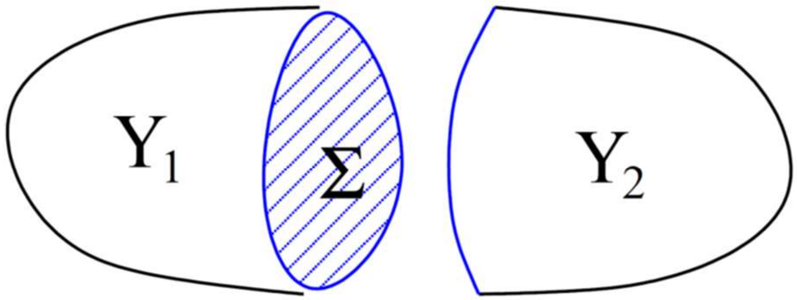
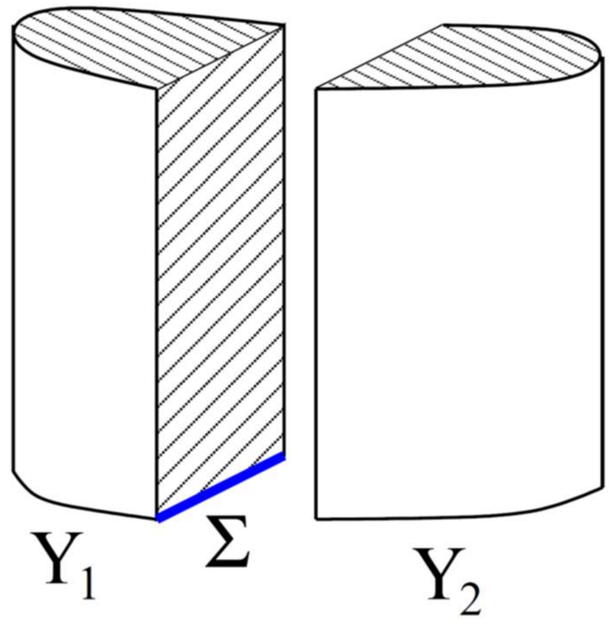
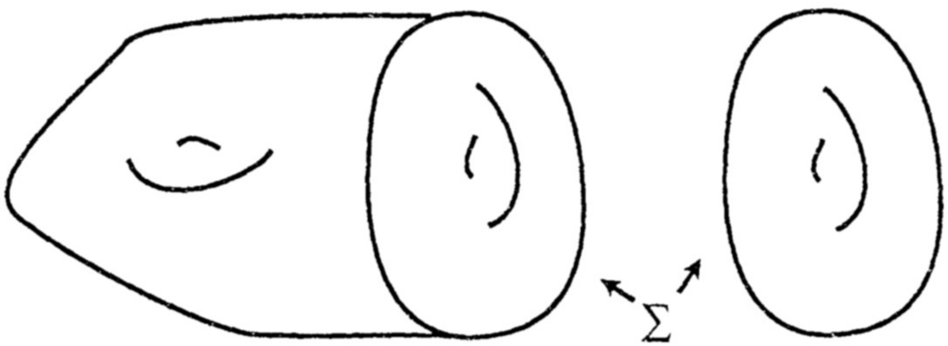




Based on: S.G. (2007)
S.G., E.Witten
T.Dimofte, S.G., L.Hollands
:
S.G., D.Pei, P.Putrov, C.Vafa
S.G., M.Marino, P.Putrov
S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G.
S.G., A.Sheshmani, S.-T.Yau
w.i.p. w/ P.-S.Hsin, D.Pei
w/ P.Putrov

Related work: :
Y.Murakami
M.Jagadale
M.Kontsevich, Y.Soibelman
L.Diogo, T.Ekholm
S.Park (2020, 2022)
A.Beliakova, C.Blanchet, A.Gainutdinov
S.Garoufalidis, J.Gu, M.Marino
J.Chae (2020, 2021, 2022)
S.Ganatra, J.Pardon, V.Shende
M.Jeffs

Non-perturbative Complex Chern-Simons



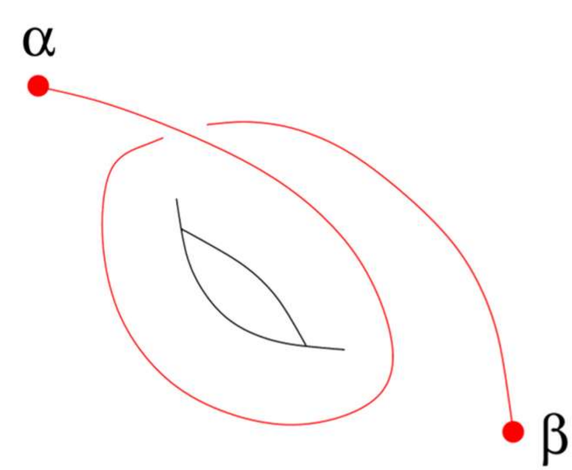
cf. E.Witten (1989)

Non-perturbative Complex Chern-Simons

$$\int \frac{DA}{\text{gauge}} e^{-\frac{1}{\hbar}S[A]}$$

Contains contributions of all complex flat connections $\alpha \in \pi_0(\mathcal{M}_{\text{flat}}(M_3, G))$:

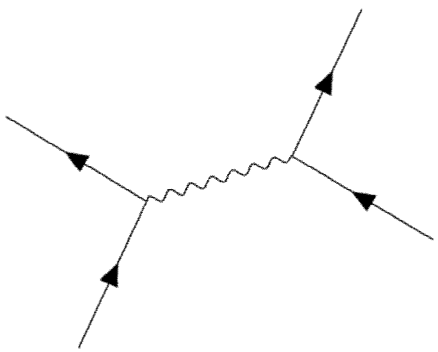
$$\mathcal{Z}_\alpha^{\text{pert}}(\hbar) = \sum_{n=0}^{\infty} a_n^\alpha \hbar^{n+c_\alpha}$$



Lift $\tilde{\alpha} \in \pi_0(\mathcal{M}_{\text{flat}}(M_3, G)) \times \mathbb{Z}$

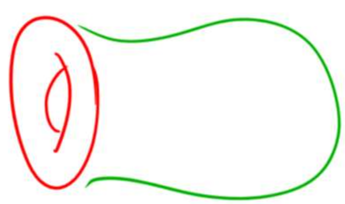
- Perturbative complex Chern-Simons well understood in early 2000s (conceptually and computationally)

- even the direct calculation leads to finite integrals under good analytical control

