Spectral summability of 1D oscillators and Fourier analysis on Carnot groups

Davide Barilari,
Dipartimento di Matematica "Tullio Levi-Civita",
Universitá degli Studi di Padova

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Joint work with

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This is based on a joint work with

- Hajer Bahouri (LJLL, CNRS & Sorbonne Univ)
- Isabelle Gallagher (DMA, École Normale Supérieure)
- Matthieu Léautaud (IMO, Univ. Paris Saclay)

→ Main references:

BBG-21 H.Bahouri, D.Barilari, I.Gallagher, *Strichartz estimates and Fourier restriction theorems in the Heisenberg group*, Journal of Fourier Analysis and Applications, 2021

BBGM-23 H.Bahouri, D.Barilari, I.Gallagher, M.Léautaud *Spectral summability for the quartic oscillator with applications to the Engel group*, Journal of Spectral Theory, 2023

Outline

1 Strichartz estimates in the Heisenberg group

2 Spectral summability of quartic oscillators and the Engel group

3 Comments and generalizations

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1 Strichartz estimates in the Heisenberg group

2 Spectral summability of quartic oscillators and the Engel group

3 Comments and generalizations

The Schrödinger sequation on $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

From the explicit expression of the solution, using Fourier analysis:

$$u(t,\cdot) = \frac{e^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0 \,.$$

one obtains the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \tag{1}$$

Strichartz estimates

From the basic dispersive estimate (using so-called TT^* argument)

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following Strichartz estimate

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{2}$$

where (p, q) satisfies the scaling admissibility condition

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \qquad q \geq 2,\ (n, q, p) \neq (2, 2, \infty)$$

- Similar dispersive inequality for the inhomogeneous Schrödinger equation $i\partial_t u - \Delta u = f$
- crucial in the study of semilinear and quasilinear Schrödinger equations

No dispersion in Heisenberg

The linear Schrödinger equations on $\mathbb{H}$ associated with the sublaplacian

$$(S_{\mathbb{H}}) \quad \begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = u_0 \,, \end{cases}$$

Theorem (Bahouri-Gérard-Xu 2000)

There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H})$ such that the solution to the free Schrödinger equation $(S_{\mathbb{H}})$ satisfies

$$u(t, x_1, x_2, x_3) = u_0(x_1, x_2, x_3 + t) \,.$$

In particular for all $1 \leq p \leq \infty$

$$\|u(t,\cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

$\rightarrow$ no dispersion

The Heisenberg group $\mathbb{H}$

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$\mathbb{H} \sim \mathbb{R}^3$

$$X_1 := \partial_1 - \frac{x_2}{2}\partial_3 \ , \quad X_2 := \partial_2 + \frac{x_1}{2}\partial_3 \, , \quad X_3 := \partial_3 \, .$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \end{pmatrix}$$

The Haar measure is equal to the Lebesgue measure.

Convolution product $f \star g(x) := \int_{\mathbb{H}} f(x \cdot y^{-1}) g(y) \, dy \, .$

Homogeneous dimension
$Q = \sum_j j \dim \mathfrak{g}_j = 4 \, , \quad \left| B_{\mathbb{H}}(x, r) \right| = r^Q \left| B_{\mathbb{H}}(0, 1) \right|$

Carnot groups

The Lie algebra $\mathfrak{g}$ of a Carnot (stratified Lie) group of step r admits the following stratification

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i] \, .$$

A sub-Riemannian structure is given by a scalar product on $\mathfrak{g}_1$

Heisenberg group $\mathbb{H}$ (step 2)

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \, , \quad \overbrace{X_1, X_2}^{\mathfrak{g}_1} , \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}$$

Engel group $\mathbb{E}$ (step 3)

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \, , \quad \overbrace{X_1, X_2}^{\mathfrak{g}_1} , \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2} , \quad \overbrace{X_4 = [X_1, X_3]}^{\mathfrak{g}_3}$$

- Goal: prove (some) Strichartz estimates in the Heisenberg group

$\rightarrow$ the original approach of Strichartz, 1977

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RESTRICTIONS OF FOURIER TRANSFORMS TO QUADRATIC SURFACES AND DECAY OF SOLUTIONS OF WAVE EQUATIONS

