

# THE RESURGENT STRUCTURE OF TOPOLOGICAL STRINGS

Marcos Mariño  
University of Geneva

Jie Gu, M.M. [2211.0143]

J. Gu, A. Kashani-Poor, A. Klemm and M.M. [2305.19916]

M.M., Ramon Miravitllas [2306.01104]

The search for a non-perturbative understanding of (topological) string theory has been going on for many years.

In 2006-2008 it was proposed in [M.M., M.M.-Schiappa-Weiss] to look at this problem by using the theory of resurgence, i.e. by using traditional tools which had been very successful in QM and QFT.

An key aspect of this approach:

**“make use as much as possible of the important pieces of information contained in the coupling constant expansion”**

(G. 't Hooft, 1979)

# Resurgent structure

A precise mathematical formulation of non-perturbative sectors: **resurgent structure**.

Let us consider a (simple) resurgent function, given by a formal, factorially divergent series


$$\varphi(z) = \sum_{n \geq 0} a_n z^n \quad a_n \sim n!$$

and its Borel transform

$$\hat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

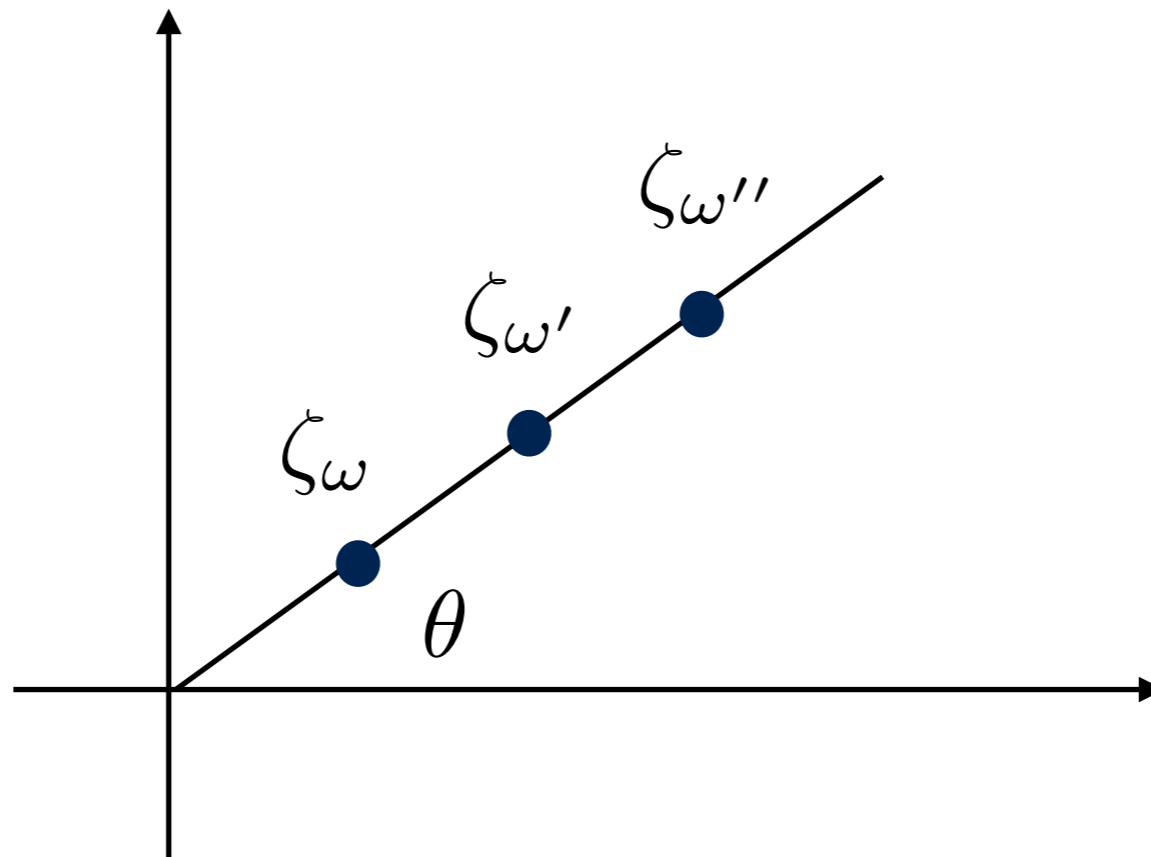
Given a singularity  $\zeta_\omega$  of the Borel transform, one can associate a formal trans-series to it through the so-called pointed alien derivative

$$\dot{\Delta}_{\zeta_\omega} \varphi(z) = S_\omega e^{-\zeta_\omega/z} \varphi_\omega(z)$$


