Resurgence and quantum modularity

Wall-Crossing Structures, Analyticity, and Resurgence

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In the 80s, Jones discovered a remarkable polynomial invariant of links. Witten interpreted the Jones polynomial in terms of quantum field theory. In particular, for $\mathrm{SU}(2)$ connections on a three manifold $\mathcal{A}_M$,

$$Z_M(\hbar) = \int_{\mathcal{A}_M/\mathcal{G}_M} \exp\left(\frac{CS(A)}{2\pi i\hbar}\right) DA$$

where

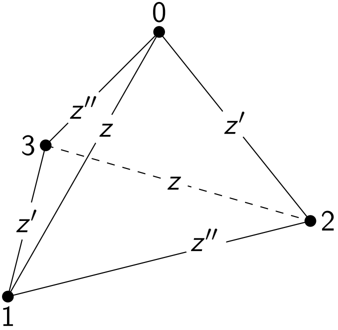
$$CS(A) = \int_M \mathrm{Tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{C} + (2\pi i)^2 \mathbb{Z}\,.$$

For $\hbar \in 1/\mathbb{Z}$ Witten related this integral to the invariants of Jones evaluated at certain roots of unity. However if defined, this invariant Z_M is given as an exponential integral over an infinite dimensional space. Therefore, we expect asymptotics as $\hbar \to 0$. The critical points of $CS(A)$ are flat connections,

$$\mathcal{A}^{\mathrm{flat}}_{M,\mathbb{C}}/\mathcal{G}_M \cong \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})\,.$$

Ideal tetrahedra

Thurston used ideal triangulations to construct hyperbolic structures on 3–manifolds. More generally, they can be used to construct flat connections on a three manifold. Hyperbolic ideal tetrahedra are determined by a one dimensional moduli space given by $\mathfrak{h}/(z \mapsto z' \mapsto z'')$ where $z' = 1/(1-z)$ and $z'' = 1 - 1/z$. These symmetries relate to a choice of edge and the angles at the edges of the ideal tetrahedra are determined by the arguments of these numbers.



The volume of such a tetrahedron is given by $D(z) = D(z') = D(z'')$ where D is the Bloch–Wigner dilogarithm function.

Gluing equations

By lifting a hyperbolic 3–manifold to the universal cover in hyperbolic three space, one can glue together tetrahedra along face using explicit elements of $\mathrm{SL}_2(\mathbb{C})$. These elements are determined by the shapes of the tetrahedra z_i. The vanishing of the monodromy around an edge of the triangulation then leads to algebraic equations of the form

$$\prod_{j=1}^{N} z_j^{A_{ij}} z_j''^{B_{ij}} \;=\; \prod_{j=1}^{N} z_j^{A_{ij}} (1 - z_j^{-1})^{B_{ij}} \;=\; (-1)^{\nu_i}\,,$$

for some integral A, B, ν. These algebraic equations are call gluing equations and the matrices A, B are called Neumann–Zagier data. These matrices satisfy the symplectic properties that

$$AB^t \;=\; BA^t\,, \quad \text{and} \quad (A\,|\,B) \quad \text{is full rank over} \quad \mathbb{Q}\,.$$

A reduction to finite dimensions

From work of Kashaev in the mid 90s, we expect that Witten's integral should be able to be reduced. In particular, using a triangulation one should be able to express this kind of invariant as a finite dimensional integral where the integrand gets some kind of quantum dilogarithm associated to each tetrahedron. The quantum dilogarithm is given

$$\Phi_{\mathsf{b}}(x) \;=\; \frac{(-q^{\frac{1}{2}}\,e^{2\pi \mathsf{b} x};q)_\infty}{(-\tilde{q}^{\frac{1}{2}}\,e^{2\pi \mathsf{b}^{-1} x};\tilde{q})_\infty}\,,$$

where we note that

$$(ze^{x\hbar^{1/2}};e^{\hbar})_\infty^{-1} \;\sim\; \widehat{\Psi}_\hbar(x;z) \;=\; \exp\left(-\sum_{k,\ell\in\mathbb{Z}_{\geq 0}} \frac{B_k\,x^\ell\,\hbar^{k+\frac{\ell}{2}-1}}{\ell!\,k!}\mathrm{Li}_{2-k-\ell}(z)\right).$$

This was further explored by Hikami and then formalised by Andersen–Kashaev.

The state integrals

The state integrals are integrals of a Gaussian measure times a product of Faddeev quantum dilogarithms. Explicitly,

$$\int \cdots \int \exp\left(\frac{1}{2} x^t B^{-1} A x/2 + \mu x \mathsf{b} + \nu x \mathsf{b}^{-1}\right) \prod_{j=1}^{N} \Phi_{\mathsf{b}}(x_j) dx$$

where to be an invariant we need to choose an ordered triangulation and use Andersen–Kashaev's choice of contour.