ROBERT S. STRICHARTZ

§1. Introduction

Let S be a subset of $\mathbb{R}^n$ and $d\mu$ a positive measure supported on S and of temperate growth at infinity. We consider the following two problems:

Problem A. For which values of p, $1 \leq p < 2$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\hat{f}$ has a well-defined restriction to S in $L^2(d\mu)$ with

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \leq c_p \|f\|_p? \tag{1.1}$$

- The Fourier dual of $\mathbb{R}^n$ is $\mathbb{R}^n$
- The Fourier dual of $\mathbb{H}^d$ is not $\mathbb{H}^d$

$\rightarrow$ anisotropic norms due to the no-dispersion effect

The result

A function f on $\mathbb{H}^1$ is said to be *radial* if $f(x, y, z) = \phi(x^2 + y^2, z)$.

Theorem (Bahouri, DB, Gallagher, '19)

Given (p, q) *belonging to the admissible set*

$$\mathcal{A} = \left\{(p, q) \in [2, \infty]^2 \,/\, p \leq q \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}\right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ *with radial data satisfies*

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1}\left(\|u_0\|_{L^2(\mathbb{H}^d)}\right).$$

- very restrictive due to $p \leq q$ then $p = q = 2$
- we stress that $L^\infty_z\ L^q_t\ L^p_{x,y} \neq L^\infty_t\ L^q_z\ L^p_{x,y}$
- similar for inhomogeneous and wave

The result

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A function f on $\mathbb{H}^1$ is said to be *radial* if $f(x, y, z) = \phi(x^2 + y^2, z)$.

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Given (p, q) *belonging to the admissible set*

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the solution to the Schrödinger equation $(S_{\mathbb{H}})$ *with radial data satisfies*

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{H^\sigma(\mathbb{H}^d)} \right).$$

- $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$ is the loss of derivatives, $\sigma = 0$ forces $p = q$
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

Fourier restriction problem

Stein, Fefferman, Tomas, etc:

Can we restrict the Fourier transform of an L^p function to hypersurfaces ?

- f in $L^1(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ continuous $\rightarrow$ OK.
- f in $L^2(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ in $L^2(\widehat{\mathbb{R}}^n)$ $\rightarrow$ arbitrary on a zero meas set

Tomas and Stein

For which $1 \leq p \leq 2$ then $\mathcal{F}(f)$ can be restricted to a hypersurfaces $\widehat{S}$ and is in L^q?

S should be "sufficiently curved" since

$$f(x) = \frac{e^{-|x'|^2}}{1 + |x_1|} \qquad x = (x_1, x') \in \mathbb{R}^n, \tag{3}$$

is in L^p for $p > 1$ but cannot be restricted to hyperplane.

Tomas-Stein

Theorem (Tomas-Stein, 1975)

Let $\widehat{S}$ be a smooth compact hypersurface in $\widehat{\mathbb{R}}^n$ with non vanishing Gaussian curvature at every point, $d\sigma$ a smooth measure on $\widehat{S}$.

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^2(\widehat{S},d\sigma)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}\,.$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$ and every $p \leq (2n+2)/(n+3)$,

- A necessary condition $p \leq (2n+2)/(n+3)$ Knapp counterexample
- A similar result is possible for surfaces with vanishing Gaussian curvature (that are not flat).
- In this case the range of p is smaller depending on the order of tangency of the surface to its tangent space.
- for $q \neq 2$ not completely solved

From restriction to Strichartz estimates

The classical Schrödinger equation in $\mathbb{R}^n$: taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi \,. \tag{4}$$

Consider the paraboloid $\widehat{S}$ in the space of frequencies $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$

$$\widehat{S} = \left\{ (\alpha, \xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2 \right\} .$$

- Given $\widehat{u}_0 : \widehat{\mathbb{R}}^n \to \mathbb{C}$ define $g : \widehat{S} \to \mathbb{C}$ as $g(|\xi|^2, \xi) = \widehat{u}_0(\xi)$. Then

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z) d\sigma(z)$$

where $y = (t,x)$ and $z = (\alpha, \xi)$.

Some obervations

1. Prove a Fourier restriction on the Heisenberg group

- a (spectral) restriction result of D.Müller $\rightarrow$ specific for the "sphere"
- what is the sphere? what about paraboloid?