 Stokes constant

Roughly, the alien derivative can be obtained from the local expansion of the Borel transform at the singularity

$$\hat{\varphi}(\zeta) = -S_\omega \hat{\varphi}_\omega(\zeta - \zeta_\omega) \frac{\log(\zeta - \zeta_\omega)}{2\pi i} + \text{regular}$$



The exponentiation of the alien derivatives associated to the singularities along a ray gives the **Stokes automorphism**

$$\mathfrak{S}_\theta = \exp \left( \sum_{\omega \in \Omega_\theta} \dot{\Delta}_{\zeta_\omega} \right)$$

The resurgent structure associated to a formal power series is **the set of all its alien derivatives at all its singularities** (i.e. trans-series plus Stokes constants)

From a physical point of view, the resurgence structure gives **all the non-perturbative sectors which can be obtained from the knowledge of perturbation theory only**

**Warning:** we know from explicit examples in quantum theory that there can be non-perturbative sectors **not** included in the resurgent structure

# Resurgence and asymptotics

The “largest” exponentially small trans-series, associated to the closest Borel singularity to the origin

$$\Phi_{\mathcal{A}} = e^{-\mathcal{A}/z} (c_0 + c_1 z + \cdots)$$

turns out to determine the asymptotic behavior of the perturbative series

$$a_n \sim \frac{\mathcal{A}^{-n}}{2\pi} \Gamma(n) \left( c_0 + \frac{c_1 \mathcal{A}}{n-1} + \cdots \right)$$

The trans-series **“resurges”** in the perturbative series

This is the famous connection between perturbative and non-perturbative sectors predicted by resurgence, which goes back to the pioneering work by Bender and Wu

## Large-Order Behavior of Perturbation Theory

Carl M. Bender\*

*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

and

Tai Tsun Wu†

*Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138, ‡ and*

*Deutsches Elektronen-Synchrotron, Hamburg, Germany*

(Received 28 June 1971)

This connection is very useful to obtain “experimental” tests of the results for non-perturbative sectors.

# Topological string theory

Let  $M$  be a Calabi-Yau (CY) threefold. For each genus  $g$  we can associate to it the topological string free energy  $F_g(t)$ , which depends on the Kahler modulus  $t$  (for simplicity I will often write down things for one-modulus CYs)

At large  $t$  this free energy has an expansion encoding Gromov-Witten invariants of  $M$ , which “count” holomorphic curves of genus  $g$ :

$$F_g(t) = \sum_{d \geq 1} N_{g,d} e^{-dt}$$

I will use mirror symmetry throughout. In the mirror manifold  $M^*$  one can calculate **periods** by integrating the holomorphic 3-form over a symplectic basis of 3-cycles

$$X^I = \int_{\alpha^I} \Omega \qquad \mathcal{F}_I = \int_{\beta_I} \Omega = \frac{\partial F_0}{\partial X^I}$$

The  $X^I$  are projective coordinates of the CY moduli space,  
and the mirror map is

$$t^a(z) = \frac{X^a(z)}{X^0(z)}$$

The  $z_a$  are arbitrary complex coordinates of the CY  
moduli space

String perturbation theory tells us that the **total free energy** is a formal power series involving a small parameter, a.k.a. the string coupling constant, and we have to sum over all genera:

$$F(t, g_s) = \sum_{g \geq 0} F_g(t) g_s^{2g-2}$$

General arguments [Gross-Periwal, Shenker] indicate that this series grows doubly-factorially, at fixed  $t$

$$F_g(t) \sim (2g)!, \quad g \gg 1$$

What is the resurgent structure associated to this series?