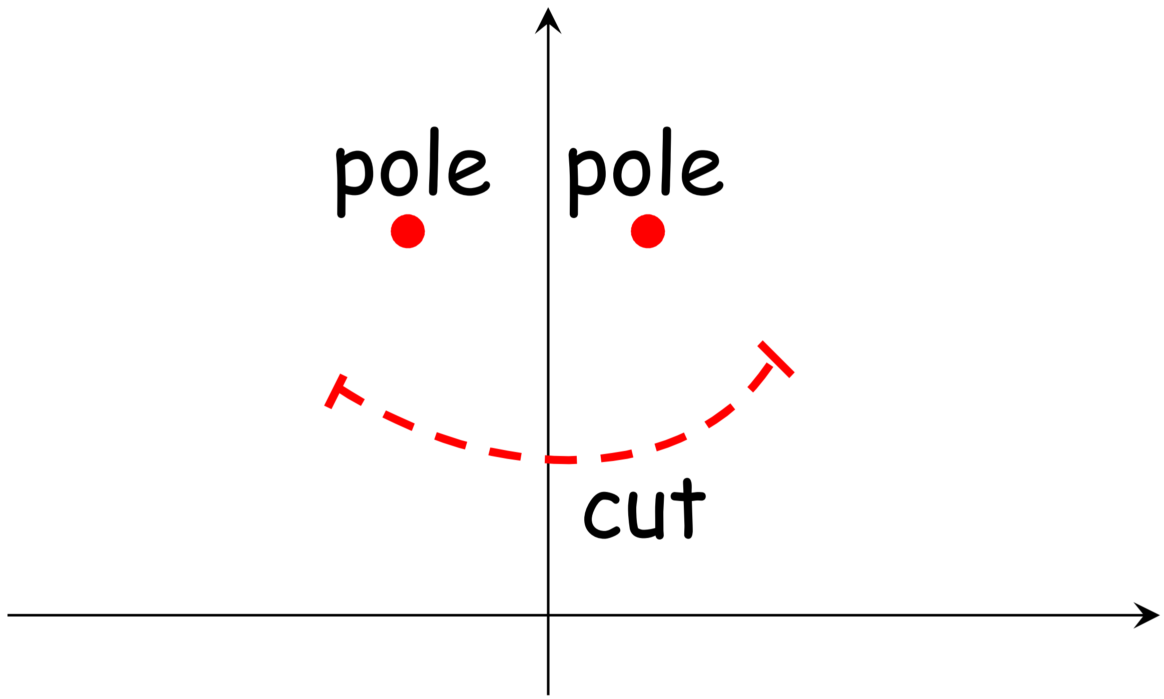


cf. S.Axelrod, I.Singer

- Many partition functions from 3d-3d correspondence ... can not completely close the boundary until 2013



Non-perturbative Complex Chern-Simons



perturbative → Borel plane → non-perturbative

$$\sum a_n \hbar^n \qquad B(S) = \sum \frac{a_n}{n!} S^n \qquad \frac{1}{\hbar}\int_0^\infty B(S) e^{-S/\hbar} dS$$

Non-perturbative Complex Chern-Simons

$U_q(\mathfrak{g})$

at generic q

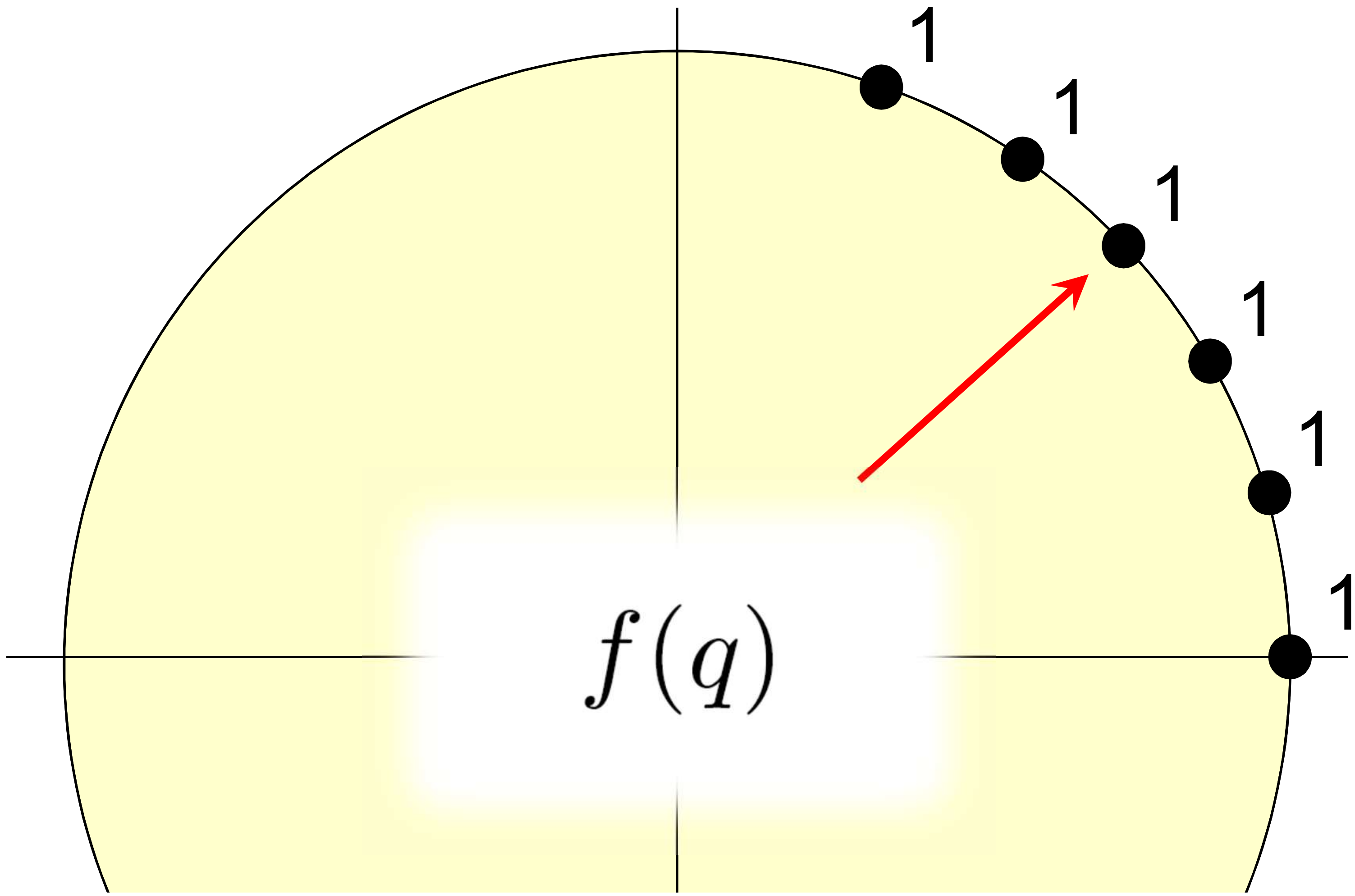
$q = e^{\hbar} \to 1$

$$\mathcal{Z}_\alpha^{\text{pert}}(\hbar) \;=\; \sum_{n=0}^{\infty} a_n^\alpha \hbar^{n+c_\alpha}$$

function of q

Borel resum

$q = e^{\hbar}$ complex, continuous



$$\mathfrak{g} = \mathfrak{sl}_2 \; : \qquad KEK^{-1} = q^2 E\,, \;\; KFK^{-1} = q^{-2} F$$

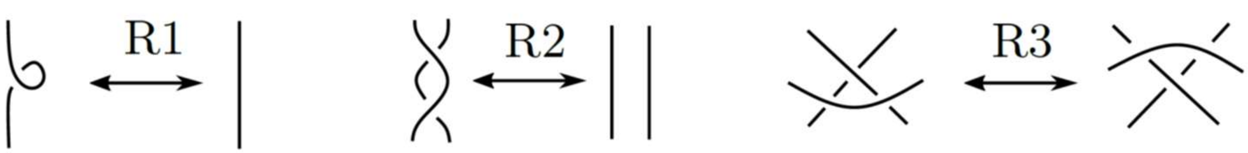
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

$S^3_{-1}($  $) = S^3_{+1}($  $)$

<u>Theorem:</u> Using the R-matrix for Verma modules, for all links of unknots (plumbings), torus links, positive braid links, fibered knots up to 10 Xs, and homogeneous braid links the two-variable series

$$F_K(x,q) := \sum_{b\in\mathbb{Z}} x^b\, \widehat{Z}_b\left(S^3 \setminus K\right)$$

is well defined and invariant under the required braid moves (cf. Reidemeister moves).