Saddle points

These state integrals are finite dimensional and therefore their asymptotics can be studied via stationary phase. This was studied by Dimofte–Gukov–Lenells–Zagier and then formalised by Dimofte–Garoufalidis. This leads to a definition of an asymptotic series for each hyperbolic manifold from the Neumann–Zagier data. In particular, from $M = [A, B, \nu, f, f'', z]$ where $Af + Bf'' = \nu$, $\det(B) \neq 0$ and z satisfies the gluing equations for the geometric connection, Dimofte–Garoufalidis defined

$$\widehat{\Phi}_M(\hbar) \;=\; \left\langle \exp\left(\frac{\hbar^{1/2}}{2} x^t(1 - B^{-1}\nu) + \frac{\hbar}{8} f^t B^{-1} A f\right) \prod_{j=1}^{N} \widehat{\Psi}_\hbar(x_j; z_j) \right\rangle$$

where $\langle\ \rangle$ represents Gaussian integration with respect to the variables x.

Theorem:[Garoufalidis–Storzer–W.,2023]

$\widehat{\Phi}_M(\hbar)$ is a topological invariant of M.

Example: figure eight knot

The NZ datum for 4_1 is given by

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \nu = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad f'' = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $z_1 = z_2 = \mathbf{e}(1/6)$ and using a Fourier transform identity of the Faddeev quantum dilogarithm we find that

$$\Phi_{4_1}(\hbar) = \Big\langle \exp\left(\frac{x}{2}\hbar^{\frac{1}{2}}\right)\Psi_\hbar(x, \mathbf{e}(1/6))^2 \Big\rangle_{x,\sqrt{-3}}$$

this can then be computed to show

$$\begin{aligned}\Phi_{4_1}(\hbar) = \; & 1 - \frac{11}{216}\sqrt{-3}\,\hbar - \frac{697}{31104}\hbar^2 + \frac{724351}{100776960}\sqrt{-3}\,\hbar^3 + \frac{278392949}{29023764480}\hbar^4 - \frac{244284791741}{43883931893760}\sqrt{-3}\,\hbar^5 \\ & - \frac{1140363907117019}{94789292890521600}\hbar^6 + \frac{212114205337147471}{20474487264352665600}\sqrt{-3}\,\hbar^7 + \frac{367362844229968131557}{117933046642671353856 00}\hbar^8 \\ & - \frac{44921192873529779078383921}{126094013470344211542835200 0}\sqrt{-3}\,\hbar^9 - \frac{3174342130562495575602143407}{231095937414739936791232512 00}\hbar^{10} + O(\hbar^{11})\,.\end{aligned}$$

Neumann–Zagier with no manifold

The asymptotic series just needs $\det(B) \neq 0$ and the critical point around which we expand to be non–degenerate. Therefore, we can define these series more generally for (A, B, ν, f, f'', z) and they will remain invariant under the various moves between these Neumann–Zagier data such as the 2-3 move.

If the quadratic form is degenerate at the critical point, which happens when the critical point is in a component of dimension greater than 0, then more work is needed to define the asymptotic series.

Remark:

Some experiments were done related to examples whose critical points come with positive dimensional components in work with Garoufalidis (Periods, the mero... arXiv:2209.02843). There numerically, asymptotic series were found with coefficients given by periods of the positive dimensional components, in that case the zero locus of the A–polynomial of a knot.

State integrals and q–hypergeometric sums

As asymptotics of q–hypergeometric sums

The asymptotic series $\widehat{\Phi}_M(\hbar)$ appear as asymptotics of elements of the Habiro ring and q–series. We can replace the integral by a sum and the Faddeev quantum dilogarithms for Pochhammer symbols. In particular, of

$$\sum_{k\in\mathbb{Z}_{\geq 0}^N} (-1)^{\nu k} q^{k^t B^{-1} A k/2+\mu k} (q^{-1};q^{-1})_k \quad \text{and} \quad \sum_{k\in\mathbb{Z}_{\geq 0}^N} (-1)^{\nu k} \frac{q^{k^t B^{-1} A k/2+\mu k}}{(q;q)_k}$$

where for $v \in \mathbb{Z}_{\geq 0}^N$

$$(a;q)_v = \prod_{j=1}^{N} \prod_{\ell=0}^{v_j-1} (1-aq^\ell)$$

Remark:

These kind of asymptotics can be proved in examples but I don't know a complete proof in general. One needs to apply a summation method, justify exponentially smaller boundary terms, and justify the use of the saddle point method.