2. We do not exactly need restriction theorems for $\mathbb{H}^d$

- we applied the result to a surface in the space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$

$\rightarrow$ the paraboloid for the Schrödinger eq. (the cone for the wave equation) is in $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$,

- which is not related to $\mathbb{H}^{d'}$ for some d'.

It is defined using irreducible unitary representations : for any integrable function u on $\mathbb{H}$ (Kirillov theory)

$$\forall \lambda \in \mathbb{R}^*, \quad \widehat{u}(\lambda) := \int_{\mathbb{H}} u(x) \mathcal{R}_x^\lambda dx \,,$$

with $\mathcal{R}^\lambda$ the group homomorphism between $\mathbb{H}$ and the unitary group $\mathcal{U}(L^2(\mathbb{R}))$ of $L^2(\mathbb{R})$ given for all x in $\mathbb{H}$ and ϕ in $L^2(\mathbb{R})$, by

$$\mathcal{R}_x^\lambda \phi(\theta) := \exp\Big(i\lambda x_3 + i\lambda\theta x_2\Big)\phi(\theta + x_1)\,.$$

Then $\widehat{u}(\lambda)$ is a family of bounded operators on $L^2(\mathbb{R})$, with many properties similar to $\mathbb{R}^d$: $\underbrace{\text{inversion formula}}_{Trace}$, $\underbrace{\text{Fourier-Plancherel identity}}_{Hilbert-Schmidt}$

The Fourier transform of the sublaplacian on $\mathbb{H}$

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The sub-Laplacian

$$\Delta_{\mathbb{H}} = X_1^2 + X_2^2$$

There holds

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda) = \widehat{u}(\lambda) \circ P_\lambda \,, \quad \text{with} \quad P_\lambda := -\frac{d^2}{d\theta^2} + \lambda^2\theta^2 \,.$$

The spectrum of the **rescaled harmonic oscillator** is

$$\mathrm{Sp}(P_\lambda) = \big\{ |\lambda|(2m+1) \,, m \in \mathbb{N} \big\}$$

and the eigenfunctions are the Hermite functions ψ_m^λ. So for all $m \in \mathbb{N}$,

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda)\psi_m^\lambda = E_m(\lambda)\widehat{u}(\lambda)\psi_m^\lambda \,.$$

The frequency space on $\mathbb{H}$

Set $\widehat{x} := (n, m, \lambda) \in \widehat{\mathbb{H}} = \mathbb{N}^2 \times \mathbb{R}^*$, and

$$\begin{aligned}\mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda) &:= (\widehat{u}(\lambda)\psi_m^\lambda | \psi_n^\lambda)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{H}} \mathcal{W}(\widehat{x}, x) u(x) dx\end{aligned}$$

where $\mathcal{W}(\widehat{x}, x) := e^{i\lambda x_3} e^{-|\lambda|(x_1^2+x_2^2)} \underbrace{L_m(2|\lambda|(x_1^2 + x_2^2))}_{\text{Laguerre polynomial}}$.

Then

$$\mathcal{F}_{\mathbb{H}}(-\Delta_{\mathbb{H}} u)(n, m, \lambda) = \underbrace{E_m(\lambda)}_{\text{frequency}} \mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda)\,.$$

Bahouri, Chemin, Danchin

First observation in the Heisenberg group

Let u_0 in $\mathcal{S}(\mathbb{H}^d)$ be radial and consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Taking the partial Fourier transform with respect to the variable w

$$\begin{cases} i\frac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = -|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} = \mathcal{F}_{\mathbb{H}} u_0 \,. \end{cases}$$

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(n,m,\lambda)\delta_{n,m} \,.$$

$\rightarrow$ Notice that if we set $n = m = 0$ we see the "transport" part

$$\mathcal{F}_{\mathbb{H}}(u)(t,0,0,\lambda) = e^{it|\lambda|d}\mathcal{F}_{\mathbb{H}}(u_0)(0,0,\lambda) \,.$$

What we proved is the following restriction theorem

Theorem (Bahouri, DB, Gallagher, '19)

If $1 \leq q \leq p \leq 2$*, then for* f *radial*

$$\|\mathcal{F}_{\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d}(f)|_\Sigma\|_{L^2(d\Sigma)} \leq C_{p,q}\|f\|_{L^1_z L^q_t L^p_{x,y}} , \tag{5}$$

where Σ is the paraboloid

$$\Sigma = \Big\{\big(\alpha,(n,n,\lambda)\big) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \,/\, \alpha = |\lambda|(2|n|+d)\Big\}.$$