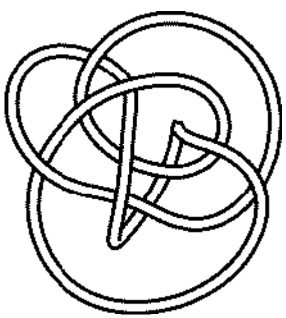


S.G., D.Pei, P.Putrov, C.Vafa
S.G., C.Manolescu
S.Park (2020, 2021)
J.Chae
:

$$M_3 = S^3_{-1/2}($$  $$):$$

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212q^{500} + \ldots)$$

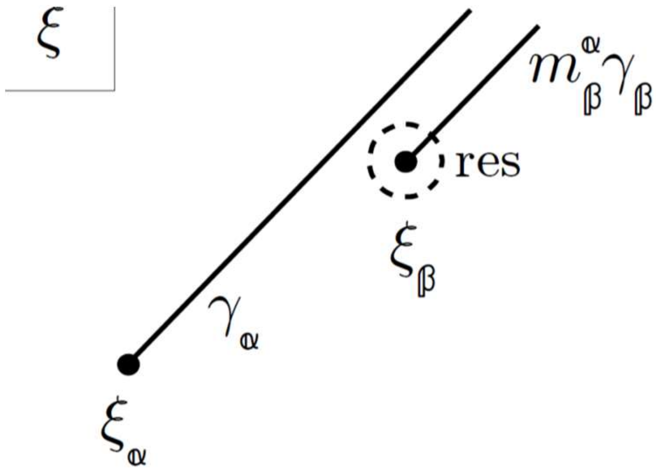
$$M_3 = -S^3_{+5}(\mathbf{10_{145}}):$$



$$\begin{aligned}
b = 2:& \quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right)\\
b = 1:& \quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right)\\
b = 0:& \quad 2q^4 + 2q^5 + 4q^6 + 8q^7 + 14q^8 + \ldots\\
b = -1:& \quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right)\\
b = -2:& \quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right)
\end{aligned}$$

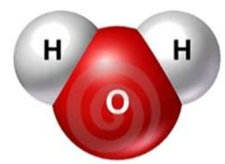
$$Z_\alpha(q=e^{\hbar}) \longrightarrow Z_a(q) \longrightarrow \widehat{Z}_b(M_3;q)$$

thimble integral



labeled by complex flat connection

Borel resummation, function of q



labeled by $a \in H_1(M_3;\mathbb{Z})$

integral q-series, functoriality, cutting-and-gluing

labeled by $b \in \mathrm{Spin}^c(M_3)$

$$\widehat{Z}_b = \sum_{\substack{a \\ \text{abelian}}} S^{ab}\Big(\mathcal{S}Z_a^{\text{pert}} + \sum_{\substack{\beta \\ \text{non-abelian}}} n_a^{\beta}\,\mathcal{S}Z_{\beta}^{\text{pert}}\Big)$$

Example: Lens spaces & some mapping tori

$$\widehat{Z}(M_3) = q = e^{\hbar}$$

S.Chun, S.G., S.Park, N.Sopenko
J.Andersen, W.Mistegaard

→ trivial Borel plane,
"almost abelian" flat connections

Theorem:

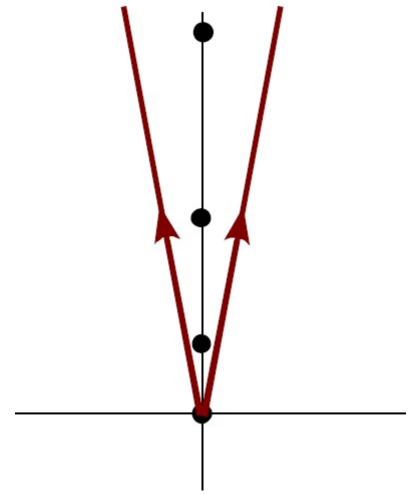
$$n_{\beta}^{\alpha} = 0 \qquad \begin{aligned} \alpha &= \text{any} \\ \beta &= \text{abelian} \end{aligned}$$

S.G., M.Marino, P.Putrov

Stokes / trans-series coefficients are
not symmetric!

Definition:

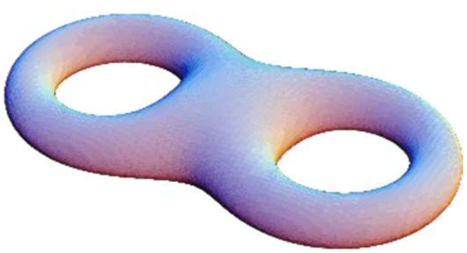
$$\mathcal{I}^{\alpha}_{\beta}(M_3) := \sum_{\beta \text{ (fixed } \beta)} n^{\alpha}_{\beta}\, \widetilde{q}^{\,\mathrm{CS}_{\alpha}-\mathrm{CS}_{\beta}}$$



Example:

$$\mathcal{I}^{\alpha_1}_{0} = \widetilde{q}^{\frac{1}{4}} \left(1 + \widetilde{q}^{2} + \widetilde{q}^{6} + \widetilde{q}^{12} + \widetilde{q}^{20} + \ldots\right)$$

