q-series, resurgence and modularity

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Wall-Crossing Structures, Analyticity and Resurgence

Introduction

How much resurgence knows about modularity

Resurgence

Study **resurgence** of a divergent power series, which are related to **q-series** (or q-functions) and try to understand if their resummation is related with a **quantum modular form (QMF)**.

q-series

Is there a *general* resurgent structure from which we can construct QMF?

Modularity

In the main example of today we know already there is a QMF related with a q-series. Does modularity help to prove resurgence?

Introduction

How much resurgence knows about modularity and vice-versa

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Study **resurgence** of a divergent power series, which are related to **q-series** (or q-functions) and try to understand if their resummation is related with a **quantum modular form (QMF)**.

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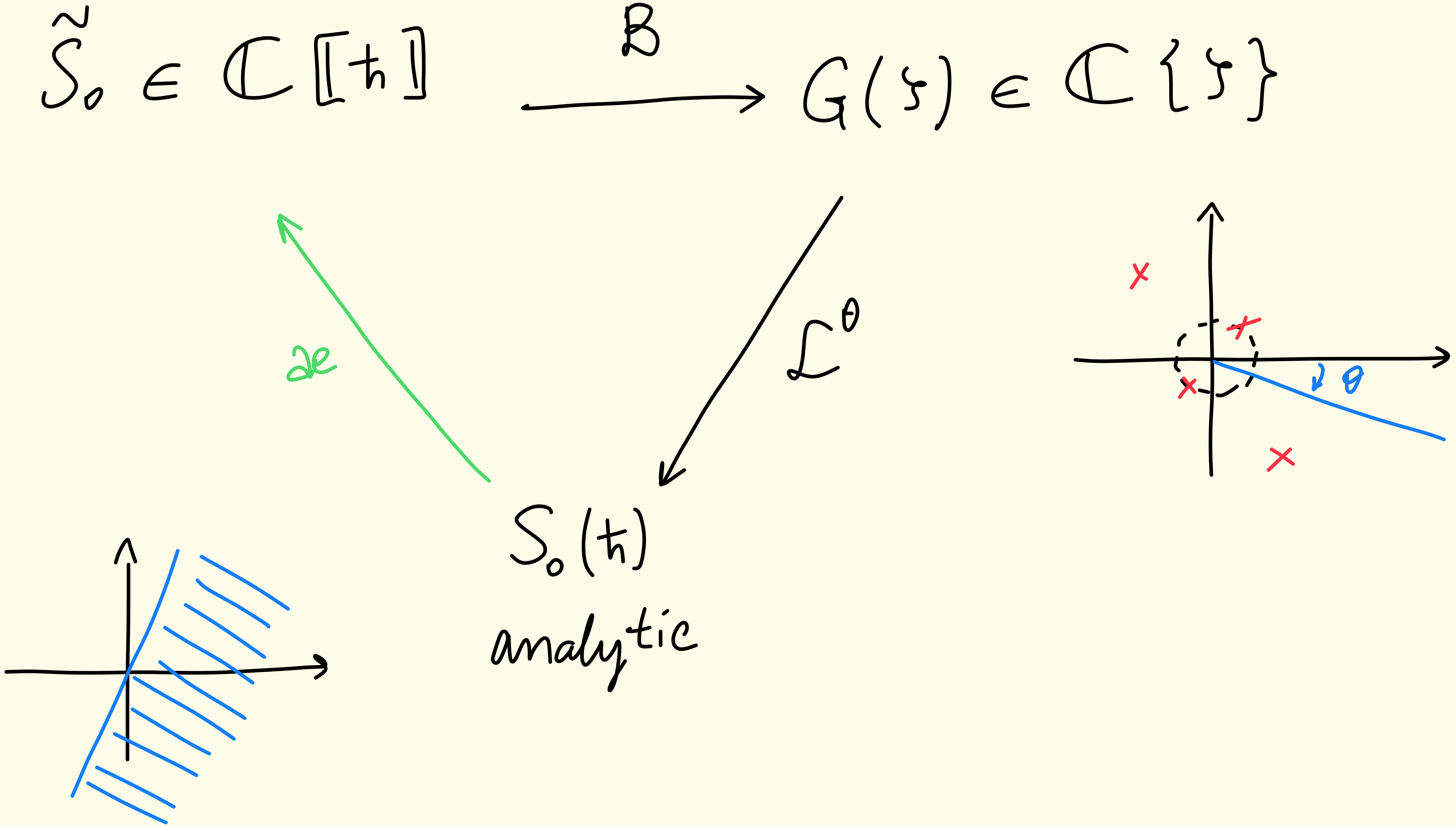
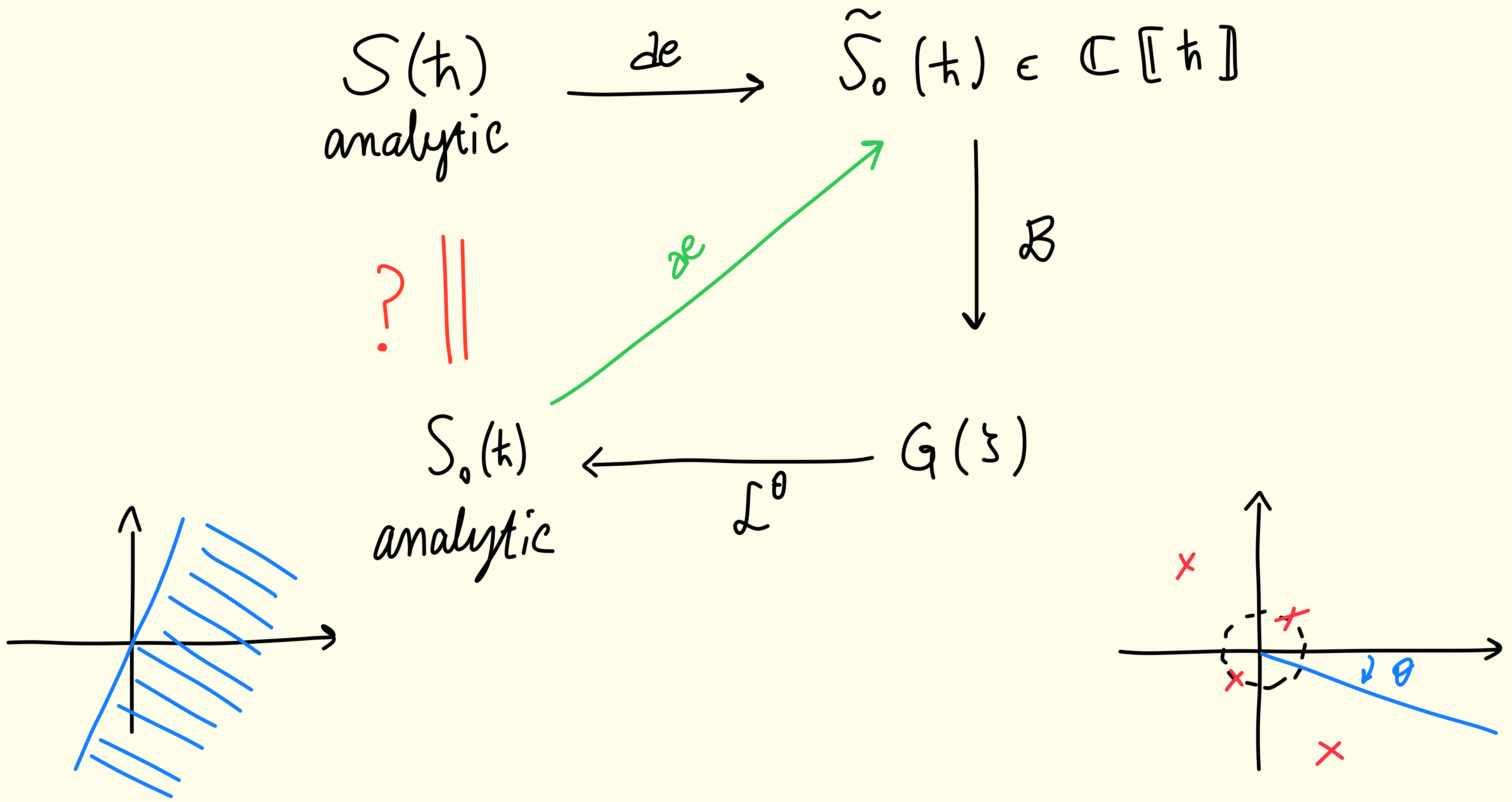
Borel-Laplace summability