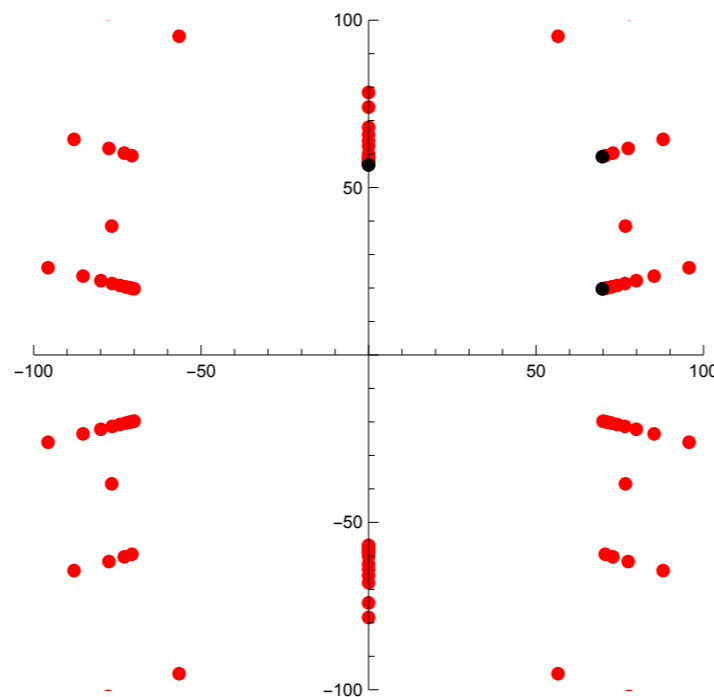
This is a difficult problem. Note that in this case the resurgent structure depends on the moduli of the CY manifold, parametrized by  $z$  or  $t$ .

I will present a conjecture on the possible location of Borel singularities, and an exact description of the trans-series associated to singularities.

# Borel plane and CY periods

**Conjecture:** the Borel singularities for the series of free energies are **integral periods** of the mirror CY (in the local case, this was conjectured in 2011 in [Drukker-M.M.-Putrov])

Determining which periods are actual singularities is much harder and only partial information is available (often based on numerical calculations)



# Trans-series for topological strings

How do we determine the trans-series associated to the singularities?

In the case of ODEs, a simple way to obtain trans-series is to use an ansatz involving exponentially small terms [Ecalte, Costin, ...]

Euler equation:  $x^2 y'(x) - y(x) = -x$

perturbative solution:  $y_p(x) = \sum_{n \geq 0} n! x^{n+1}$

trans-series solution:  $y(x) = y_p(x) + C e^{-1/x}$

# The holomorphic anomaly equations

In the case of the topological string, and in contrast to e.g. non-critical strings, there is no ODE in the string coupling constant.

However, we have a PDE governing the total free energy: the famous **holomorphic anomaly equations** (HAE) of BCOV.

I will present a simplified version of these equations which is valid for toric CYs with one modulus.

The total free energy of the topological string satisfies a partial differential equation involving a **propagator**  $S$  and the complex modulus  $z$  of the CY

$$\frac{\partial F}{\partial S} = \frac{g_s^2}{2} D_z^2 F + \frac{1}{2} (D_z F)^2$$

One can solve this equation perturbatively to obtain the genus  $g$  free energies as polynomials in the propagator, and involving known functions of  $z$  [BCOV, Yamaguchi-Yau, Grimm-Klemm-M.M.-Weiss, Alim-Lange, Klemm et al., ...]

$$F_2(S, z) = \frac{5Y(z)^2 S^3}{24} + \dots$$

$Y(z)$  : Yukawa coupling (third derivative of  $F_0$  )

In this formulation,  $S$  is essentially an arbitrary variable. The conventional topological string free energies are recovered in the so-called **holomorphic limit**, where  $S$  becomes a (known) function of  $z$ .

By using this method, one can calculate the genus expansion efficiently, up to an integration constant independent of  $S$ , the **holomorphic ambiguity**, which is a function of  $z$ .

Fixing this ambiguity is the main problem of the method. In some local examples it can be done to all genus, and for some compact CYs one can do it up to genus  $\sim 60$

[Klemm et al.].

# The CESV ansatz

In two remarkable papers in 2013-4, CESV [Couso-Edelstein-Schiappa-Vonk] proposed to use the holomorphic anomaly equations of BCOV to calculate trans-series.

We then solve 
$$\frac{\partial F}{\partial S} = \frac{g_s^2}{2} D_z^2 F + \frac{1}{2} (D_z F)^2$$

with a **trans-series ansatz**

$$F = \sum_{g \geq 0} F_g(S, z) g_s^{2g-2} + g_s^{-b} e^{-\mathcal{A}/g_s} \sum_{k \geq 0} F_k^{(1)}(S, z) g_s^k$$

perturbative series

“instanton” correction

# Exact solutions

In recent work [Gu-M.M, Gu-Kashani-Poor-Klemm-M.M.] we obtained all-orders, **exact solutions** for these trans-series, for **arbitrary** CY

One consequence of our exact results is that these trans-series, in the holomorphic limit, can be written in closed form in terms of **perturbative** topological string free energies and their derivatives.