Using the dual inequality and assuming that $\mathcal{F}_{\mathbb{H}} u_0$ is localized in a ball

For any $2 \leq p \leq q \leq \infty$

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C\|\mathcal{F}_{\mathbb{H}} u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C\|u_0\|_{L^2(\mathbb{H}^d)} ,$$

Comments

On the positive side:

- an interpretation of Müller result
- extension to other surfaces in the dual of Heisenberg group
- some new Strichartz estimates for linear Shrödinger/wave equations

Still to do ($\rightarrow$ a lot!):

- remove the radial assumption on the initial data ?
- extend this analysis to more general groups ? every 2-step ?
- obtain applications to sub-Riemannian NLS ? (seems difficult)
- what about 3 steps?

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3 Comments and generalizations

The Engel group

$\mathbb{E} \sim \mathbb{R}^4$

$$X_1 := \partial_1 \,, \quad X_2 := \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4 \,, \quad X_3 := \partial_3 + x_1\partial_4 \,, \quad X_4 := \partial_4 \,.$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_1 y_2 \\ x_4 y_4 + x_1 y_3 + \frac{x_1^2}{2} y_2 \end{pmatrix}$$

Homogeneous dimension: $Q = \sum_j j \dim \mathfrak{g}_j = 7$

$$\delta_\varepsilon(x_1, x_2, x_3, x_4) = (\varepsilon x_1, \varepsilon x_2, \varepsilon^2 x_3, \varepsilon^3 x_4)$$

The sublaplacian

In general

$$\Delta := \sum_{X_j \in \mathfrak{g}_1} X_j^2$$

so on $\mathbb{H}$ and $\mathbb{E}$

$$\Delta = X_1^2 + X_2^2 \,.$$

Homogeneous and inhomogeneous Sobolev spaces are defined by

$$\|u\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} \,, \quad \|f\|_{H^s} = \|(\mathrm{Id} - \Delta)^{\frac{s}{2}} u\|_{L^2} \,.$$

Questions :

- "Space of frequencies" for Fourier Analysis
- Summation formula
- Some applications

The Fourier transform on $\mathbb{E}$

For any integrable function u on $\mathbb{E}$

$$\forall(\nu,\lambda)\in\mathbb{R}\times\mathbb{R}^*\,,\quad \widehat{u}(\nu,\lambda):=\int_{\mathbb{E}}u(x)\mathcal{R}_x^{\nu,\lambda}dx\,,$$

- $\mathcal{R}^{\nu,\lambda}$ the group homomorphism between $\mathbb{E}$ and $\mathcal{U}(L^2(\mathbb{R}))$
- for all x in $\mathbb{E}$ and ϕ in $L^2(\mathbb{R})$, by

$$\mathcal{R}_x^{\nu,\lambda}\phi(\theta):=\exp\left(i\lambda x_4+i\lambda\theta x_3-i\frac{\nu}{\lambda}x_2+i\lambda\frac{\theta^2}{2}x_2\right)\phi(\theta+x_1)\,.$$

- λ is dual to the center X_4 (homogeneous of degree 3)
- ν is representing the operator (homogeneous of degree 4)

$$X_4X_2-\frac{1}{2}X_3^2$$

$$\widehat{-\Delta_{\mathbb{E}} u}(\nu,\lambda) = \widehat{u}(\nu,\lambda) \circ P_{\nu,\lambda}\,, \quad \text{with} \quad P_{\nu,\lambda} := -\frac{d^2}{d\theta^2} + \left(\lambda\frac{\theta^2}{2} - \frac{\nu}{\lambda}\right)^2 .$$

- $\mathrm{Sp}(P_{\nu,\lambda}) = \{E_m(\nu,\lambda), m \in \mathbb{N}\}$ not explicit!
- $\psi_m^{\nu,\lambda}$ the eigenfunctions of $P_{\nu,\lambda}$ associated with $E_m(\nu,\lambda)$.