Remark on generality

Remark:

Using the q–binomial theorem we see that proper q–hypergeometric q–series can be written in the form above as

$$(q;q)_k = \frac{(q;q)_\infty}{(q^{k+1};q)_\infty} = (q;q)_\infty \sum_{\ell=0}^{\infty} \frac{q^{\ell(k+1)}}{(q;q)_\ell}\,,$$

and

$$\frac{1}{(q;q)_k} = \frac{(q^{k+1};q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} (-1)^\ell \frac{q^{\ell(\ell+1)/2+\ell k}}{(q;q)_\ell}\,,$$

Factorisation of state integrals

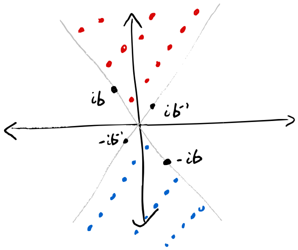
These state integrals can often be factorised into products bilinear combinations of functions in $q = \mathbf{e}(\tau)$ and $\tilde{q} = \mathbf{e}(-1/\tau)$ where $\mathbf{e}(x) = \exp(2\pi i x)$ and $\mathsf{b}^2 = \tau$. This was known in the physics literature (for example the work of Beem–Dimofte–Pasquetti) and was explicitly proved for a family of examples by Garoufalidis–Kashaev.

There are two different places one can factorise a state integral. Either when $\tau \in \mathbb{C} - \mathbb{R}$ or when $\tau \in \mathbb{Q}$.

Factorisation in the upper half plane

One way to factorise a state integral when τ is in the upper half plane is to use the pole structure of the Faddeev quantum dilogarithm. Then one deforms the contour of integration to infinity in a good direction which reduces the integral to a sum of the residues captured on the way. This sum then decouples or can be made to resulting in a bilinear combination.

The poles and zeros of the Faddeev quantum dilogarithm are located on cones as can easily be seen from the product formula $\frac{(-q^{\frac{1}{2}}e^{2\pi \mathsf{b}x};q)_\infty}{(-\tilde{q}^{\frac{1}{2}}e^{2\pi \mathsf{b}^{-1}x};\tilde{q})_\infty}$. This is depicted below.



Example: figure eight knot

The state integrals

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathrm{b}}(x)^2 \mathbf{e}(-x^2/2) dx\,, \quad \text{and} \quad \int_{\mathbb{R}+\epsilon} \Phi_{\mathrm{b}}(x)^2 \frac{\mathbf{e}(-x^2/2)}{1+q^{1/2}\mathbf{e}(-i\mathrm{b}x)} dx$$

factorises as elementary functions times

$$G^{(1)}(q)G^{(0)}(\tilde{q}) - \tau^{-1}G^{(0)}(q)G^{(1)}(\tilde{q})\,, \quad \text{and} \quad G^{(2)}(q) + \tau^{-1}G^{(1)}(q)L^{(0)}(\tilde{q}) - \tau^{-2}G^{(0)}(q)L^{(1)}(\tilde{q})$$

where

$$G^{(0)}(q) = \sum_{n\geq 0}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} = 1 - q - 2q^2 - 2q^3 - 2q^4 + \dots$$

$$2G^{(1)}(q) = 2\sum_{n\geq 0}\left(n + 1/2 - 2E_1^{(n)}(q)\right)(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2} = 1 - 7q - 14q^2 - 8q^3 - 2q^4 + \dots$$

$$12G^{(2)}(q) = 12\sum_{n\geq 0}\left(\frac{1}{2}\left(n+1/2-2E_1^{(n)}(q)\right)^2 - E_2^{(n)}(q) - \frac{1}{24}E_2(q)\right)(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}$$

$$= 1 - 25q - 38q^2 + 58q^3 + 178q^4 + \dots$$

with additional series

$$L^{(0)}(q) = 2E_0^{(1)}(q) + \sum_{n=1}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q;q)_n^2}\frac{q^n}{1-q^n}$$

$$L^{(1)}(q) = \frac{1}{8} - 2E_1^{(0)}(q)^2 - E_2^{(0)}(q) + \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)/2}}{(q)_n^2}\frac{q^n}{1-q^n}\left(n+1/2-2E_1^{(n)}(q) + \frac{1}{1-q^n}\right)$$

Example: a trapped example

The state integral

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)\mathbf{e}(-2x^2)dx$$

is what we colloquially call "trapped". This means we can't push the contour to infinity. However, in unpublished work of Garoufalidis–Kashaev (Garoufalidis gave a talk on at MPIM on in 2018) these integrals could sometimes be dealt with using an untrapping procedure using an analogue of the q–binomial theorem for the Faddeev quantum dilogarithm. There the trapped part of the integral would be turned into a Gaussian integral which could be explicitly evaluated giving an expression of the form

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)\mathbf{e}(-3x^2/8)dx\,.$$

Example: a trapped example as q–series

Using the q–binomial theorem I rediscovered this method for q–series. For hypergeometric q–series one would like convergence of sums for $|q| \neq 1$. In this example, the q–series

$$\sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q;q)_k}$$

can be rewritten as a bilinear combination of θ–functions and sums of the form

$$\sum_{k=0}^{\infty} i^k \frac{q^{3k^2/8}}{(q;q)_k} .$$

Then one can often make sense of a θ–function when $|q| \neq 1$ (or at least choose such an extension).