Let us consider a Borel singularity located at

$$\mathcal{A} = c^I \mathcal{F}_I + d_I X^I$$

If all  $c^I$  vanish, the trans-series is of the Pasquetti-Schiappa form

$$\mathcal{F}^{(1)} = \frac{1}{2\pi g_s} (\mathcal{A} + g_s) e^{-\mathcal{A}/g_s}$$

If at least one  $c^I$  is different from zero, the trans-series is non-trivial, and of the form

$$\mathcal{F}^{(1)} = \frac{1}{2\pi} \left( 1 + g_s c^J \frac{\partial F}{\partial X^J} (X^I - c^I g_s) \right) e^{F(X^I - c^I g_s) - F(X^I)}$$

There are also explicit formulae for “multi-instanton” trans-series, associated to the singularities

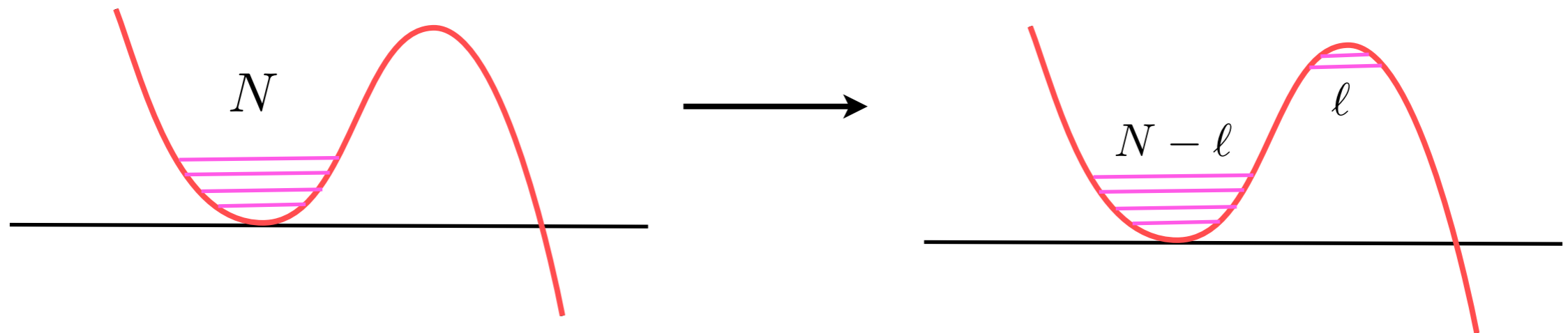
$$\ell \mathcal{A}, \quad \ell \in \mathbb{Z}_{>0}$$

$$\mathcal{F}^{(\ell)} = \{ \dots \} e^{F(X^I - \ell c^I g_s) - F(X^I)} + \dots$$

# Quantized moduli?

These formulae suggests that the “big moduli space” coordinates of the CY  $X^I$  are **quantized**, as postulated in large  $N$  dualities! A surprising result in the compact case.

The resulting amplitudes are also similar to large  $N$  instanton effects in matrix models, which are obtained by “eigenvalue tunneling”



# Resurgent structure

We **conjecture** (and have tested in many cases) that these trans-series give the alien derivatives of the total topological string free energy

$$\dot{\Delta}_{\ell\mathcal{A}} F(t, g_s) = S_{\ell} \mathcal{F}^{(\ell)}$$

The Stokes constants appearing here are known in some special cases, but not in general.

In some examples, Stokes constants turn out to be related to BPS invariants. This is the case in WKB theory and in complex Chern-Simons theory [Gaiotto-Moore-Neitzke, Garoufalidis-Gu-M.M.]

We expect a similar picture in our CY case.

