

Wall-crossing and resurgence in susy QFT

Lotte Hollands

[Gaiotto - Moore - Neitzke]

[Nekrasov - Shatashvili - ...]

[M - Neitzke]

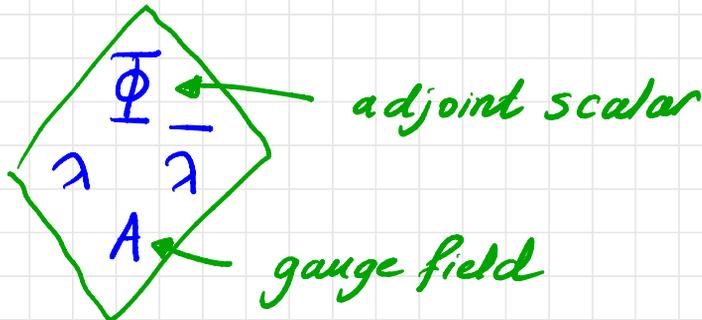
[M - Rüter - Szabo]

[Alim - M - Tulli]

4d $N=2$ quantum field theories

4d Yang-Mills theory :

$$Z = \int DA \exp \left(\int_{\mathbb{R}^4} F_A \wedge F_A + \dots \right)$$

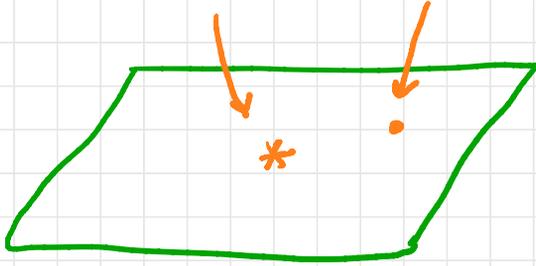


Low energy description:

[Seiberg-Witten]



$\Sigma =$ Seiberg-Witten curve



$\mathcal{B} =$ quantum Coulomb branch

special class: theories of class S

[Gaiotto, Gaiotto-Moore-Neitzke]

For these, the more

fundamental object is a (punctured)

Riemann surface C , called the UV curve

Coulomb branch \mathcal{B} is the space of tuples of k -differentials on C

e.g. $G = SU(k)$ a vacuum is par'd by $(\varphi_1, \varphi_2, \dots, \varphi_k)$
" "

$$\varphi_k \sim u_k(x) dx^{\otimes k}$$

Schubert-Witten curve $\Sigma \subset T_y^* C_x$:

$$\lambda^k - \varphi_2 \lambda^{k-2} + \dots - \varphi_k = 0$$

λ tautological 1-form on T^*C
" $y dx$

Given any (punctured) C , a Lie algebra of
there is a yd $N=2$ theory $\mathcal{X}(C)$

family of
Coulomb
vacua

Σ_u

$\downarrow K: 1$

C

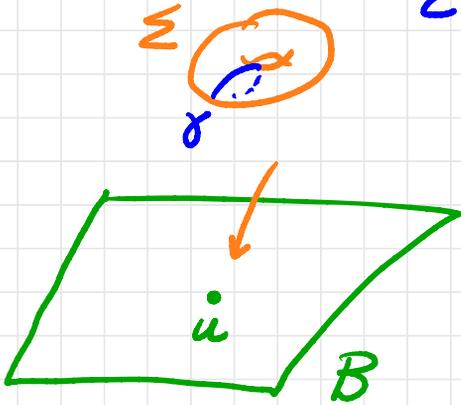
Two properties:

① SW prepotential F_0

$N=2$ SUSY algebra \leadsto

holomorphic central charge f/n

$$Z : H_1(\Sigma_g) \rightarrow \mathbb{C}$$



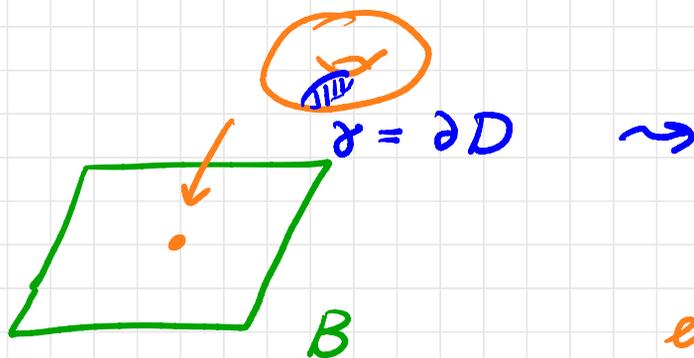
$$Z(\gamma) = \oint_{\gamma} \lambda$$

polarization of $\bar{\Sigma}$, i.e. choice $\{\alpha^I, \beta_I\}$

$$\text{then } Z(\alpha^I) \equiv a^I$$

$$Z(\beta_I) \equiv a_{0,I} = \frac{\partial F_0}{\partial a^I}$$

② particle BPS spectrum



in
yd theory
we find
a particle

e-m charge = γ
mass $\propto |\lambda|$
 γ

if particle BPS

then $|\frac{\propto |\lambda|}{\gamma}| = \frac{\propto |\lambda|}{\gamma}$

\parallel
 \cong

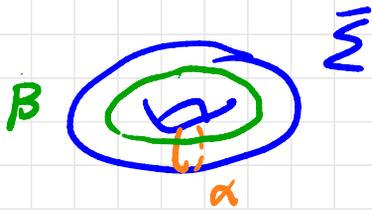
$$\mathcal{H}_{u, \gamma}^{\text{BPS}} = \{ \psi \in \mathcal{H}_{u, \gamma} \mid H\psi = |\mathcal{E}_\gamma(u)| \psi \}$$