degee-1 circle bundle over a genus-2 surfaces



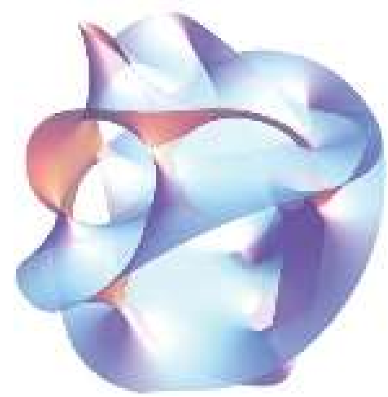
$$\vdots$$

$$b_{10} = 0$$

$$b_{8} = 15$$

$$b_{6} = 2$$

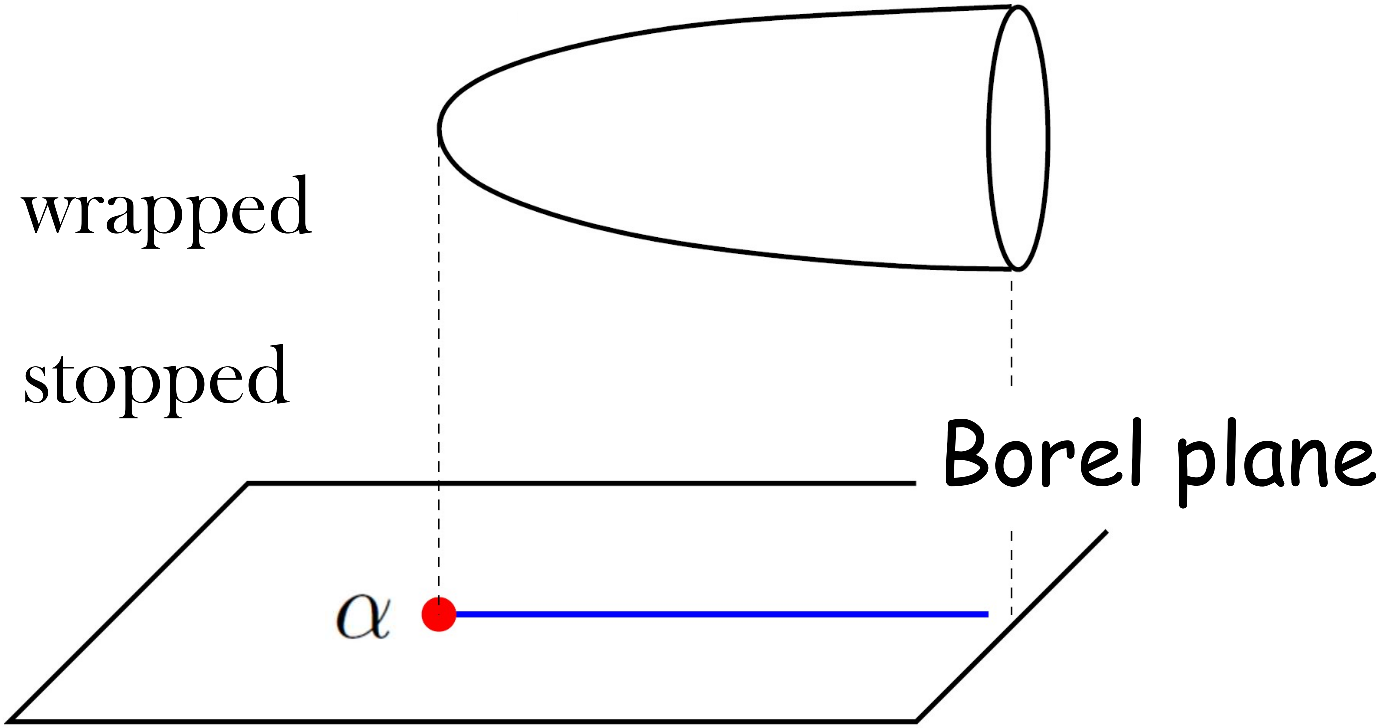
$$\vdots$$

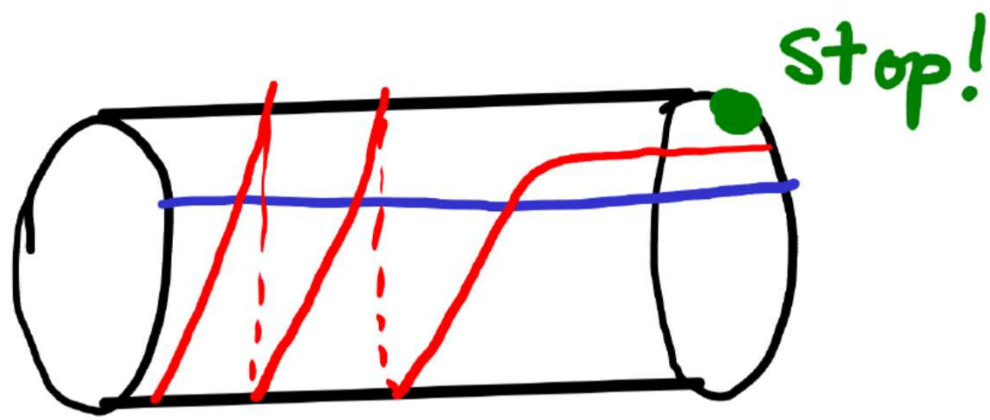



α
β

Landau-Ginzburg *(X, W)*

- Fiberwise wrapped
- Fiberwise stopped

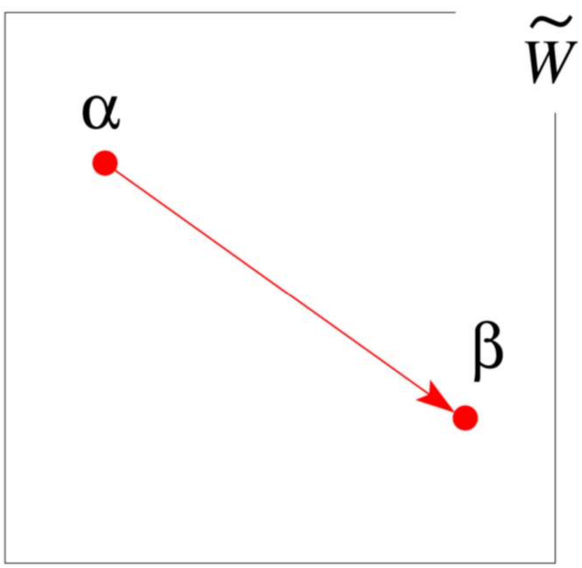




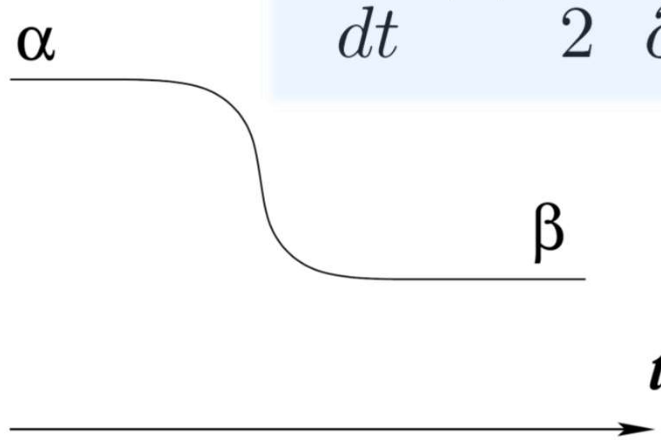
Landau-Ginzburg model	complex Chern-Simons
$\widetilde{W}$-plane	Borel plane
critical points	flat connections
solitons	instantons

S.G., M.Marino, P.Putrov

$$\mathrlap{\overbrace{\phantom{\mathbb{R} \quad \times \quad \mathbb{R}}}^{\text{LG model}}}\mathbb{R} \quad \times \quad \underbrace{\mathbb{R} \times M_3}_{\text{YM instantons}}$$



$$\frac{d\overline{z}_j}{dt} = \frac{e^{i\theta}}{2} \frac{\partial \widetilde{W}}{\partial z_j}$$