Homogeneity reduces to the study

$$P_\mu := -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2$$

Setting $T_\alpha \varphi := \alpha^{\frac{1}{2}}\varphi(\alpha\,\cdot)$ and $\mu = \frac{\nu}{|\lambda|^{4/3}}$ then $P_{\nu,\lambda} = |\lambda|^{2/3} T_{|\lambda|^{1/3}} \mathrm{P}_\mu T_{|\lambda|^{-1/3}}$

$$E_m(\nu,\lambda) = |\lambda|^{2/3}\mathrm{E}_m(\mu) \quad \text{and} \quad \psi_m^{\nu,\lambda} = T_{|\lambda|^{1/3}}\varphi_m^\mu$$

The Lai-Robert, Colin de Verdière-Letrouit,
Helffer, Helffer-Léautaud...

The frequency space on $\mathbb{E}$

Set $\widehat{x} := (n, m, \nu, \lambda) \in \widehat{\mathbb{E}} = \mathbb{N}^2 \times \mathbb{R} \times \mathbb{R}^*$, and

$$\mathcal{F}_{\mathbb{E}}(u)(n, m, \nu, \lambda) := \left(\widehat{u}(\lambda)\psi_m^{\nu,\lambda} | \psi_n^{\nu,\lambda}\right)_{L^2(\mathbb{R})}$$
$$=: \int_{\mathbb{H}} \mathcal{W}(\widehat{x}, x) u(x) dx$$

where

$$\mathcal{W}((n, m, \nu, \lambda), x) := e^{i(\lambda x_4 - \frac{\nu}{\lambda} x_2)} \int_{\mathbb{R}} e^{i\lambda(\theta x_3 + \frac{\theta^2}{2} x_2)} \psi_m^{\nu,\lambda}(\theta + x_1) \psi_n^{\nu,\lambda}(\theta) d\theta \, .$$

Then

$$\mathcal{F}_{\mathbb{E}}(-\Delta_{\mathbb{E}} u)(n, m, \nu, \lambda) = \underbrace{E_m(\nu, \lambda)}_{\text{frequency}} \mathcal{F}_{\mathbb{E}}(u)(n, m, \nu, \lambda) \, .$$

Spectral summability

Theorem (Bahouri-DB-Gallagher-Léautaud 2023)

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathsf{E}_m(\mu)^\gamma} d\mu < \infty \iff \gamma > 2$$

Moreover assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{5}{2}} dr)$

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} \Phi\big(E_m(\nu,\lambda)\big)\, d\nu d\lambda = \mathsf{C} \int_0^\infty \Phi(r) r^{\frac{5}{2}}\, dr\,.$$

where

$$\mathsf{C} = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{3}{\mathsf{E}_m(\mu)^{\frac{7}{2}}} d\mu\,.$$

- it splits the contribution of the spectrum and the one of F
- it is a summability result for all the spectra

Theorem (Bahouri-DB-Gallagher-Léautaud 2023)

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathrm{E}_m(\mu)^\gamma} d\mu < \infty \iff \gamma > 2$$

Moreover assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{Q-2}{2}} dr)$

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} \Phi\big(E_m(\nu,\lambda)\big)\, d\nu d\lambda = \mathrm{C} \int_0^\infty \Phi(r) r^{\frac{Q-2}{2}}\, dr\,.$$

where

$$\mathrm{C} = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{3}{\mathrm{E}_m(\mu)^{\frac{Q}{2}}} d\mu\,.$$

- it splits the contribution of the spectrum and the one of F
- it is a summability result for all the spectra

On the summability of the spectrum

It relies on a refined analysis of the spectrum of P_μ: recall

$$\mathrm{P}_\mu = -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2, \quad \mu \in \mathbb{R}$$

The behavior of the potential depends on the sign of the parameter μ:

- It admits a single well when $\mu < 0$
- It admits a double well when $\mu > 0$.
- need combination of microlocal and semiclassical analysis along with known spectral results.