From divergent series to analytic functions



Borel-Laplace summability

From analytic function to divergent series and back to analytic functions



Resurgence

The importance of the Borel plane

DEFINITION: A function $\hat{\phi}(\zeta) \in \mathbb{C}\{\zeta\}$ is resurgent if it can be endlessly analytically continued, i.e. for every L>0 there exists a finite subset $\Omega_L \subset \mathbb{C}$ such that $\hat{\phi}(\zeta)$ can be analytically continued along every path of length less than L which avoids Ω_L.

- The Borel plane contains the main informations
- *Alien calculus, median resummation, Hankel contour Laplace transform,…* are alternatives to the usual Laplace transform, well defined to work with resurgent functions

Resurgence

Simple resurgent functions

Let ω be a singular point of $G(\zeta) \in \mathbb{C}\{\zeta\}$

$$G(\zeta) = \frac{C_\omega}{\zeta - \omega} + \frac{S_\omega}{2\pi i} \log(\zeta - \omega)\hat{\phi}_\omega(\zeta - \omega) + \quad \text{h.f.}$$

where $S_\omega, C_\omega \in \mathbb{Z}$ and $\hat{\phi}_\omega(\zeta) \in \mathbb{C}\{\zeta\}$.

In fact, looking at $\hat{\phi}_\omega(\zeta)$ we can continue to study the singularities in the Borel plane. If $G(\zeta)$ is resurgent, the $\hat{\phi}_\omega$ *know each other*.

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Resurgence

Resurgent structure

DEFINITION: A **resurgent structure** consists of the following data

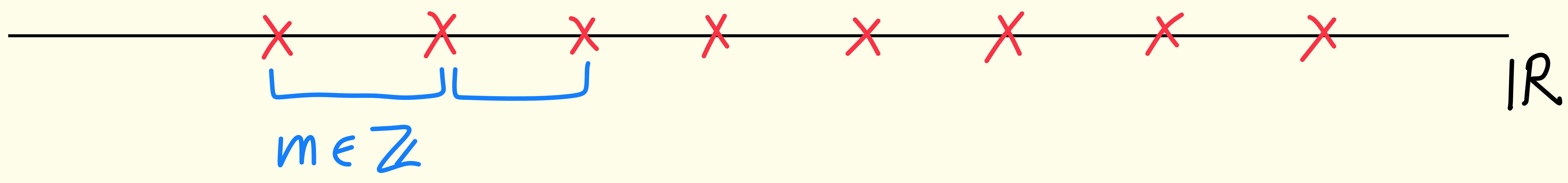
- set of singularities $\{\omega \in \Omega\}$
- collection of germs $\hat{\phi}_\omega(\zeta) \in \mathbb{C}\{\zeta\}$
- Stokes constants C_ω, S_ω

Resurgence

"Modular" resurgent structure

DEFINITION: A "modular" resurgent structure is a resurgent structure such that

- tower of singularities $\{\rho_k \in \mathbb{R} \,|\, k \in \Omega\}$, $\Omega \subset \mathbb{Z}$
- collection of germs $\hat{\phi}_k(\zeta) = 1$
- Stokes constants $S_k \in \mathbb{Z}$ in a suitable normalisation, and $L(s) = \sum_{k \in \Omega} \frac{S_k}{|k|^s}$ is an L-function



Resurgence

Divergent series with "modular" resurgent structure

Let $\tilde{S}_0(\hbar) = \sum_{n=1}^{\infty} c_n \hbar^n$ be Gevrey-1 and

$$c_n = \text{const} \cdot (n-1)! \sum_{k \in \Omega} \frac{S_k}{\rho_k^n}$$

with $\{\rho_k \in \mathbb{R} \,|\, k \in \Omega\}$ and $\Omega \subset \mathbb{Z}$ and $L(s) = \sum_{k \in \Omega} \frac{S_k}{|k|^s}$ is an L-function. Then $G(\zeta) = \mathscr{B}\tilde{S}_0$ has a "modular" resurgent structure.

In fact, $G(\zeta) = \text{const} \sum_{n=1}^{\infty} \sum_{k \in \Omega} \frac{S_k}{\rho_k^n} \frac{\zeta^n}{n} = \text{const} \sum_{k \in \Omega} S_k \log(\zeta - \rho_k) \cdot 1 + \text{ h.f.}$

■

Resurgence

Divergent series with "modular" resurgent structure

Let $\tilde{S}_0(\hbar) = \sum_{n=0}^{\infty} c_n \hbar^n$ be a Gevrey-1 series and

$$c_n = \text{const} \cdot \Gamma(n+\alpha) \sum_{k\in\Omega} \frac{S_k}{\rho_k^{n+\alpha}} \quad \text{for } \alpha \in \mathbb{Q}_{\geq 0} \text{ not integer}$$

with $\{\rho_k \in \mathbb{R} \,|\, k \in \Omega\}$ and $\Omega \subset \mathbb{Z}$ and $L(s) = \sum_{k\in\Omega} \frac{S_k}{|k|^s}$ is an L-function. Then $G(\zeta) = \mathscr{B}\tilde{S}_0$ has a "modular" resurgent structure.