Using this one can factorise the original trapped state integral as bilinear combinations of these sums with coefficients given by modular forms.

Factorisation at rationals

Factorising at rationals is done via a different approach. This uses the following lemma of Garoufalidis–Kashaev.

Lemma:[Garoufalidis–Kashaev]

If $U \subseteq \mathbb{C}$ is open, $U + a = U$ and $f : U \to \mathbb{C}$ is an analytic function such that for

$$g(z) = \frac{f(z+a)}{f(z)} \quad \text{we have} \quad g(z+a) = g(z)\,,$$

then if γ is a contour such that $g(z) \neq 1$ on γ then

$$\int_\gamma f(z)\,dz = \Big(\int_\gamma - \int_{\gamma+a}\Big)\frac{f(z)}{1-g(z)}\,dz\,.$$

This has a higher dimensional analogue. We use these formulas and evaluate the state integrals using the residue theorem. A beautiful consequence of this lemma is that for the state integrals the equation $g(z) = 1$ is the the same as the gluing equations (or critical point equations).

Example: figure eight knot

For the state integral we saw before,

$$\int_{\mathbb{R}+\epsilon} \Phi_{\mathsf{b}}(x)^2 \frac{\mathbf{e}(-x^2/2)}{1+q^{1/2}\mathbf{e}(-i\mathsf{b}x)} dx$$

we can factorise when $\tau = N/M \in \mathbb{Q}_{>0}$ using the fundamental lemma to find an elementary function times

$$\tau^{3/2} J(q) + \mathbf{e}(V_1/NM(2\pi i)^2) J_1(q) LJ_2(\tilde{q}) + \mathbf{e}(V_2/NM(2\pi i)^2) J_2(q) LJ_1(\tilde{q})$$

where

$$J(q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2} (q;q)_k^2$$

is the Kashaev invariant of 4_1 and for $X_j = \frac{1}{2} + (-1)^j \frac{\sqrt{-3}}{2}$

$$LJ_i(q) = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2} X_i^{k/M} X_i^{1/2M}}{(1-X_i^{1/M} q^k) \prod_{j=0}^{N-1} (1-q^{1+k+j} X_i^{1/M})^{2(1+j+k)/M-1}}$$

Quantum modularity

Modular forms

Vector valued modular forms on $\mathrm{SL}_2(\mathbb{Z})$ are functions satisfying symmetries. In particular, for $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_N(\mathbb{C})$ a function $f : \mathfrak{h} \to \mathbb{C}^N$ is a modular form for ρ of weight k if (push growth at ∞)

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(\tau)(c\tau + d)^k\,.$$

Example:

A standard example is the ϑ functions

$$\vartheta(\tau) = \sum_{k\in\mathbb{Z}} q^{k^2/2} \begin{pmatrix} 1 \\ (-1)^k \\ q^{k/2+1/4} \end{pmatrix}$$

which satisfy equations

$$\vartheta(\tau+1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vartheta(\tau)\,, \quad \vartheta(-1/\tau) = \mathbf{e}(-1/8) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sqrt{\tau}\vartheta(\tau)\,.$$

Original definition of quantum modular forms

Zagier introduced the notion of quantum modular forms to explain phenomenon that was observed in various special functions. The original idea is as follows: a function $f : \mathbb{Q} \to \mathbb{C}$ is a quantum modular form of weight k if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) - (c\tau + d)^k f(\tau)$$

is "better behaved" than f. For example the difference could be the restriction of an analytic function on $\mathbb{R}$ minus some points. The most original example of this phenomenon was for the log of the Kashaev invariant of the figure eight knot

$$\log J(q) = \log\left(\sum_{k=0}^{\infty}(-1)^k q^{-k(k+1)/2}(q;q)_k^2\right).$$

