

Thank you for your attention