In fact, $G(\zeta) = \text{const} \sum_{n=0}^{\infty} \sum_{k\in\Omega} \frac{S_k}{\rho_k^{n+\alpha}} \frac{\Gamma(n+\alpha)}{n!} \zeta^n = \frac{\text{const}}{(\alpha-1)\Gamma(\alpha-1)} \sum_{k\in\Omega} \frac{S_k}{(\rho_k - \zeta)^{\alpha}}$ ∎

Resurgence

Taking *generalised* Laplace transform

Let $\tilde{S}_0 \in \mathbb{Q}[[\hbar]]$ be a formal series such that it is Gevrey-1 and its Borel transform $G(\zeta)$ admits a simple "modular" resurgent structure.

Then the *generalised* Laplace transform (for some path $\mathscr{C}$)

$$S_0(\hbar) := \int_{\mathscr{C}} e^{-\zeta/\hbar} G(\zeta) d\zeta$$

defines a QMF $f(\tau) := S_0(2\pi i \tau)$, i.e.

Resurgence

Taking *generalised* Laplace transform

$f\colon \mathbb{Q} \to \mathbb{C}$ such that

$$f(\tau+1) = e^{2\pi i/m} f(\tau) \qquad\qquad (c\tau+d)^{-1} f(\gamma\tau) = e^{2\pi i/m} f(\tau) + h_\gamma(\tau)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$ and $h_\gamma\colon \mathbb{R} \to \mathbb{C}$ is C^∞ and real-analytic except at $\tau = \gamma^{-1}(\infty)$.

The main example

The main example

after Andrews, Cohen, Zagier,…

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n} = 1 + \sum_{n=0}^{\infty} (-1)^n q^{n+1} (q)_n$$

$$\sigma^*(q) := 2 \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}}{(q;q^2)_n} = -2 \sum_{n=0}^{\infty} q^{n+1} (q^2;q^2)_n$$

- $\sigma(q), \sigma^*(q)$ are convergent for $|q| < 1$,
- they make sense when q is a root of unity: if $q = \exp(2\pi i \tau)$ with $\tau \in \mathbb{Q}$ then $\sigma(q) = -\sigma^*(q^{-1})$
- Define the coefficients $\{T(k)\}_{k \in 24\mathbb{Z}+1}$ by

$$q\sigma(q^{24}) = \sum_{k \geq 0} T(k) q^k, \qquad q^{-1}\sigma^*(q^{24}) = \sum_{k<0} T(k) q^{|k|}$$

then $L(s) = \sum_{k \in 24\mathbb{Z}+1} T(k) |k|^{-s} = \zeta_{\mathbb{Q}\left(\sqrt{3+\sqrt{3}}\right)}(s) / \zeta_{\mathbb{Q}\left(\sqrt{3}\right)}(s)$.

The main example
after Zagier, Cohen, Andrews,…

Define $f: \mathbb{Q} \to \mathbb{C}$

$$f(\tau) = q^{1/24}\sigma(q) = -q^{1/24}\sigma^*(q^{-1}) \qquad \tau \in \mathbb{Q}, q = \exp(2\pi i\tau)$$

PROPOSITION [Zagier,10; Kontsevich]. The function $f(\tau)$ satisfies

$$f(\tau+1) = e^{2\pi i/24}f(\tau), \qquad \frac{1}{|2\tau+1|}f\left(\frac{\tau}{2\tau+1}\right) = e^{2\pi i/24}f(\tau) + h(\tau)$$

where $h: \mathbb{R} \to \mathbb{C}$ is C^∞ on $\mathbb{R}$ and real-analytic except at $\tau = 0, -1/2$.

The main example

Divergent series

Let $\hbar \in \mathbb{C}$ and define the formal power series

$$\tilde{S}_0(\hbar) := \hbar e^{\hbar/24}\left(1 + \sum_{n=1}^{\infty}(-1)^{n+1}e^{n\hbar}\prod_{k=0}^{n}(1 - e^{k\hbar})\right)$$

$$= \hbar e^{\hbar/24}(1 + e^{\hbar} - e^{2\hbar}(1 - e^{\hbar}) + \ldots) = \sum_{n=1}^{\infty} c_n \hbar^n \in \mathbb{Q}[[\hbar]]$$

CONJECTURE 1:

$$\frac{c_n}{(n-1)!} = \sqrt{2}\sum_{k \in 24\mathbb{Z}+1}\frac{T(k)}{\rho_k^n} \quad \text{with } \rho_k := \frac{\pi^2}{12}k$$

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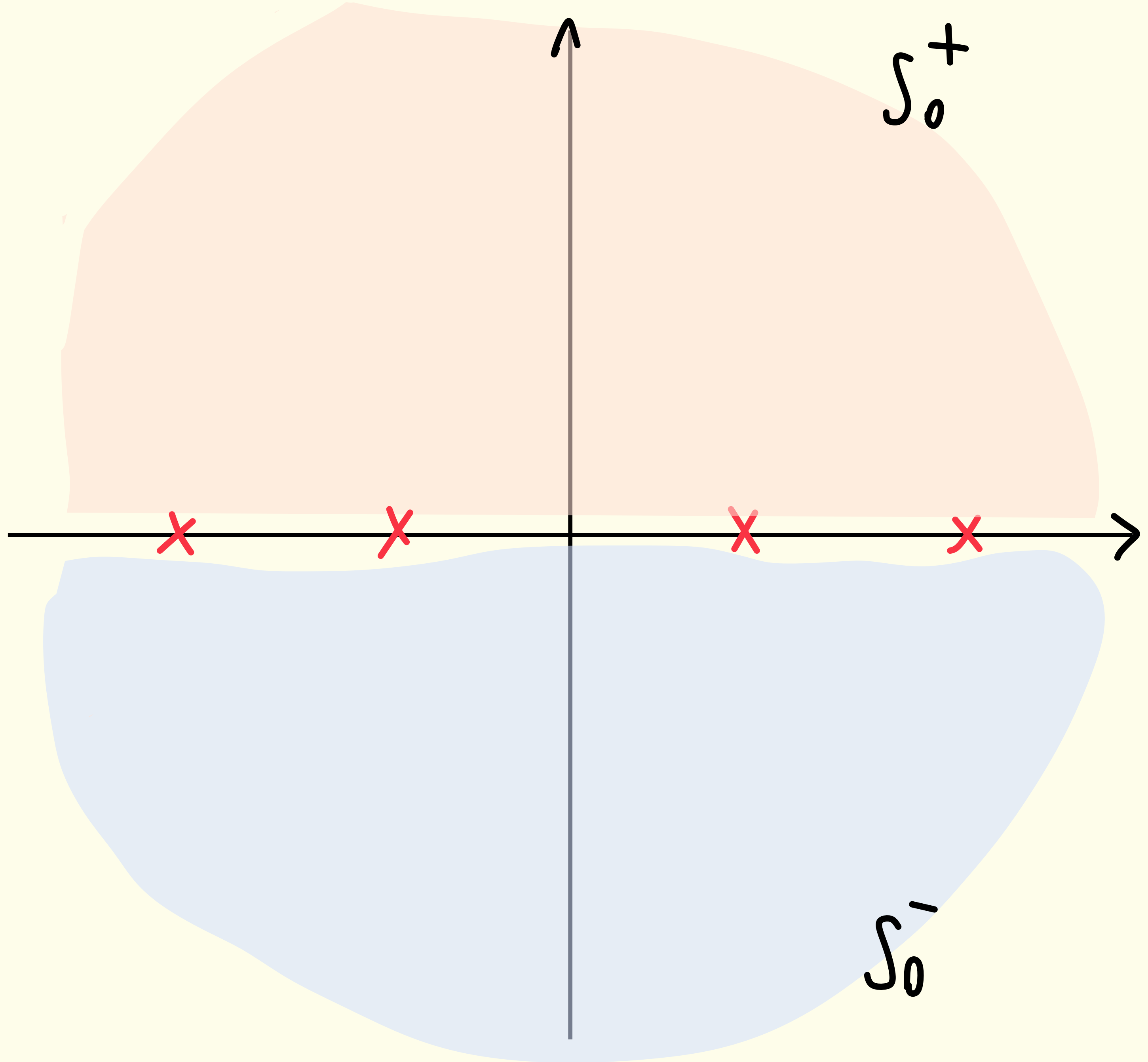
The main example