Another observation for later

- it is the square of a polynomial of degree 2 (with no 1st order term)

Formula in simpler situations

Analogue in Heisenberg $\mathbb{H}^d$

$$\sum_{m\in\mathbb{N}^d}\int_0^\infty \Phi\big(|\lambda|(2|m|+d)\big)|\lambda|^d d\lambda = \left(\sum_{m\in\mathbb{N}^d}\frac{2}{(2|m|+d)^{d+1}}\right)\int_0^\infty \Phi(r)r^d\,dr\,.$$

- notice the Plancherel measure in LHS and $d=(Q-2)/2$, $d+1=Q/2$.
- the convergence in this case is easy

Analogue in $\mathbb{R}^n$ would be the spherical coordinate formula

$$\int_0^\infty \Phi\big(|\xi|^2\big)d\xi = |S^{d-1}|\int_0^\infty \Phi(r)r^{\frac{n-2}{2}}\,dr\,.$$

As for instance some Sobolev embeddings. Remember here $Q = 7$.

Proposition

For $s > Q/2$, then $H^s(\mathbb{E})$ embeds in $L^\infty(\mathbb{E})$.

Start from the inversion formula

$$u(x) = (2\pi)^{-3} \int_{\widehat{E}} \mathcal{W}(\widehat{x}, x^{-1}) \mathcal{F}_{\mathbb{E}}(u)(\widehat{x})\, d\widehat{x}$$

so that

$$|u(x)| \leq \int_{\widehat{E}} |\mathcal{W}(\widehat{x}, x)||\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})|\, d\widehat{x}$$

Recall that

$$\|u\|^2_{H^s(\mathbb{E})} := \int_{\widehat{E}} |\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})|^2 (1 + E_m(\nu, \lambda))^s\, d\widehat{x}$$

Sobolev embeddings

Multiplying/dividing $(1+E_m(\nu,\lambda))^{s/2}$ and using Cauchy-Schwartz

$$|u(x)| \leq \|u\|_{H^s} \left(\int_{\widehat{E}} |\mathcal{W}(\widehat{x}, x^{-1})|^2 (1+E_m(\nu,\lambda))^{-s} \, d\widehat{x} \right)^{1/2}$$

Since $\sum_{n\in\mathbb{N}} |\mathcal{W}(\widehat{x}, x^{-1})|^2 = 1$ due to the fact that representation are unitary it remains to estimate

$$\left(\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} (1+E_m(\nu,\lambda))^{-s} d\lambda d\nu \right)^{1/2}$$

which thanks to the summation formula is finite for $s > Q/2$

$$\leq \left(\int_0^\infty (1+r)^{-s} r^{\frac{Q-2}{2}} \, dr \right) \left(\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathrm{E}_m(\mu)^{\frac{Q}{2}}} d\mu \right)$$

An application

We are interested in the assumptions on Φ giving,

$$\Phi(-\Delta_{\mathbb{E}})u = u \star k_\Phi\,, \quad \text{for all } u \in \mathcal{S}(\mathbb{E})\,, \tag{6}$$

Theorem (BBGL, 23)

Assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{5}{2}}dr)$. *Then*

- *For any* $u \in \mathcal{S}(\mathbb{E})$, *the operator* $\Phi(-\Delta_{\mathbb{E}}) : \mathcal{S} \to L^\infty$ *is well-defined through*

$$\Phi(-\Delta_{\mathbb{E}})u \overset{\text{def}}{=} \mathcal{F}_{\mathbb{E}}^{-1}\Big(\Phi\big(E_m(\nu,\lambda)\big)\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})\Big)\,.$$

- *Moreover, there is* k_Φ *in* $\mathcal{S}'(\mathbb{E})$ *such that* $\Phi(-\Delta_{\mathbb{E}})u = u \star k_\Phi$ *and we have the continuous map*

$$\begin{aligned} L^1(\mathbb{R}_+, r^{\frac{5}{2}}dr) &\longrightarrow \mathcal{S}'(\mathbb{E}) \\ \Phi &\longmapsto k_\Phi \end{aligned}$$

- Indeed k_Φ belongs to $C^0 \cap L^\infty(\mathbb{E})$ and there holds

$$\|k_\Phi\|_{L^\infty(\mathbb{E})} \le (2\pi)^{-3} \mathrm{C} \int_0^\infty r^{5/2} |\Phi(r)| dr \quad \text{and}$$
$$k_\Phi(0) = (2\pi)^{-3} \mathrm{C} \int_0^\infty r^{5/2} \Phi(r) dr \,,$$

where

$$\mathrm{C} \stackrel{\text{def}}{=} \sum_{m \in \mathbb{N}} \int_{\mathbb{R}} \frac{3}{\mathrm{E}_m(\mu)^{\frac{7}{2}}} d\mu < \infty \,.$$