⋮

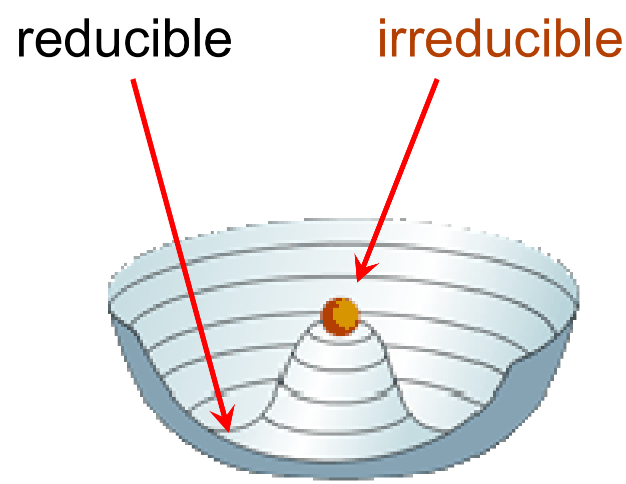
S.Cecotti, C.Vafa

K.Hori, A.Iqbal, C.Vafa

⋮

Landau-Ginzburg

S.G., M.Marino, P.Putrov



$$\Gamma_{\alpha,\theta} \cap \Gamma_{\beta,\theta+\pi} = \mathcal{M}_{\text{soliton}}(\widetilde{W}, \alpha, \beta)$$

$$= \mathcal{M}_{\text{inst}}(M_3 \times \mathbb{R}, \alpha, \beta; SL(2,\mathbb{C})))$$

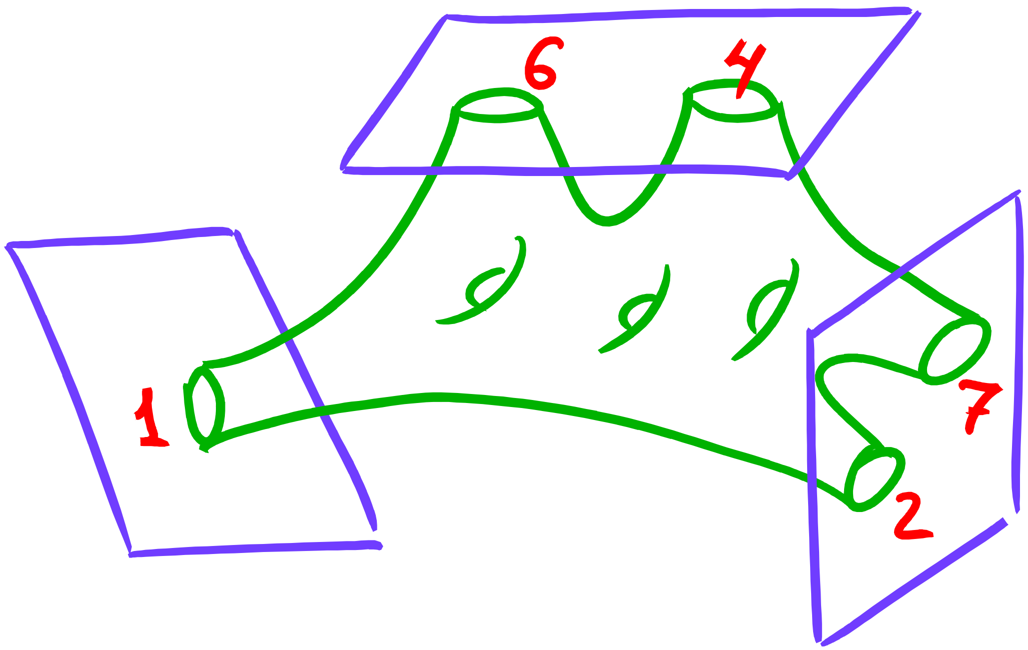
$$Z = \int_\Gamma \frac{dz}{2\pi i z}\, e^{\frac{1}{\hbar}\widetilde{W}(z)+\ldots}$$

Conjecture: $\widehat{Z}_b(M_3;q)$ should admit a definition via moduli spaces in curve counting

S.G., D.Pei, P.Putrov, C.Vafa

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow (X, L)$$

Σ genus g, with n boundary components



$$\partial\Sigma = \gamma_1 \sqcup \ldots \sqcup \gamma_n$$

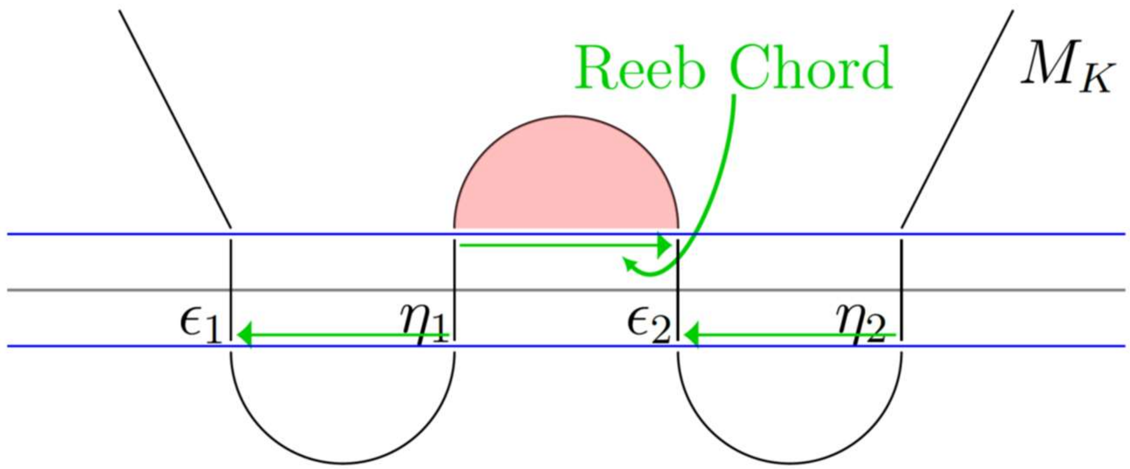
$$\beta = \phi_*[\Sigma] \in H_2(X,L)$$

$$b_i = \phi_*[\gamma_i] \in H_1(L)$$

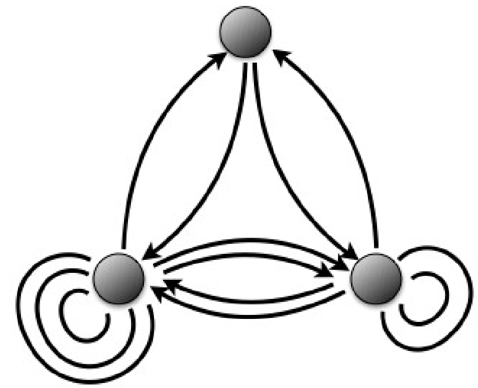
Some evidence: L.Diogo, T.Ekholm

:

➡ Landau-Ginzburg $\widetilde{\mathcal{W}}(\boldsymbol{x})$



BPS quiver



$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\widetilde{\mathcal{W}}(\boldsymbol{x}) = \sum_i \mathrm{Li}_2(x_i) + \sum_{i,j} \frac{C_{ij}}{2} \log x_i \, \log x_j$$

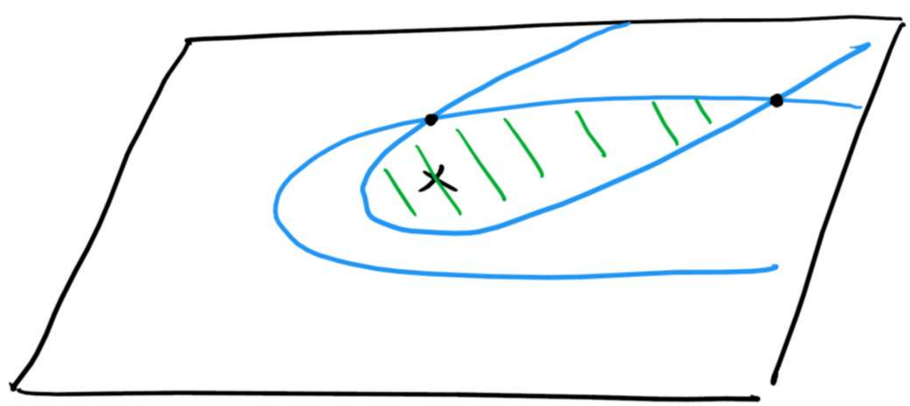
T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G., S.Harrison, D.Passaro
:

Problem: reproduce from Fukaya-Seidel category

$$\mathcal{I}_0^{\alpha_1}(S^3_{-1/r}(\mathbf{3_1})) = \Phi^{(6r-5)}_{36r+6} - \Phi^{(6r+7)}_{36r+6}$$

where
$$\Phi_p^{(a)}(q) \ := \ \sum_{n=0}^{\infty} \phi_p^{(a)}(n) q^{\frac{n^2}{4p}} \qquad \in q^{\frac{a^2}{4p}}\,\mathbb{Z}[[q]]$$

$$\phi_p^{(a)}(n) = \begin{cases} (-1)^n, & n \equiv \pm a \bmod p\,, \\ 0, & \text{otherwise.} \end{cases}$$

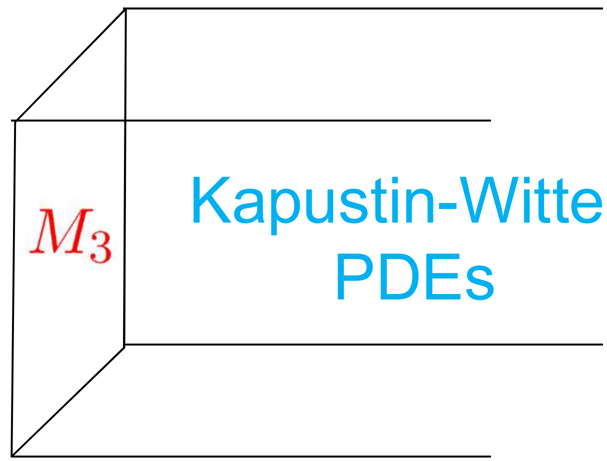


Conjecture: $\widehat{Z}_b(M_3;q)$ should admit a definition via moduli spaces in gauge theory *

S.G., D.Pei, P.Putrov, C.Vafa

$$\mathcal{B}_{\text{Nahm}}$$

$$\frac{\sum_{i=1}^{3} e^i_\mu t_i}{y} + \ldots$$



$$\mathcal{B}_b$$

$$\cap$$

$$\text{Spin}^c(M_3)$$

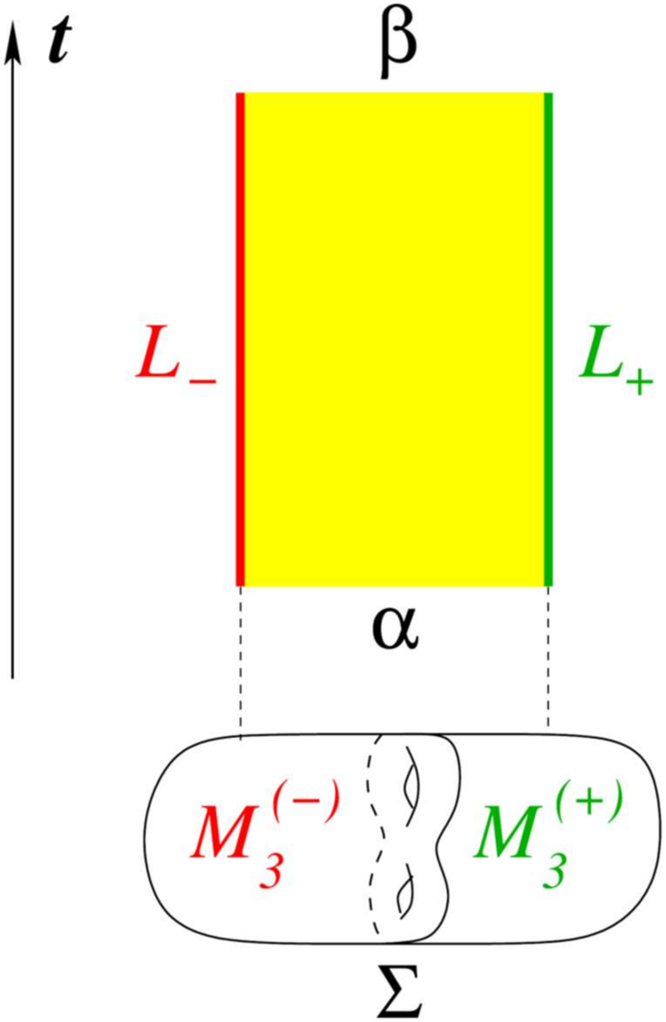
---

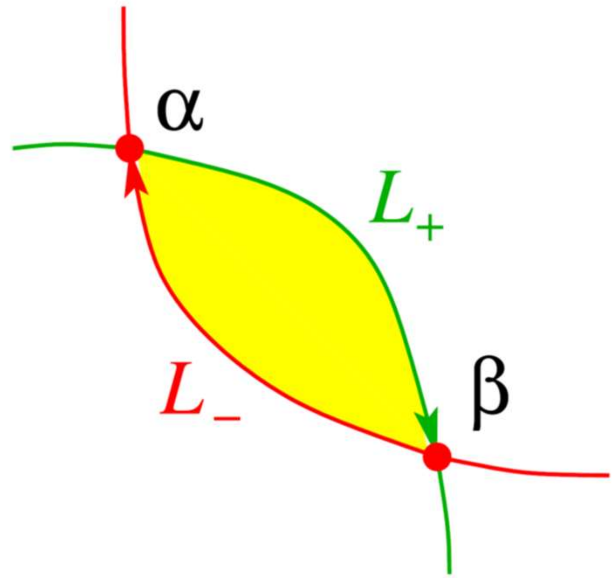
* Compactification and choice of chamber: K-theory / multiplicative / caloron version

Early clues: S.Chun, S.G., S.Park, N.Sopenko

More detailed analysis: S.G., P.-S.Hsin, D.Pei (to appear soon)