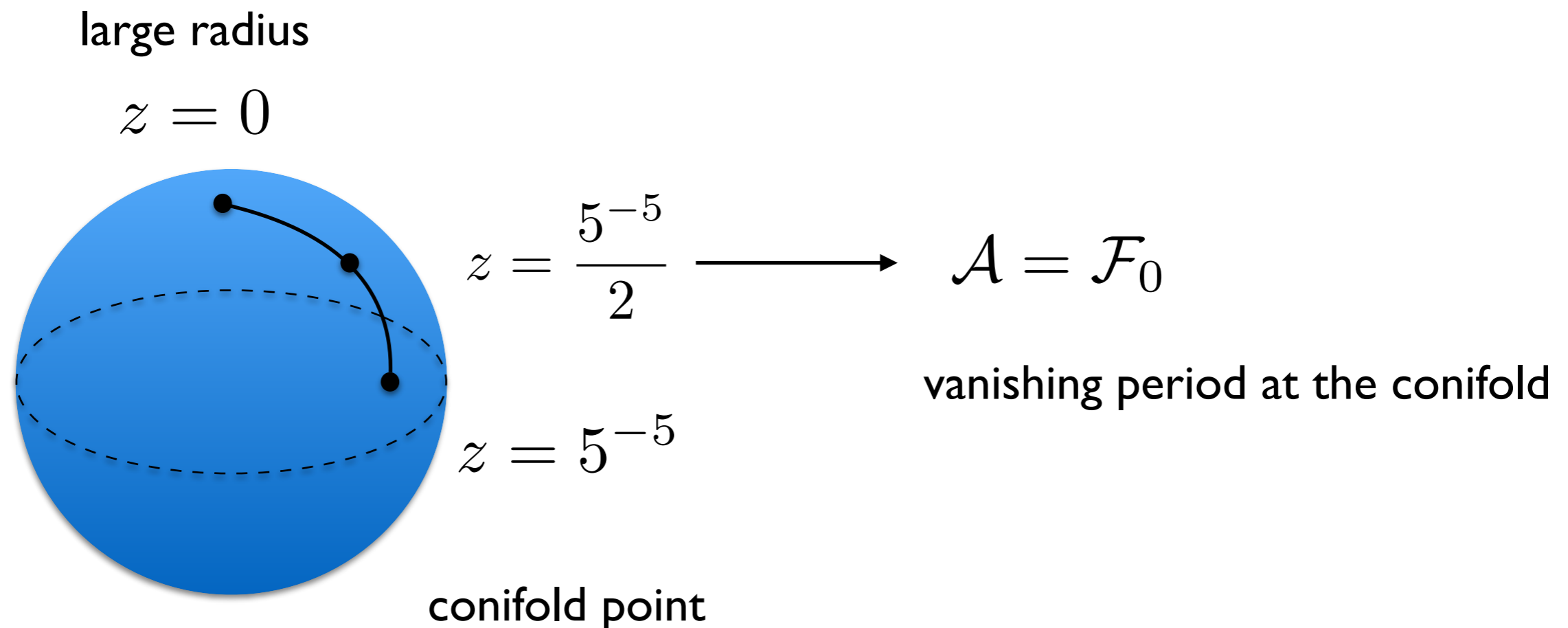
We can show e.g. that the genus zero Gopakumar-Vafa invariants  $n_{g=0,p_a}$  arise as Stokes constants associated to towers of Borel singularities at