$\tilde{S}_0(\hbar)$ has a "modular" resurgent structure

$$S_0^+(\hbar) := \sqrt{2}\int_0^{+i\infty} e^{-s/\hbar} \sum_{k\in 24\mathbb{Z}+1} \frac{T(k)}{\rho_k - s} ds, \quad \Im\hbar > 0$$

$$S_0^-(\hbar) := \sqrt{2}\int_0^{-i\infty} e^{-s/\hbar} \sum_{k\in 24\mathbb{Z}+1} \frac{T(k)}{\rho_k - s} ds, \quad \Im\hbar < 0$$



In fact S_0^+ and S_0^- extend analytically to the right and the left half-planes $\Re\hbar > 0, \Re\hbar < 0$

$$\left(S_0^+(\hbar) - S_0^-(\hbar)\right)\Big|_{\Re\hbar>0} = 2\pi i\sqrt{2} \sum_{k\in 24\mathbb{Z}_{>0}+1} T(k)e^{-\rho_k/\hbar}$$

$$\left(S_0^+(\hbar) - S_0^-(\hbar)\right)\Big|_{\Re\hbar<0} = -2\pi i\sqrt{2} \sum_{k\in 24\mathbb{Z}_{<0}+1} T(k)e^{-\rho_k/\hbar}$$

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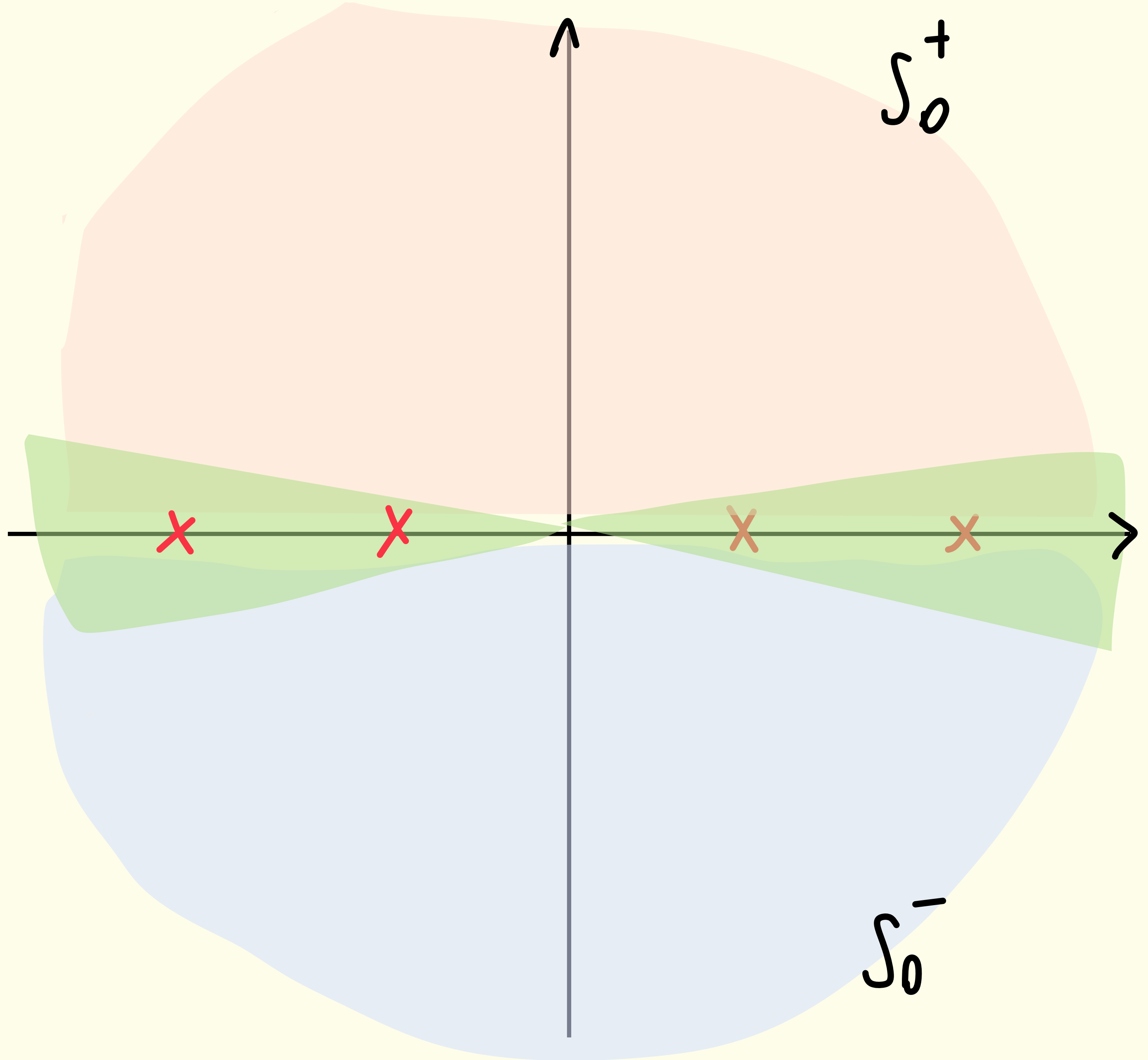