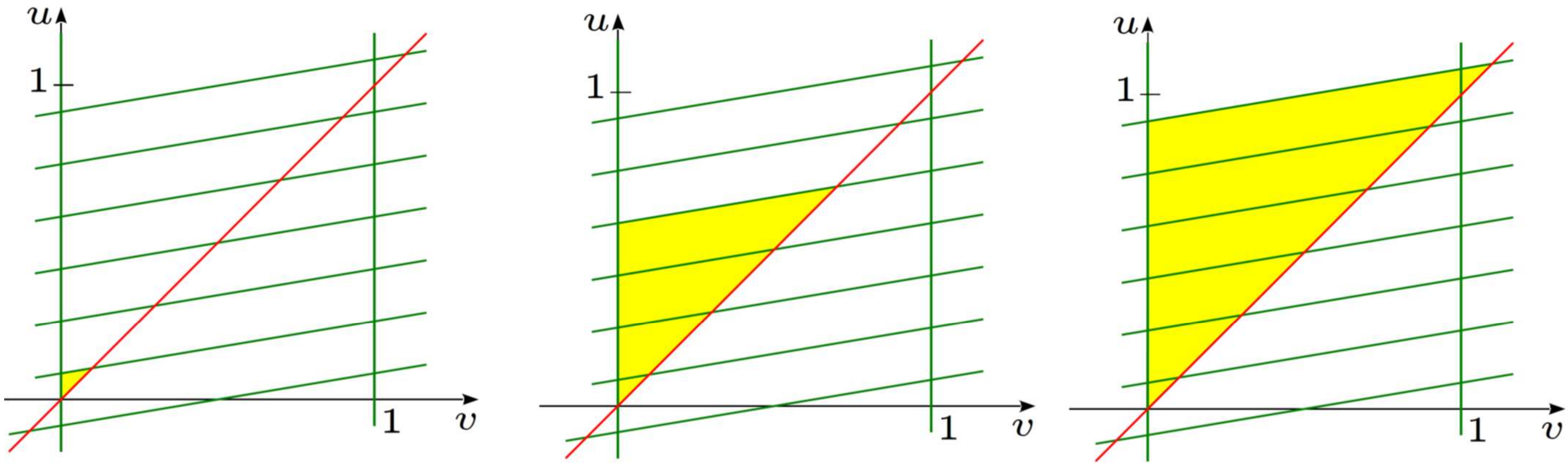
Heegaard decomposition $\quad M_3 = M_3^{(+)} \cup_\Sigma M_3^{(-)}$





S.G., M.Marino, P.Putrov

A-model $\quad \phi : \mathbb{R} \times I \to \mathcal{M}_H(G, \Sigma) \cong \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, \Sigma)$



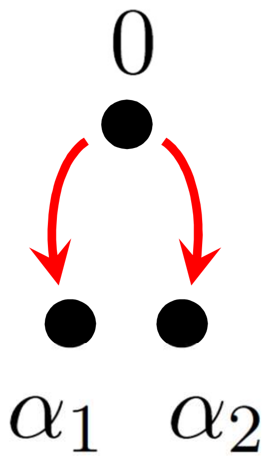
1/120 49/120 121/120

Figure 9. Disk instantons for the Poincaré sphere $M_3 = \Sigma(2,3,5)$.

S.G., M.Marino, P.Putrov

$$\mathcal{I}^{\alpha}_{\beta}(M_3) := \sum_{\mathbb{\beta}\ (\text{fixed } \beta)} n^{\mathbb{\alpha}}_{\mathbb{\beta}}\, \widetilde{q}^{\mathrm{CS}_{\mathbb{\alpha}} - \mathrm{CS}_{\mathbb{\beta}}}$$

Two functions of Lawrence and Zagier:



$$q^{\frac{1}{120}}\left(1+q+q^3+q^7-q^8-q^{14}-q^{20}+\ldots\right)$$

$$q^{\frac{49}{120}}\left(1+q+q^2+q^4-q^{11}-q^{15}-q^{18}+\ldots\right)$$

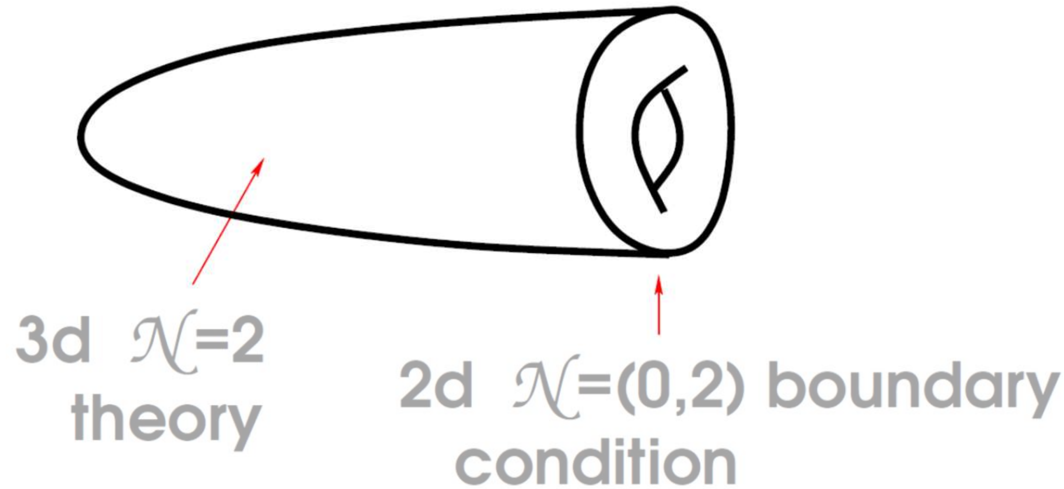
generating series of Stokes coefficients !

S.G., M.Marino, P.Putrov

$$\mathcal{I}^{\alpha}_{\beta}(M_3) := \quad \alpha$$

β

$$\widehat{Z}_{b}(M_3;q) =$$



$$\mathcal{I}^{\alpha}_{\beta}(M_3,\widetilde{q}) = \alpha$$

$$\beta$$

Puzzle: Explain why, for Brieskorn spheres,

$$? \quad \mathcal{B}_{\alpha_*} \overset{?}{=} \mathcal{B}_{\mathrm{Nahm}}$$

On $M_4 = \mathbb{R} \times M_3$ same as VW equations

$$\begin{array}{llll}
\textbf{VW:} & A & B \in \Omega^{2,+}(M_4, \mathfrak{g}) & \overbrace{\phi,\ \overline{\phi}}^{\text{“u-plane”}} \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}}) \\
 & & C \in \Omega^0(M_4, \mathfrak{g}) & \\
 & \updownarrow & \updownarrow & \updownarrow \\
\textbf{KW:} & A & \phi \in \Omega^1(M_4, \mathfrak{g}) & \underbrace{\sigma,\ \overline{\sigma}} \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})
\end{array}$$

play an important role in the gauge theory approach to the geometric Langlands program

- Vafa-Witten theory

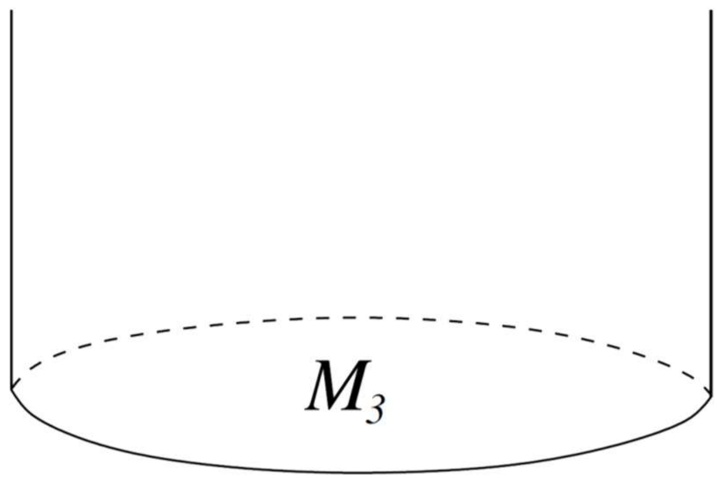
$$\begin{cases} F_A^+ - \frac{1}{2}[B \times B] + [C, B] = 0 \\ d_A^* B - d_A C = 0 \end{cases}$$

$$A \in \mathcal{A}_P$$
$$B \in \Omega^{2,+}(M_4; \mathrm{ad}_P)$$
$$C \in \Omega^0(M_4; \mathrm{ad}_P)$$

How close is it to an Atiyah-style TQFT ?