example pure $SU(2)$ theory

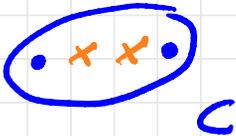
$$C = \text{circle with two dots} = \mathbb{P}^1_{0, \infty}$$

$$B = \xi \varphi_2(x) = \left(\frac{1}{x^3} - \frac{2u}{x^2} + \frac{1}{x} \right) dx^{\otimes 2}$$

$$\Sigma: y^2 = \varphi_2(x)$$



$\downarrow 2:1$



$$2a^2$$

$$F_0 = F_0 d$$

$$+ F_0^{1-loop}$$

$$a^2 \log a^2 + F_0^{inst}$$

$$\sum_{k=1}^{\infty} F_k(a) \wedge^{4k}$$

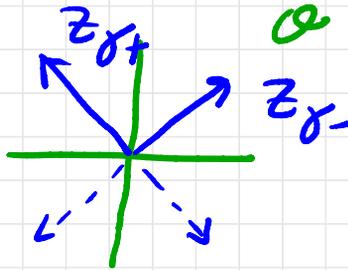


$$\arg z_{\gamma_+} = \arg z_{\gamma_-}$$

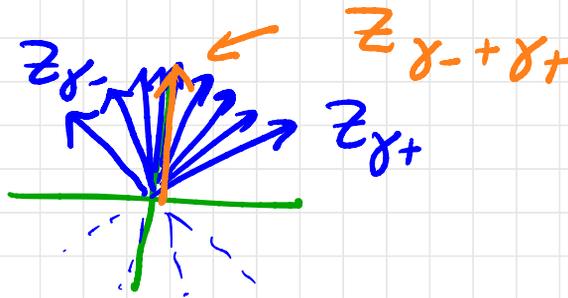
$$u \rightarrow \infty : \text{1-cycle } \alpha \rightarrow 0$$

$$u \rightarrow \pm 1 : \text{1-cycle } \gamma_{\pm} \rightarrow 0$$

STRONG:



WEAK



KS wall-crossing formula:

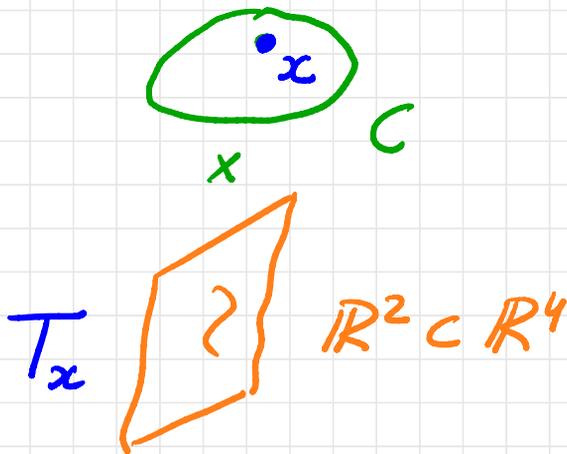
$$K_{\gamma_+} K_{\gamma_-} = K_{\gamma_-} K_{\gamma_+ + 2\gamma_-} \dots K_{\gamma_+ + \gamma_-}^2 \dots K_{2\gamma_+ + \gamma_-} K_{\gamma_+}$$

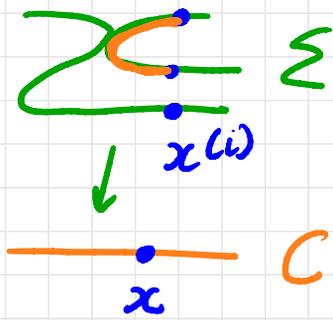
$$K_{\gamma} : \chi_{\gamma} \rightarrow \chi_{\gamma} (1 - (-)^{\langle \gamma', \gamma \rangle} \chi_{\gamma})^{\langle \gamma', \gamma \rangle}$$

Goal: \swarrow deformed version of
Relation between F_0 and BPS spectrum
of 4d $\mathcal{N}=2$ theories in the so-called
 Ω -background

1 more ingredient:

UV curve C is also moduli space of
"canonical surface defect"





$x^{(i)}$ parametrize
vacua of the
2d theory T_x

T_x has 2d BPS states with
charge γ_{ij}

$$|\oint_{\gamma_{ij}} \lambda| = \oint_{\gamma_{ij}} |\lambda| \quad \text{[Lecotti-Vafa]}$$