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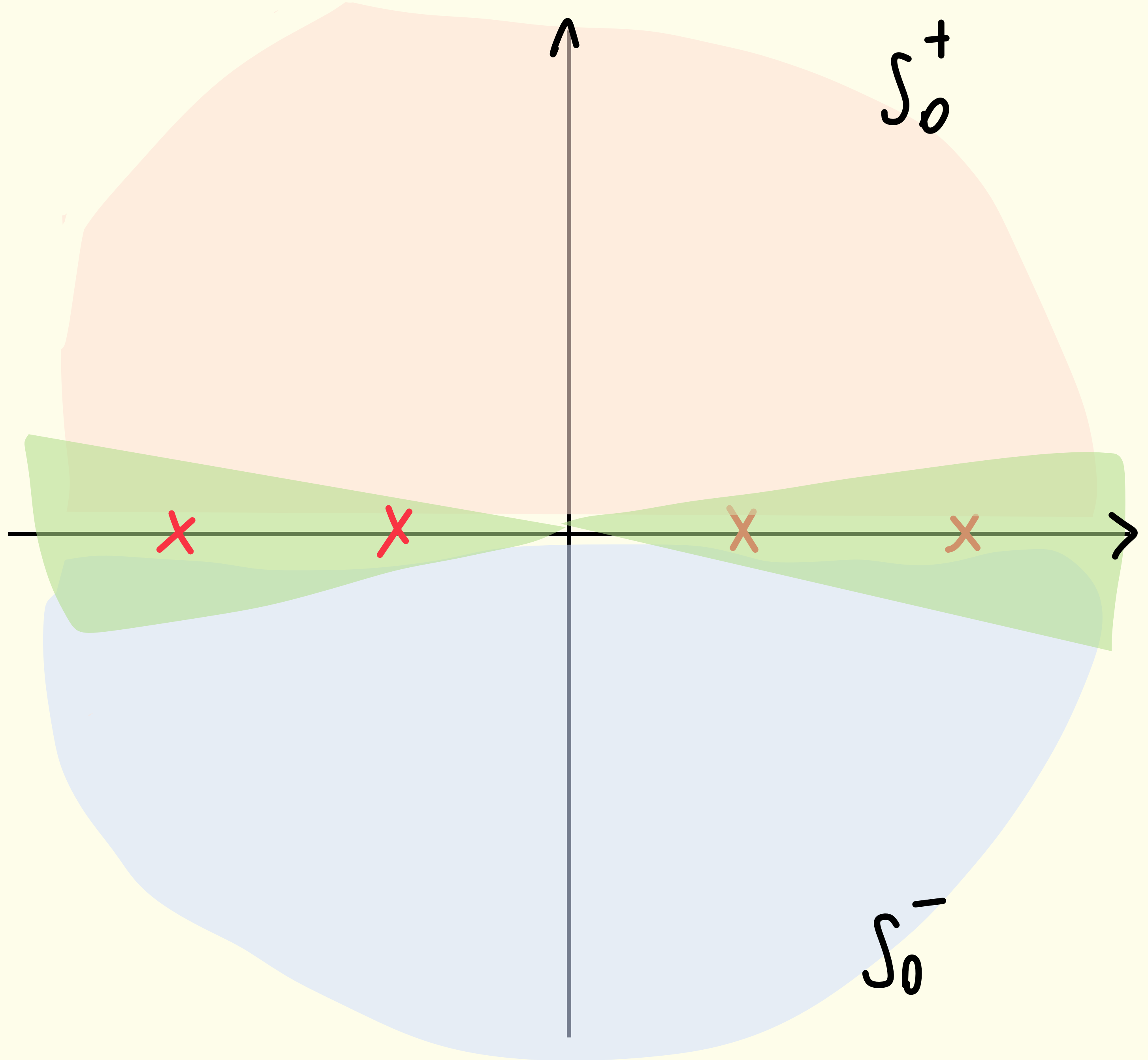

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$$\left(S_0^+(\hbar) - S_0^-(\hbar)\right)\Big|_{\Re\hbar<0} = -2\pi i\sqrt{2} \sum_{k\in 24\mathbb{Z}_{<0}+1} T(k)e^{-\rho_k/\hbar} = 2\pi i\sqrt{2}\ \tilde{q}^{1/48}\sigma^*(\tilde{q}^{-1/2})$$

The main example

Quantum modularity

CONJECTURE 2

If $\hbar = \dfrac{2\pi i}{N}$, then

$$\frac{1}{N} f\left(\frac{1}{N}\right) = \frac{1}{2\pi i} \begin{cases} S_0^{+}(\hbar) - \sqrt{2}\, 2\pi i\, \mathbf{e}_0\left(\frac{25}{48}N\right) & N = 1,2,3,\ldots \\ S_0^{-}(\hbar) + \sqrt{2}\, 2\pi i\, \mathbf{e}_0\left(\frac{25}{48}N\right) & N = -1, -2, -3,\ldots \end{cases}$$

REMARK: $\tau f(\tau) \neq \dfrac{1}{2\pi i} S_0^{\pm}(2\pi i \tau) \pm \sqrt{2}\ \mathbf{e}_0\left(\dfrac{25}{48\cdot\tau}\right)$ for $\tau \in \mathbb{Q}\backslash\{0\}$ and $\tau \neq \pm\dfrac{1}{N}$

The main example

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The main example

Quantum modularity: the cocycle $h(\tau)$

COROLLARY 1

Let $h(\tau) := \frac{1}{|2\tau+1|} f\left(\frac{\tau}{2\tau+1}\right) - e^{\frac{2\pi i}{24}} f(\tau)$, then

$$\forall \tau > -\frac{1}{2}, \tau \neq 0 \quad h(\tau) = \frac{\sqrt{2}}{\tau} \sum_{k \in 24\mathbb{Z}+1} T(k) \left[\mathbf{e}_0\left(\frac{1}{24}\right) \mathbf{e}_1\left(\frac{k}{\tau \cdot 48}\right) - \mathbf{e}_1\left(\frac{(2\tau+1)k}{\tau \cdot 48}\right)\right]$$

$$\forall \tau < -\frac{1}{2}, \qquad h(\tau) = \frac{\sqrt{2}}{\tau} \sum_{k \in 24\mathbb{Z}+1} T(k) \left[\mathbf{e}_0\left(\frac{1}{24}\right) \mathbf{e}_1\left(\frac{k}{\tau \cdot 48}\right) + \mathbf{e}_1\left(\frac{(2\tau+1)k}{\tau \cdot 48}\right)\right]$$

The main example

COROLLARY 2

For any non-zero rational number $\tau \in \mathbb{Q}\backslash\{0\}$

$$\boxed{\sum_{k\in 24\mathbb{Z}+1} T(k)\mathbf{e}_1\left(\frac{k\tau}{48}\right) = -\frac{\operatorname{sgn}(\tau)}{2} f(\tau/2) - \frac{f(1/\tau)}{\sqrt{2}\tau}}$$

$$\Rightarrow \tau f(\tau) = \frac{1}{2\pi i} S_0^{\pm}(2\pi i\tau) - \sqrt{2}\ \frac{\operatorname{sgn}(\tau)}{2} f\left(\frac{1}{2\tau}\right) \text{ for } \tau \in \mathbb{Q}\backslash\{0\}$$

Kontsevich-Zagier

Kontsevich-Zagier q-series

after Zagier and Costin-Garoufalidis

Consider the q-series

$$\phi(q) := \sum_{n\geq 0} (q)_n$$

It is NOT an analytic function of q inside or outside the unit disk but it is well defined at q equal to root of unity

$$\textit{Strange identity}\ \ \phi(q) = -\frac{1}{2}\sum_{n=1}^{\infty} n\chi(n) q^{(n^2-1)/24}$$

where $\chi(n) = \begin{cases} 1 & n \equiv 1,11 \mod 12 \\ -1 & n \equiv 5,7 \mod 12 \\ 0 & \text{otherwise} \end{cases}$

RHS is convergent for $|q| < 1$ and its limit as q goes to roots of unity is ϕ.