Pictures from Zagier's article

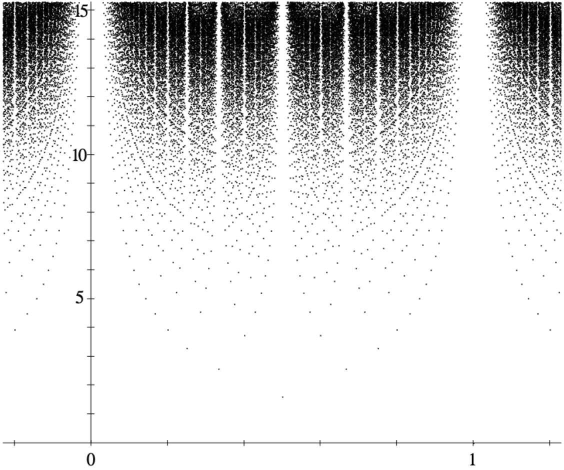


Figure 3. Graph of $f(x) = \log(\mathbf{J}(x))$

Pictures from Zagier's article

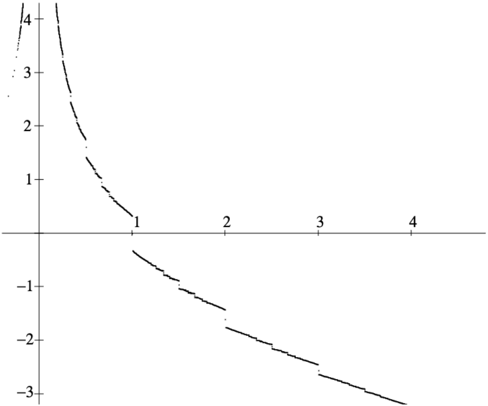


Figure 4. Graph of $h(x) = \log(\mathbf{J}(x)/\mathbf{J}(1/x))$

Quantum modularity conjecture

This leads to Zagier's quantum modularity conjecture that the Kashaev invariants of hyperbolic knots are quantum modular forms. The better behaved functions are now just functions at $\mathbb{Q}$ with a full asymptotic expansions from the left and right of each rational point.

These are conjectured to be versions of the same asymptotic series $\Phi_M(\hbar)$.

It then becomes interesting as to how this relate to the state integrals. To understand this fully, we need to refine the modularity conjecture, which we will come to later.

Remark: A q-series analogue

The original definition of quantum modularity can also be used for q–series. Suppose that $\tau = x + iy$ and $x \to \infty$ and $y > 0$ fixed. Then a quantum modular form in this sense is a function with asymptotics given by for example

$$f(\mathbf{e}(-1/\tau)) \sim f(\mathbf{e}(\tau))\Phi(2\pi i/\tau).$$

Proving this kind of modularity is similar to the other and will be discussed now.

Modularity of the Pochhammer symbol

For the figure eight knot Garoufalidis–Zagier, give a proof using certain positivity of the sum however this was generalised in work of Bettin–Drappeau for the first 10 or so hyperbolic knots. The method of Bettin and Drappaeu uses the properties of the Pochhammer symbol. In particular, the analytic properties of the Faddeev quantum dilogarithm.

Theorem:[Woronowicz]

Take

$$\frac{\left(\mathbf{e}\left(\frac{u-1}{\tau}\right);\mathbf{e}\left(-\frac{1}{\tau}\right)\right)_\infty}{(\mathbf{e}(u);\mathbf{e}(\tau))_\infty} = \frac{1}{(\mathbf{e}(u);\mathbf{e}(\tau))_{-\lfloor \Re\left(\frac{u}{\tau}\right)\rfloor}\sqrt{1-\mathbf{e}(u/\tau)}}$$

$$\times \exp\Bigg(-\frac{\tau}{2\pi i}\mathrm{Li}_2\left(\mathbf{e}\left(\frac{u}{\tau}\right)\right) + i\tau\int_0^\infty \frac{\log\left(1-\mathbf{e}\left(-ix+\frac{u}{\tau}\right)\right)-\log\left(1-\mathbf{e}\left(ix+\frac{u}{\tau}\right)\right)}{1-\mathbf{e}(-i\tau x)}dx\Bigg).$$

Proving the original form of quantum modularity

Using this we can express q–hypergeometric sums as

$$\sum_{k=0}^{\infty} \mathbf{e}(nk) \frac{\tilde{q}^{\frac{A}{2}k^2+km}}{(\tilde{q};\tilde{q})_k} = \sum_{k=0}^{\infty} \mathbf{e}(nk) \frac{\tilde{q}^{\frac{A}{2}k^2+km}}{(q;q)_{\lfloor k\Re(1/\tau)\rfloor}} \exp(\phi(\tau,k))$$

$$= \sum_{\ell=0}^{\infty} \sum_{k\in\mathbb{Z}}^{\ell\leq k\Re(1/\tau)<\ell+1} \mathbf{e}(nk) \frac{\tilde{q}^{\frac{A}{2}k^2+km}}{(q;q)_\ell} \exp(\phi(\tau,k))$$

$$= \sum_{\ell=0}^{\infty} \mathbf{e}(m\ell) \frac{q^{\frac{A}{2}\ell^2+n\ell}}{(q;q)_\ell} \sum_{x\in\mathbb{Z}-\ell\tau}^{0\leq\Re(x/\tau)<1} \mathbf{e}\Big(\frac{A}{2}x^2/\tau + xm/\tau + nx\Big) \exp(\phi(\tau,x))$$

$$= \sum_{\ell=0}^{\infty} \mathbf{e}(m\ell) \frac{q^{\frac{A}{2}\ell^2+(n+j)\ell}}{(q;q)_\ell} \sum_{x\in\mathbb{Z}-\ell\tau}^{0\leq\Re(x/\tau)<1} \mathbf{e}\Big(\frac{A}{2}x^2\tau + xm\tau + (n+j)x\Big) \exp(\phi(\tau,x))$$

Then we apply a summation method and stationary phase to the second sum. To be able to apply stationary phase we need to choose a particular j.