S.G., A.Sheshmani, S.-T.Yau

$$M_4 = \mathbb{R} \times M_3 =$$



Homology *a la* Floer

$$\mathcal{H}_{\mathrm{VW}}(M_3) \ = \ ?$$

Special family:

$M_3 = S^1 \times \Sigma_g$

$g = 0:$ Gluck twist

$g = 1:$ knot surgeries, log-transforms, ...

and degree-p circle bundles

Expect modular SL(2,$\mathbb{Z}$) action S.G., P.Putrov, C.Vafa

THE SELF-DUALITY EQUATIONS ON A RIEMANN SURFACE

N. J. HITCHIN

[Received 15 September 1986]

Introduction

In this paper we shall study a special class of solutions of the self-dual Yang–Mills equations. The original self-duality equations which arose in mathematical physics were defined on Euclidean 4-space. The physically relevant solutions were the ones with finite action—the so-called 'instantons'. The same equations may be dimensionally reduced to Euclidean 3-space by imposing invariance under translation in one direction. These equations also have physical relevance—the

<u>Proposition:</u> in Vafa-Witten theory on $M_3 = S^1 \times \Sigma$

$$\mathcal{M}_{\mathrm{VW}}(G, M_3) \cong \mathcal{M}_H^{\mathcal{E}}(G, \Sigma)$$

moduli space of $\mathcal{E}$-valued Higgs bundles with $R_i = (2, 0, 0)$.

S.G., A.Sheshmani, S.-T.Yau

$\mathcal{M}_H^{\mathcal{E}}(G,\Sigma)$:

$$L_i = K^{R_i/2} \quad (R_i \in \mathbb{Z})$$

$L_1 \otimes \mathrm{ad}_P \oplus L_2 \otimes \mathrm{ad}_P \oplus L_3 \otimes \mathrm{ad}_P$ valued Higgs bundles

or, $\mathcal{E} = L_1 \oplus L_2 \oplus L_3$ valued, for short

Moreover, $U(1)_t \circlearrowright \mathcal{M}_H(G,\Sigma)$

$(A,\Phi) \mapsto (A, e^{i\theta}\Phi)$ circle action

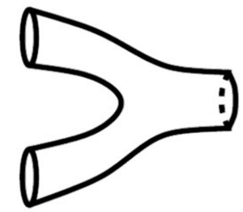
Similarly, $U(1)_x \times U(1)_y \times U(1)_t \circlearrowright \mathcal{M}_H^{\mathcal{E}}(G,\Sigma)$

⇨ explicit expression for $\mathcal{H}_{\mathrm{VW}}(\Sigma_g \times S^1)$ and its equivariant character

S.G., A.Sheshmani, S.-T.Yau
cf. V.Munoz

explicit expression for $\mathcal{H}_{\mathrm{VW}}(\Sigma_g \times S^1)$
and its equivariant character

cf. equivariant Verlinde formula



2d TQFT on Σ_g

cf. J.Andersen, S.G., D.Pei

Equivalently, a Rozansky-Witten theory with non-compact Coulomb branch

the answer is very large, even for $G = SU(2)$

spectral sequences

Theorem:

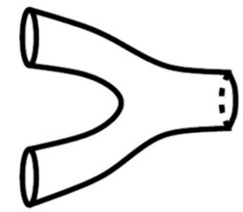
P.Ozsvath, Z.Szabo

$$HF^+(S^2 \times S^1, \mathfrak{s}) \quad = \quad \mathcal{T}^+_{-1/2} \oplus \mathcal{T}^+_{1/2}, \qquad \mathfrak{s} = \mathfrak{s}_0$$

$$HF^+(\Sigma_g \times S^1, \mathfrak{s}_h) \quad = \quad \bigoplus_{i=0}^{d} \Lambda^i H^1(\Sigma_g; \mathbb{Z}) \otimes \mathcal{T}_0^+ / (U^{i-d-1}), \qquad h \neq 0$$

where $d = g - 1 - |h|$ *and* $\mathfrak{s}_h$ *is the* spin^c *structure with* $c_1(\mathfrak{s}_h) = 2h[S^1]$.

explicit expression for $\mathcal{H}_{\mathrm{VW}}(\Sigma_g \times S^1)$
and its equivariant character
cf. equivariant Verlinde formula



<u>Claim:</u> when $\pi_1(G) = 1$ the Gluck involution acts trivially on $\mathcal{H}_{\mathrm{VW}}(S^2 \times S^1)$ and $Z_{\mathrm{VW}}(M_4, G)$ can not detect the Gluck twist.

<u>Remark:</u> the moduli spaces are different for

$$M_4 = \mathbb{R} \times S^1 \times \Sigma_g \cong \mathbb{C}^* \times \Sigma_g$$ ("K-theoretic")

and

$$M_4 = \mathbb{R} \times \mathbb{R} \times \Sigma_g \cong \mathbb{C} \times \Sigma_g$$ ("homological")

Surprise: New 4-manifold invariant

S.G., P.-S.Hsin, D.Pei

$$Z(M_4; q_1, q_2)$$

$q_2 \to 1$

$q_1 = q_2 = q$

$$Z_{\text{VW}}(M_4) = q_1^{\frac{\chi}{12}} \sum_{n \in \mathbb{Z}_{\geq 0}} a_n q_1^n$$

$$Z_{T[M_4]}(q)$$

Elliptic genus of $T[M_4]$

6d theory on $T^2 \times M_4$

C.Vafa, E.Witten

A.Gadde, S.G., P.Putrov

B.Feigin, S.G.

:

for $G = SU(2)$ has modular weight $w = -\frac{\chi}{2}$

$$\frac{\chi}{2} + \frac{3}{2}\sigma$$

Also different as "decorated TQFTs"

(have different gradings)

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:

$$\frac{\chi}{2} + \frac{3}{2}\sigma$$

G - valued

$$B \in \Omega^{2,+}(M_4, \mathfrak{g}) \qquad C \in \Omega^0(M_4, \cancel{\mathfrak{g}}) \qquad \phi,\ \overline{\phi} \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})$$

- Similar lesson for the GPPV conjecture.

- Moreover, the role of σ, $\overline{\sigma}$, e.g. for $M_3 = L(p,1)$:

$$\widehat{Z}_b(q) = \int_{|x|=1} \frac{dx}{2\pi i x} \underbrace{\frac{(x;q)_\infty}{(xq;q)_\infty} \frac{(x^{-1};q)_\infty}{(x^{-1}q;q)_\infty}} \Theta_b^{(p)}(x;q)$$

KW: A $\qquad \phi \in \Omega^1(M_4, \mathfrak{g}) \qquad \sigma,\ \overline{\sigma}\ \in \Omega^0(M_4, \mathfrak{g}_{\mathbb{C}})$

- Stability conditions $\qquad H^2(\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M_3))$

Floer theory

non-compact moduli spaces, ...

Quantum algebra

Infinite-dimensional modules, cohomology, "Hilbert spaces," ...