Kontsevich-Zagier q-series

after Zagier and Costin-Garoufalidis

Define $f\colon \mathbb{Q} \to \mathbb{C}$

$$f(\tau) = q^{1/24} \sum_{n=0}^{\infty} (1-q)(1-q^2)\dots(1-q^n) \qquad \tau \in \mathbb{Q}, q = \exp(2\pi i\tau)$$

THEOREM [Zagier,99]. The function $f(\tau)$ satisfies

$$f(\tau+1) = e^{2\pi i/24} f(\tau), \qquad f(\tau) + (i\tau)^{-3/2} f(-1/\tau) = h(\tau)$$

where $h\colon \mathbb{R} \to \mathbb{C}$ is C^∞ on $\mathbb{R}$ and real-analytic except at $\tau = 0$.

Kontsevich-Zagier q-series

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Let $\hbar \in \mathbb{C}$ and define the formal power series

$$\tilde{S}_0(\hbar) := e^{\hbar/24} \sum_{n=0}^{\infty} \prod_{k=0}^{n} (1 - e^{k\hbar}) = \sum_{n=0}^{\infty} c_n \hbar^n \in \mathbb{Q}[[\hbar]]$$

THEOREM [Zagier,99] $c_n = \sqrt{3}(-1)^n \frac{(2n+1)!}{4^n n!} \sum_{k=1}^{\infty} \frac{\chi(k)}{(k^2\pi^2/6)^{n+1}}$

THEOREM [Costin-Garoufalidis,10]

- The Borel transform of $\tilde{S}_0$ is $G(s) = \frac{3\pi}{2\sqrt{2}} \sum_{k=1}^{\infty} \frac{k\chi(k)}{(k^2\pi^2/6 - s)^{3/2}}$
- $\left(S_{0,up}(\hbar) - S_{0,low}(\hbar) \right)\Big|_{\Re\hbar<0} = -(2\pi)^{3/2} \hbar^{-3/2} \sum_{k=1}^{\infty} k\chi(k) e^{-\rho_k/\hbar}$ where $\rho_k = k^2\pi^2/6$

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THEOREM [Costin-Garoufalidis,10]

- The Borel transform of $\tilde{S}_0$ is $G(\zeta) = \frac{3\pi}{2\sqrt{2}} \sum_{k=1}^{\infty} \frac{k\chi(k)}{(k^2\pi^2/6 - \zeta)^{5/2}}$ $\Rightarrow$ singularities at ρ_k, $k \in \mathbb{Z}_{\geq 1}$
- $\left(S_0^+(\hbar) - S_0^-(\hbar)\right)\Big|_{\Re\hbar > 0} = -(2\pi)^{3/2}\hbar^{-3/2} \sum_{k=1}^{\infty} k\chi(k) e^{-\rho_k/\hbar}$ where $\rho_k = k^2\pi^2/6$ $\Rightarrow$ Stokes $S_k = k\chi(k)$, $k \in \mathbb{Z}_{\geq 1}$

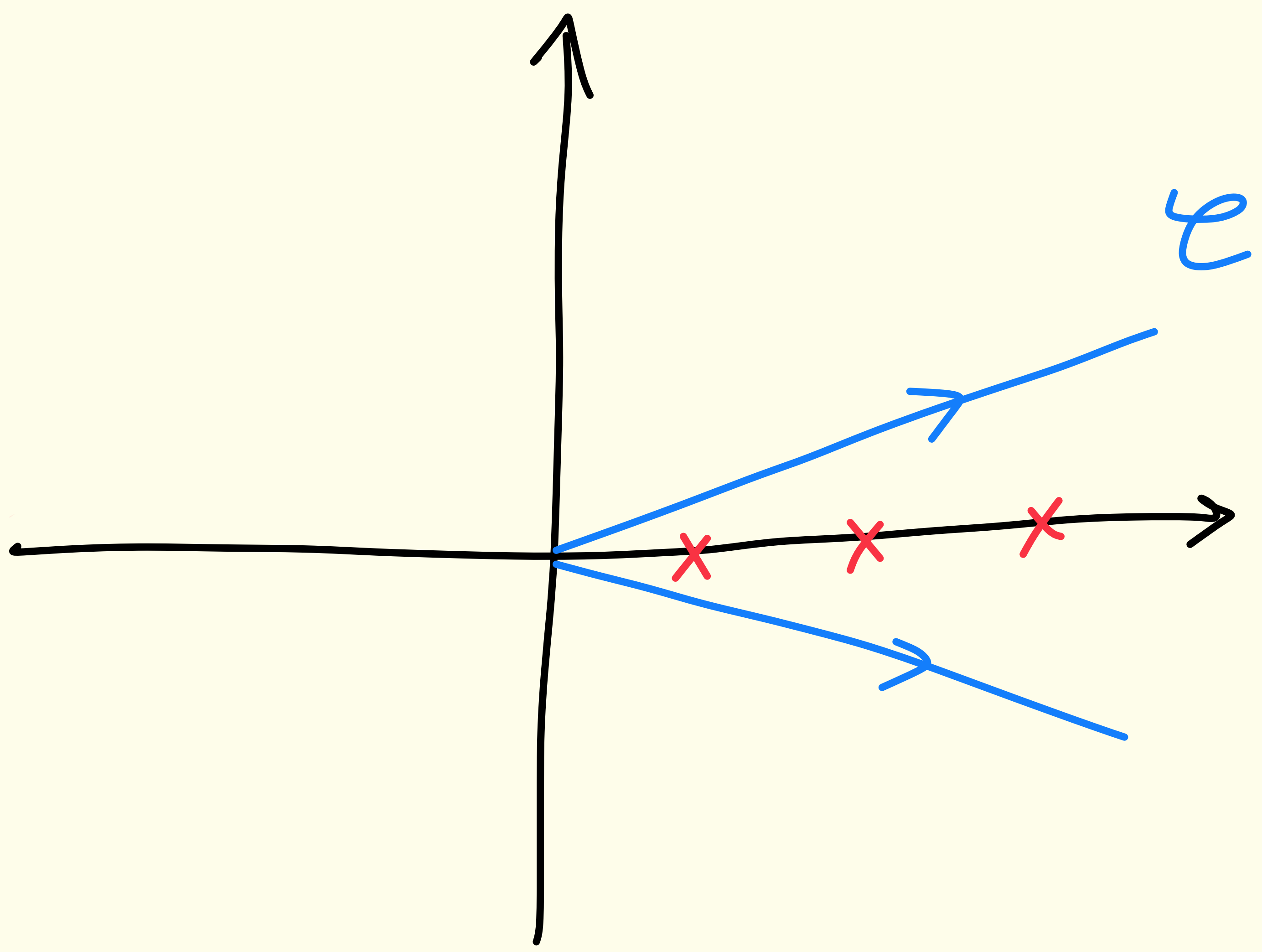
Kontsevich-Zagier q-series

after Zagier and Costin-Garoufalidis

THEOREM [Costin-Garoufalidis,10]

Let $\tau \in \mathbb{Q}\backslash\{0\}$ then

$$S_0^+(2\pi i\tau) + S_0^-(2\pi i\tau) = 2f(\tau)$$



i.e. the median resummation $S_0(2\pi i\tau) = \int_{\mathscr{C}} e^{-\zeta/\hbar} G(\zeta) d\zeta$

is a quantum modular form.