A family of examples

Theorem:[W.]

For $A \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$

$$\sum_{k=0}^{\infty} \frac{q^{Ak^2+mk}}{(q;q)_k}$$

are quantum modular forms (in the original sense) as q–series.

These kind of results are interesting however we want to gain a better understanding of the asymptotic series that appear. For this we now discuss the refinement of these conjectures.

Refining the modularity conjecture

Since Zagier's original article, Garoufalidis–Zagier have refined the original version of the quantum modularity conjecture taking into account exponentially small corrections. These can be most easily understood using Borel resummation, which we will discuss later, however they use a smoothed version of optimal truncation related to work of Dingle and Berry.

In the expression

$$J(-1/x) \sim \Phi(2\pi i/x)J(x)$$

replacing Φ with smooth optimal truncation one finds

$$J(-1/x) - \Phi^{\mathrm{smop}}(2\pi i/x)J(x) \sim \Phi^{\mathrm{new}}(2\pi i/x)J^{\mathrm{new}}(x)\,.$$

This indicated that J should be part of a vector not a single number.

A matrix of invariants

In fact, the function J comes as part of a matrix of similar functions. The rows of the matrix are indexed by a Gröbner basis of an associated q–difference equations while the columns are indexed by objects related to the solutions to the gluing equations (just the points of a 0 dimensional variety).

The matrix $\mathbf{J}$ then satisfies an expression of the form

$$\mathbf{J}(-1/x) \; = \; \Omega(2\pi i/x)\mathbf{J}(x)\mathbf{j}(x)\,.$$

where Ω is a matrix of extended asymptotic series similar to $\widehat{\Phi}_M$ and $\mathbf{j}(x)$ is an automorphy factor. However, more is expected and we call $\mathbf{J}$ a (matrix valued) holomorphic quantum modular form when Ω is the restriction of an analytic function on $\mathbb{C} - \mathbb{R}_{\leq 0}$.

One can prove this analytic property using state integrals.

Example: the figure eight knot

Recall the functions we had previously

$$J_m(q) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k+1)/2+mk} (q;q)_k^2$$

and for $X_j = \frac{1}{2} + (-1)^j \frac{\sqrt{-3}}{2}$

$$J_{i,m}(q) = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \frac{(-1)^k q^{k(k+1)/2+mk} X_i^{k/M} X_i^{1/2M+m}}{\prod_{j=0}^{N-1} (1 - q^{1+k+j} X_i^{1/M})^{2(1+j+k)/M-1}} \,.$$

These functions give the matrix

$$\mathbf{J}_m(q) = \begin{pmatrix} 1 & 0 & 0 \\ J_m(q) & J_{m,1}(q) & J_{m,2}(q) \\ J_{m+1}(q) & J_{m+1,1}(q) & J_{m+1,2}(q) \end{pmatrix}$$

Example: the figure eight knot

Theorem:[Garoufalidis, Gu, Kashaev, Mariño, W., Zagier]

The matrix $\mathbf{J}_m$ of the figure eight knot is a holomorphic quantum modular form.

Proof: Use the factorisation of the state integrals and a duality of the associated q–difference equations "quadratic relations" to write the entries of Ω as state integrals. Then use the analytic properties of the Faddeev quantum dilogarithm to prove the state integral has similar properties.

Example: trapped sums

The previous family of q–series can also be shown to give rise to quantum modular forms.

Theorem:

The vector of q-series for $A \in \mathbb{Z}_{>0}, m \in \mathbb{Z}$

$$\sum_{k=0}^{\infty} \frac{q^{Ak^2+km}}{(q;q)_k} \begin{pmatrix} 1 \\ q^k \\ \vdots \\ q^{(2A-1)k} \end{pmatrix}$$

is a holomorphic quantum modular form.

This looks better than our previous result as we now have analytic functions. However, it is really just different. Taking asymptotics of the matrix Ω would allows the use of this to prove the other but as stated these are independent.

Resurgence

What happened to the asymptotic series?

Holomorphic quantum modularity is now easily proved in examples. The question is then how these matrices of analytic functions relate to the asymptotic series. A conjectural answer was given in work of Garoufalidis–Gu–Mariño.

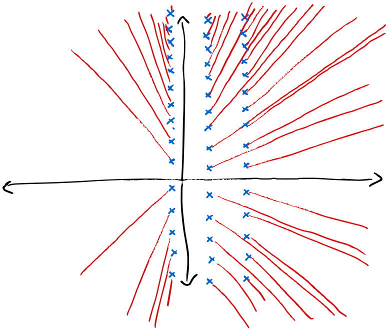
Conjecture:[Garoufalidis–Gu–Mariño]

Combinations of the state integrals associated to a q–hypergeometric sum are equal to the Borel resummation of their asymptotics.