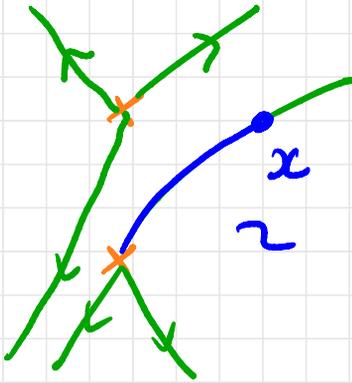
Both the 2d and 4d BPS states
can be visualized using spectral networks
[G-M-N]

Fix $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, fix vacuum
 $u \in \mathcal{B}$

real

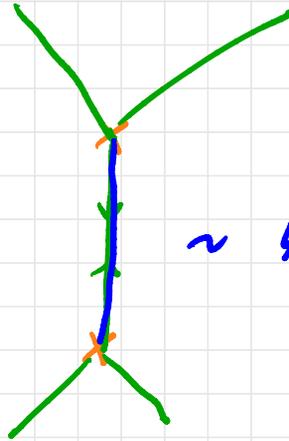
1-dim'l trajectories on \mathbb{C}

with $(\lambda_j - \lambda_k) \cdot v \in e^{i\theta} \mathbb{R}$



\sim

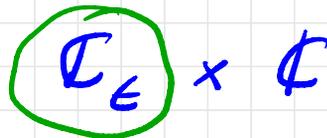
2d BPS state in T_x



\sim 4d BPS state
in $\mathcal{X}(\mathbb{C})$

Resurgence comes in when we introduce
a parameter ϵ :

$\mathbb{R}^4 \rightarrow \frac{1}{2} \Omega$ -background



$$F_0 \rightsquigarrow F_{NS} = \frac{1}{\epsilon} F_0 + \text{terms less singular}$$

↑
Nekrasov-Shatashvili
free energy

$$\mathbb{C}_\epsilon \times \boxed{\mathbb{C}}$$

$\frac{1}{2}$ Σ -background quantizes SW geom

$\Sigma \rightsquigarrow$ diff operator d_ϵ
(oper = quantum curve)

ex pure $SU(2)$ thty $\Sigma: y^2 - \varphi_2(x) = 0$

\rightsquigarrow Mathieu oper

$$d_\epsilon = \epsilon^2 \partial_x^2 - \left(\varphi_2(x) + \frac{\epsilon^2}{4x^2} \right)$$

• solns to $d_\epsilon \psi^{(i)}(x) = 0$

↑ rev of T_x in the vacuum $x^{(i)}$

• monodromies of $\psi^{(i)}(x)$ around $\gamma \in H^1(\bar{E})$

$$\left. \begin{aligned} Z_\epsilon(\alpha^I) &\equiv a_{\epsilon}^I \\ Z_\epsilon(\beta_I) &\equiv a_{\rho, I}^\epsilon \end{aligned} \right\} \frac{\partial \mathcal{F}_{NS}(a_\epsilon^I)}{\partial a_\epsilon^I}$$

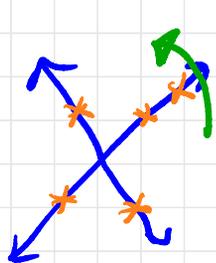
"
" ϵ
 $a_{\rho, I}$

[Mironov-Morozov]

analytic results
in ϵ

→ exact WKB formalism

$$\psi^{(i)}(x) \rightsquigarrow B_\epsilon \psi^{(i)}(x, \epsilon)$$



\mathcal{O}_c

Stokes graph

\simeq spectral network

2d BPS state

$\therefore B_{\mathcal{Q}} \gamma^{(i)}(x, \epsilon)$ non-perturbative
rev of T_x
in vacuum $x^{(i)}$

encodes 2d BPS jumps

\therefore Voros periods = spectral coord
= monodromies

jump in the presence of a 4d BPS
state

$$F_{NS}^{\mathcal{Q}}(a, \epsilon)$$

[H-Kidwai]

[H-Rüter-Szabo]

$$\downarrow \mathcal{Q} = \mathcal{Q}_{\gamma_+ + \gamma_-}$$

$$F_{NS}$$

generic jump

$$\sim \text{Li}_2(e^{\pi i a / \epsilon})$$

Remarks:

- statement $F_{NS}^{\mathcal{O}_{\gamma^+ + \gamma^-}} = F_{NS}$ is equivalent to saying that F_{NS} is the generating function of opers in Fenchel-Nielsen coord

↓
[Nekrasov-Raschke-Shatashvili]

- F_{NS} can be computed in terms of Wronskians of solutions $\psi^{(i)}$

[abelianization or exact WKB]

example: pure $SU(2)$ [H-Rüter-Szabo]

- generalization to $5d \sim$ topological strings

$d_\epsilon \rightarrow$ difference operator

example: resolved conifold

[Alim-H-Tulli]