Trace of quantum mechanical operators: local $\mathbb{P}^2$

Trace of quantum mechanical operators: local $\mathbb{P}^2$
after Rella

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \frac{1}{\sqrt{3\tau}} e^{-\frac{\pi i}{36}\tau + \frac{\pi}{12\tau} + \frac{\pi i}{4}} \frac{(q^{2/3};q)_\infty^2}{(q^{1/3};q)_\infty} \frac{(e^{2\pi i/3};\tilde{q})_\infty}{(e^{-2\pi i/3};\tilde{q})_\infty^2}$$

where $q = e^{2\pi i \tau}$, $\tilde{q} = e^{-2\pi i/\tau}$ and $\tau \in \mathbb{H}$.

- $\mathrm{Tr}(\rho_{\mathbb{P}^2})$ is convergent as a q-series and as a $\tilde{q}$-series.

In fact, taking the asymptotics as $q \to 1$ or $\tilde{q} \to 1$we get two divergent series: let $\hbar = 2\pi i \tau$

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = -\frac{\Gamma(1/3)^3}{2\pi i \hbar} \exp\left(-3\sum_{n=1}^{\infty} \frac{B_{2n}B_{2n+1}(2/3)}{2n(2n+1)!}\hbar^{2n}\right)$$

$$\mathrm{Tr}(\rho_{\mathbb{P}^2}) = \sqrt{\frac{2\pi i}{3^{3/2}\hbar}} e^{\frac{iV}{4\pi^2}\hbar} \exp\left(-\sqrt{3}i\sum_{n=1}^{\infty} \frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)} \left(\frac{12\pi^2}{\hbar}\right)^{2n-1}\right)$$

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Trace of quantum mechanical operators: local $\mathbb{P}^2$

Resurgent structure

THEOREM [Rella,22]

Let $\tilde{S}_0(\hbar) = -3\sum_{n=1}^{\infty} \frac{B_{2n}B_{2n+1}(2/3)}{2n(2n+1)!}\hbar^{2n}$ and $G_0(s) = \mathscr{B}\tilde{S}_0$, then

the singularities of its Borel transform are at $\rho_k = 4\pi^2 k,\ k \in \mathbb{Z}\backslash\{0\}$

the Stokes constants S_k satisfy $\sum_{k=1}^{\infty} \frac{S_k}{k^s} = L(s+1,\chi_{3,2})\zeta(s)$, with $\chi_{3,2}(n) = \begin{cases} 1 & n \equiv 1 \mod 3 \\ 0 & n \equiv 0 \mod 3 \\ -1 & n \equiv 2 \mod 3 \end{cases}$

Locally at $s = \rho_k$, the Borel transform is $G_0(s) = \text{const} \cdot \frac{S_k}{2\pi i} \log(s-\rho_k) \cdot 1 + \ \text{h.f.}$

Trace of quantum mechanical operators: local $\mathbb{P}^2$

Resurgent structure

REMARK [Rella,22]

- $c_{2n}^{0} = \text{const} \cdot \Gamma(2n) \sum_{k \geq 1} \frac{S_k}{\rho_k^{2n}}$ exact large order relation
- $(S_0^{+} - S_0^{-})\,|_{\Re \hbar > 0} = \text{const} \sum_{k \geq 1} S_k e^{-\rho_k/\hbar} \propto \log(w; \tilde{q})_\infty - \log(w^{-1}; \tilde{q})_\infty$ with

  $w = e^{2\pi i/3}$

Trace of quantum mechanical operators: local $\mathbb{P}^2$

Resurgent structure

THEOREM [Rella,22]

Let $\tilde{S}_\infty(\hbar) = -\sqrt{3}i \sum_{n=1}^{\infty} \frac{B_{2n}B_{2n-1}(2/3)}{(2n)!(2n-1)} \left(\frac{12\pi^2}{\hbar}\right)^{2n-1}$ and $G_\infty(s) = \mathscr{B}\tilde{S}_\infty$, then

the singularities of its Borel transform are $\rho_k = k/3,\, k \in \mathbb{Z}\backslash\{0\}$

the Stokes constants S_k satisfy $\sum_{k=1}^{\infty} \frac{S_k}{k^s} = L(s, \chi_{3,2})\zeta(s+1)$

Locally at $s = \rho_k$, the Borel transform is $G_\infty(s) = \text{const} \cdot \frac{S_k}{2\pi i} \log(s - \rho_k) \cdot 1 +$ h.f.