Well firstly one needs that these asymptotic series are Borel resummable which was conjectured by Garoufalidis about ten years prior.

The Borel plane

The structure of the Borel plane for these examples is conjectured to be related to the values of the Chern–Simons invariants V_ρ (of the three manifold or the elements of the Bloch group i.e. solutions to gluing equations). It is conjectured that there are logarithmic branch cuts in Borel plane are at the difference between these values. Given $(V_\rho - V_{\rho'})/(2\pi i)$ is only defined up to $2\pi i$ these branch cuts arrange themselves into peacock patterns.



Stokes phenomenon

Going further, Garoufalidis–Gu–Mariño gave conjectures on the behaviour of the Stokes phenomenon. They conjectured that for each V_ρ there is an collection of asymptotic series with the associated exponential singularity.

[Conjecture: Garoufalidis–Gu–Mariño]

If $\widehat{\Phi}_\rho$ is an asymptotic series with exponential singularity $\mathbf{e}(V_\rho/(2\pi i)^2/\tau)$ then if

$$\arg(\tau) = \arg\left(\frac{V_{\rho'} - V_\rho + (2\pi i)^2 k}{2\pi i}\right)$$

we have

$$s_+(\widehat{\Phi}_\rho)(2\pi i/\tau) - s_-(\widehat{\Phi}_\rho)(2\pi i/\tau) = \sum_{\rho' : \arg(\tau) = \arg(V_{\rho'}/2\pi i)} S_{\rho,\rho',k}\, q^k \widehat{\Phi}_{\rho'}\,,$$

for some $S_{\rho,\rho',k} \in \mathbb{Z}$.

Applications

These conjectures while applied originally to asymptotic series associated to knots in work of Garoufalidis–Gu–Mariño and Garoufalidis–Gu–Mariño–W. can be applied more generally to asymptotic series coming from proper q–hypergeometric functions.

To finish I will discuss a case where one can carry out computations to get conjectures for generating series of Stokes constants of the q–hypergeometric function

$$\sum_{k=0}^{\infty} \frac{q^{2k^2+mk}}{(q;q)_k}.$$

Example: generating series of Stokes constants

The function $\sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q;q)_k}$ has an asymptotic series

$$\widehat{\Phi}(\hbar) \;=\; \exp(V/\hbar)\frac{1}{\sqrt{\delta}}\sum_{k=0}^{\infty} A_k \hbar^k$$

where for $X^4 + X - 1 \;=\; 0$ and

$$V = \mathrm{Li}_2(X) - \frac{\pi^2}{6} + 2(2\pi i)^2 \log(X)^2 - (2\pi i)^2(4k + m(X))\log(X)$$
$$\delta = 4 - 3X\,,$$

with $m(X) \in \mathbb{Z}$ and

$$\begin{aligned}
A_0 &= 1\,,\\
A_1 &= \frac{-64 + 100\,X + 18\,X^2 - 54\,X^3}{24\,\delta^3}\,,\\
A_2 &= \frac{-104876 + 113812\,X + 29836\,X^2 + 17388\,X^3}{1152\,\delta^6}\,,\\
A_3 &= \frac{-79093616 - 1648464240\,X + 2928617760\,X^2 - 694542712\,X^3}{414720\,\delta^9}\,.
\end{aligned}$$

We have four series (one for each embedding of the field into $\mathbb{C}$).

Example: generating series of Stokes constants

We can consider the evaluation of the sum

$$f_m(q) \;=\; \sum_{k=0}^{\infty} \frac{q^{2k^2+km}}{(q;q)_k}\,,$$

at $\tilde{q} = \mathbf{e}(-1/\tau)$ where $\tau = 1000\,\mathbf{e}(0.0001)$ and we find that

$$f_0(\tilde{q}) \;=\; (1.4799\cdots + 1.8058\cdots i) \times 10^{67}\,.$$

we find that the quotient is given by

$$\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} \;=\; 1.0000\cdots - 2.7438\cdots \times 10^{-8}\,.$$

Then we see that

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1\Big) q^{-3} \;=\; (1.0197 - 2.4883 \times 10^{-5} \cdot i)\,,$$

and similarly,

$$\Big(\frac{f_0(\tilde{q})}{s(\widehat{\Phi}^{(3)})(2\pi i/\tau)} - 1 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - q^9\Big) q^{-10} \;=\; (2.0397\cdots - 5.0718\cdots i \times 10^{-5})\,.$$

Indeed, continuing we can identify this q–series as

$$f_1(q) \;=\; 1 + q^3 + q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + 2q^{10} + 2q^{11} + 3q^{12} + 3q^{13} + \cdots\,,$$