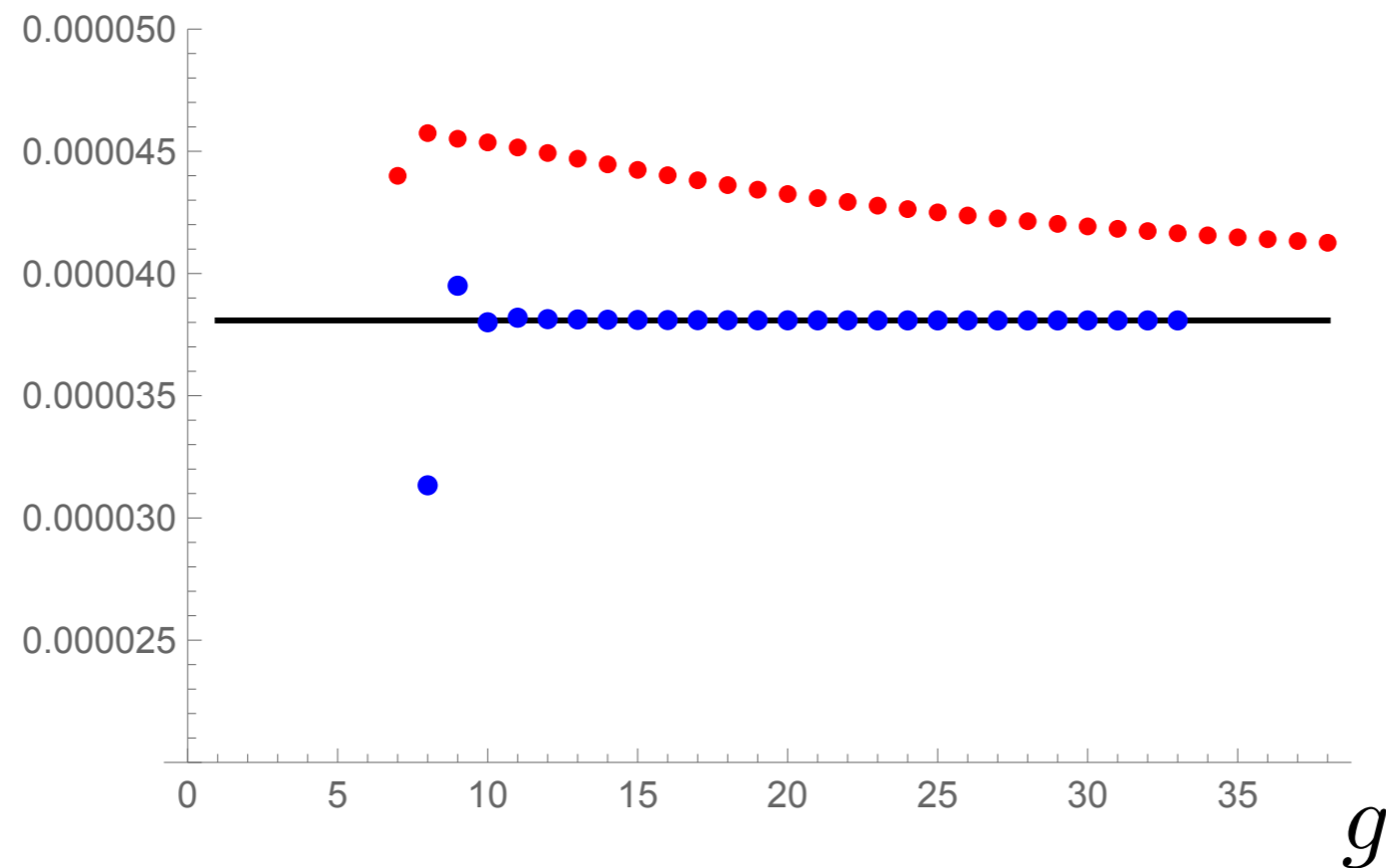
$$p_a X^a + m X^0 \quad m \in \mathbb{Z}$$

Let us note however that, in the case of topological strings, Stokes automorphisms seem to be more complicated than in the case of “quantum-periods” or in DT theory, where they are given by the Delabaere-Pham formula (see my companion paper with Jie Gu on resurgence of quantum periods).

# Experimental evidence: asymptotics in the quintic CY

We can test the explicit formulae for the trans-series against the large genus asymptotics of the free energies in e.g. the famous quintic CY





**red dots:** sequence  $\frac{\mathcal{A}^{2g-1}}{\Gamma(2g-1)} F_g$

**blue dots:** Richardson acceleration

**black line:** prediction from one-instanton formula

# Applications to large $N$ matrix models

The  $1/N$  expansion of Hermitian, multi-cut matrix models is also governed by the holomorphic anomaly equations [Huang-Klemm, Eynard-M.M.-Orantin]

The resurgent structure for this expansion is conjecturally the same that we have described for the topological string, and the multi-instanton amplitudes we have presented give the large  $N$  instantons of the matrix model.

Very explicit tests in the two-cut cubic matrix model [M.M.-Miravitllas], clarifying previous puzzles in [Klemm-M.M.-Rauch]

# Conclusions and outlook

We have obtained **explicit** and **exact** non-perturbative sectors in topological string theory, for **arbitrary** CY manifolds, by using a trans-series ansatz in the holomorphic anomaly equations.

We have conjectured that they provide the basic ingredients of the resurgent structure in topological string theory and matrix models, and checked this conjecture in many examples.

The full structure requires determining the possible Borel singularities and their Stokes constants, and we expect very rich mathematics and physics associated to this.

What is the physical interpretation of these trans-series?  
They seem to correspond to D-branes of the “wrong” type (since they are associated to periods)

Possibility: they are “renormalons” of the topological string, controlling the factorial growth of integrals over the moduli space of Riemann surface, but there are in addition “instanton” corrections associated to D-branes of the “right” type.

**Thank you for your attention!**