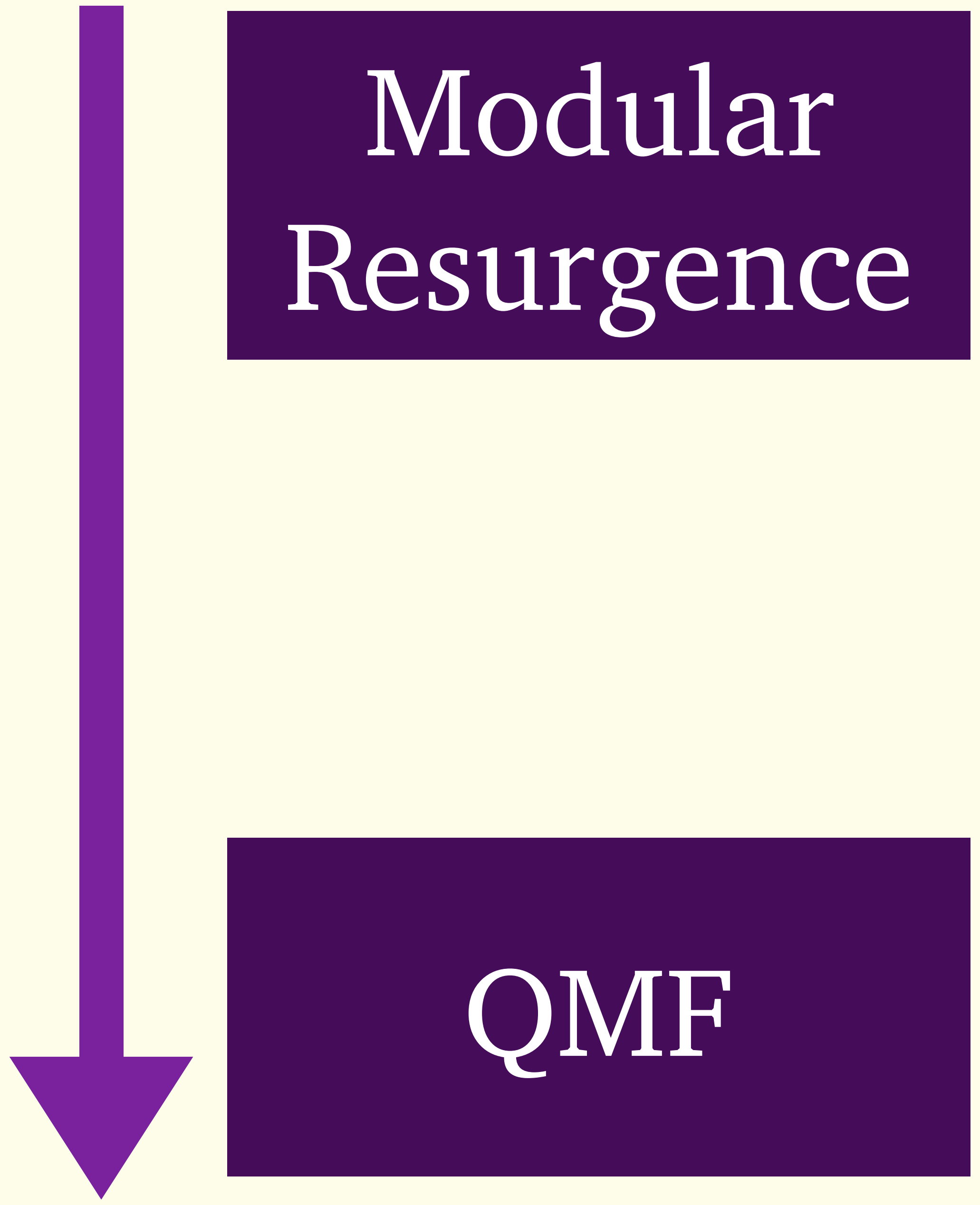
Trace of quantum mechanical operators: local $\mathbb{P}^2$

Resurgent structure

REMARK [Rella,22]

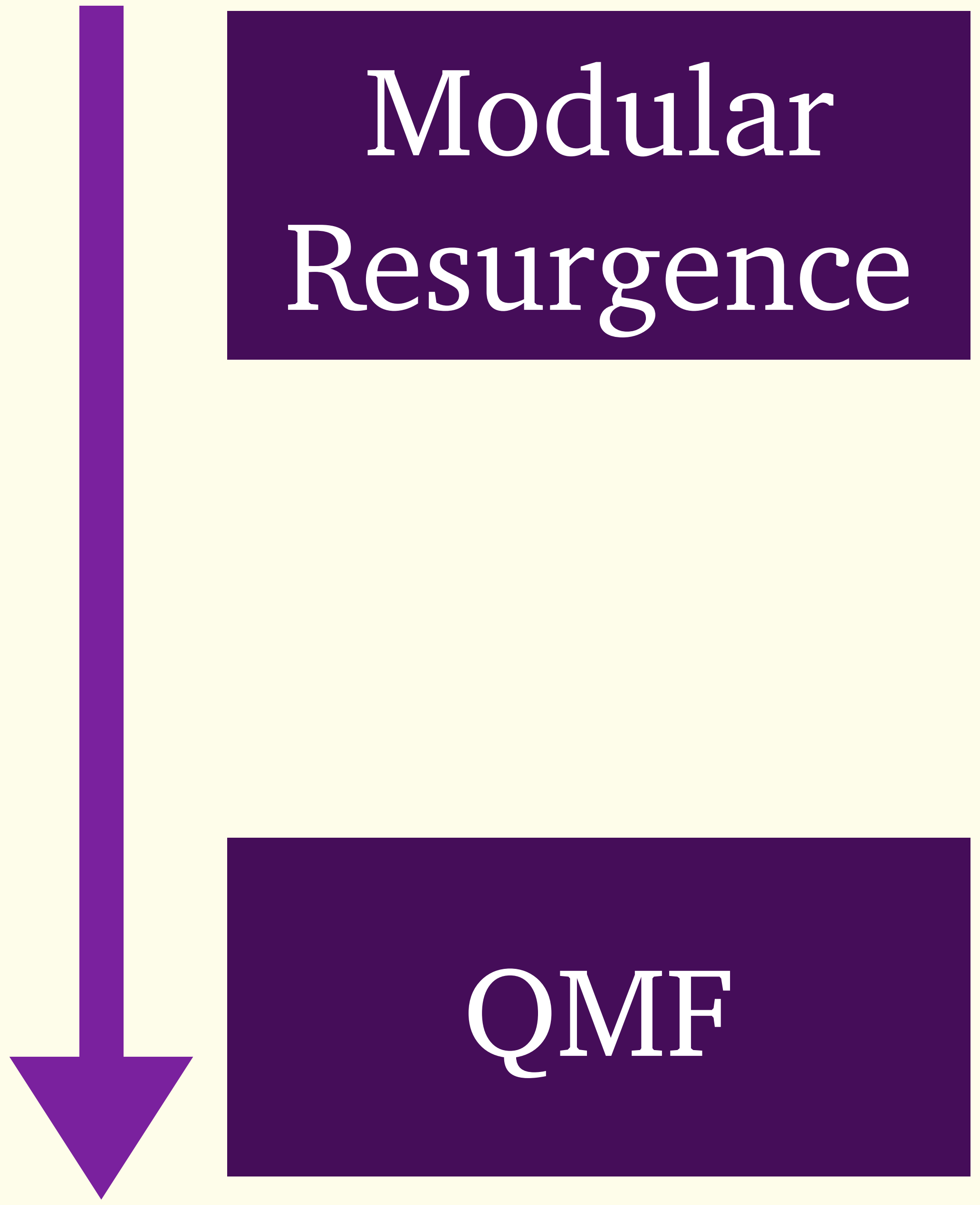
- $c_{2n}^{\infty} = \text{const} \cdot \Gamma(2n-1) \sum_{k \geq 1} \frac{S_k}{\rho_k^{2n-1}}$ exact large order relation
- $(S_0^+ - S_0^-)|_{\Re\hbar>0} = \text{const} \sum_{k\geq 1} S_k e^{-\rho_k/\hbar} \propto \log(q^{2/3};q)_\infty - \log(q^{1/3};q)_\infty$

Conclusion



- From “modular” resurgent structure we expect to find QMF by taking ***generalised* Laplace transform** (e.g. the median Laplace resummation for Kontsevich-Zagier q-series or in William’s talk)
- For simple “modular” resurgent structure the median Laplace resummation does not seem to give a QMF (as in the main example)

Conclusion



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Thank you for your attention