Example: generating series of Stokes constants

Then we find that

$$f_0(\tilde{q}) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau)f_1(q) \;=\; 0.20122\cdots + 0.68776\cdots i\,.$$

Then we can continue this kind of computation to find that numerically

$$\begin{aligned} f_0(\tilde{q}) - s(\widehat{\Phi}^{(1)})(2\pi i/\tau)f_2(q) - s(\widehat{\Phi}^{(2)})(2\pi i/\tau)f_0(q) - s(\widehat{\Phi}^{(3)})(2\pi i/\tau)f_1(q) \\ - s(\widehat{\Phi}^{(3)})(2\pi i/\tau)qf_3(q) \;=\; (5.5399\cdots - 3.7010\cdots i) \times 10^{-138}\,. \end{aligned}$$

This error is exactly the order of numerical error of the Borel resummation. This leads to the conjecture when τ is just above the positive reals

$$\begin{aligned} f_0(\tilde{q})?=?s(\widehat{\Phi}^{(1)})(2\pi i/\tau)f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau)f_0(q) \\ + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)f_1(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)qf_3(q)\,. \end{aligned}$$

Performing similar numerical checks for τ just above the negative reals, we find a similar statement

$$\begin{aligned} f_0(\tilde{q})?=?s(\widehat{\Phi}^{(1)})(2\pi i/\tau)f_2(q) + s(\widehat{\Phi}^{(2)})(2\pi i/\tau)f_0(q) \\ + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)qf_3(q) + s(\widehat{\Phi}^{(3)})(2\pi i/\tau)f_1(q)\,. \end{aligned}$$

Example: generating series of Stokes constants

Therefore, we find a q–series when we take the quotient of the two matrices of the Borel resummations that from the conjectures gives generating series for the Stokes constants. In particular, completing f_m to a matrix $F(q)$ (that appears in the factorisation of the state integral)

$$
\begin{aligned}
&s_I(\widehat{\Phi})(\tau)^{-1} s_{II}(\widehat{\Phi})(\tau) \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} F(q) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^4 \end{pmatrix} F(\tilde{q})^{-1} F(\tilde{q}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau & 0 & 0 \\ 0 & 0 & \tau^2 & 0 \\ 0 & 0 & 0 & \tau^4 \end{pmatrix}^{-1} F(q)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} F(q) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} F(q)^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \end{pmatrix}^{-1} \\
&= \mathrm{Id} + \begin{pmatrix} -q - 2q^2 + & 1 + q + q^2 + & 1 - q^2 + & -1 - q + \\ q^2 + & -q - q^2 + & -q + & q + q^2 + \\ -q - q^2 + & 1 - q^2 + & -q - 2q^2 + & 2q^2 + \\ q + & -1 + q + q^2 + & 2q + q^2 + & -q - 2q^2 \end{pmatrix} + O(q^3)
\end{aligned}
$$

Final remarks

These conjectures give powerful tools that with a little bit of numerics allows for a complete determination of the Stokes phenomenon for these proper q–hypergeometric functions.

In fact, the methods used for stationary phase and original version of quantum modularity can often be used to guess that combinations of the state integrals that are needed to numerically find the Borel resummation.

I applied these methods in my thesis to compute these Stokes constants and prove quantum modularity for the WRT and $\widehat{Z}$ invariant of the closed manifold $4_1(1,2)$ (a more specific write up to appear soon).

Thanks!

References:

- A TQFT from quantum Teichmüller theory. Andersen, Kashaev.
- Holomorphic Blocks in Three Dimensions. Beem, Dimofte, Pasquetti.
- Modularity and value distribution of quantum invariants of hyperbolic knots. Bettin, Drappeau.
- Quantum content of the gluing equations. Dimofte, Garoufalidis
- Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group. Dimofte, Gukov, Lenells, Zagier.
- Quantum dilogarithm. Faddeev, Kashaev.
- The resurgent structure of quantum knot invariants. Garoufalidis, Gu, Mariño.
- From state integrals to q–series. Garoufalidis, Kashaev.
- Evaluation of state integrals at rational points. Garoufalidis, Kashaev.
- Resurgence of Chern-Simons theory at the trivial flat connection. Garoufalidis, Gu, Mariño, W.
- Topological invariance of complex Chern-Simons and Teichmüller TQFT perturbation theory. Garoufalidis, Storzer, W.
- Periods, the meromorphic 3D-index and the Turaev–Viro invariant. Garoufalidis, W.
- Knots, perturbative series and quantum modularity Garoufalidis, Zagier.
- Knots and their related q–series. Garoufalidis, Zagier.
- Hyperbolic Structure Arising from a Knot Invariant. Hikami.
- Modular q–difference equations and quantum invariants of hyperbolic three–manifolds. W.
- Quantum exponential function. Woronowicz.
- Quantum field theory and the Jones polynomial. Witten
- Quantum modular forms. Zagier.