- Finally $k_\Phi \in L^2(\mathbb{E})$ if and only if $\Phi \in L^2(\mathbb{R}_+, r^{5/2} dr)$ and there holds

$$\|k_\Phi\|^2_{L^2(\mathbb{E})} = (2\pi)^{-3} \mathrm{C} \int_0^\infty r^{5/2} |\Phi(r)|^2 dr \,.$$

Outline

1 Strichartz estimates in the Heisenberg group

2 Spectral summability of quartic oscillators and the Engel group

3 Comments and generalizations

The filiform/Goursat group in general dimension

This is the nilpotent Lie group of dimension n and step $s = n - 1$ with a basis of the Lie algebra satisfying

$$[X_1, X_i] = X_{i+1}, \qquad i = 2, \dots, n$$

Example: Filiform/Goursat group (step 4)

$$\overbrace{X_1, X_2}^{\mathfrak{g}_1}, \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}, \quad \overbrace{X_4 = [X_1, X_3]}^{\mathfrak{g}_3}, \quad \overbrace{X_5 = [X_1, X_4]}^{\mathfrak{g}_4}$$

- dimension increase each time by 1
- s step, then $n = s + 1$ dimension
- it is always rank 2
- it is always the same vector field of $\mathfrak{g}_1$ generating the new direction

Generalization of summability

Generalization (only for this class of groups at the moment) as follows :

- the set of parameters will be $s-1=n-2$ dimensional : (ν,λ)
- $\nu=(\nu_2,\ldots,\nu_{s-1})$ a set of $s-2$ parameters
- the Plancherel measure as $f(\lambda)d\lambda d\nu$,
- $Q=1+s(s+1)/2$ be the homogeneous dimension

$$\sum_{m\in\mathbb{N}}\int\Phi\big(E_m(\nu,\lambda)\big)f(\lambda)d\lambda d\nu=c_n\left(\int r^{(Q-2)/2}\Phi(r)dr\right)\left(\sum_{m\in\mathbb{N}}\int\frac{1}{E_m(\nu,1)^{Q/2}}d\nu\right) \tag{7}$$

where $E_m(\nu,1)$ is the family of eigenvalue of a 1D oscillator of the form

$$-\frac{d^2}{d\theta^2}+\big(V_s(\nu;\theta)\big)^2$$

with $V_s(\nu;\cdot)$ polynomial of degree $s-1$ with no term of degree $s-2$

Formula for the V_s

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Better to show in dim $n+2$ (or step s, with $s = n+1$)

$$V_s(\nu;\cdot) = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{n!}\theta^n + \sum_{k=2}^{n} (-1)^{k-1} \frac{\nu_k}{(k-2)!\lambda^{k-1}} \frac{\theta^{n-k}}{n-k!} \right)^2$$

- λ is the dual variable to the center
- the ν represents the casimirs

$$\frac{1}{k} X_2^k + \sum_{\ell=1}^{k-1} (-1)^\ell \frac{(k-2)!}{(k-\ell-1)!} X_1^\ell X_2^{k-\ell-1} X_{\ell+2}$$

- explicit homogeneity

Generalization of summability

- The next case would be

$$-\frac{d^2}{d\theta^2} + \left(\frac{\lambda}{6}\theta^3 - \frac{\nu_2}{\lambda}\theta + \frac{\nu_3}{\lambda^2}\right)^2$$

  with ν_3, ν_2, λ homogeneous of degree $9, 6, 4$ respectively.

- Denoting $E_m(\nu_2, \nu_3, \lambda)$ the corresponding eigenvalues we are asking for which γ

$$\sum_{m\in\mathbb{N}} \int \frac{1}{E_m(\nu_2, \nu_3, 1)^\gamma} d\nu < \infty$$

- in progress!

Generalization of summability

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- Denoting $E_m(\nu_2, \nu_3, \lambda)$ the corresponding eigenvalues we are asking for which γ

$$\sum_{m\in\mathbb{N}}\int\frac{1}{E_m(\nu_2,\nu_3,1)^\gamma}d\nu<\infty$$

- in progress!
- thanks for your attention!

THE – END – OF – MY – TALK

Wave equation

The wave equation on $\mathbb{R}^n$

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1)\,, \end{cases}$$

The classical dispersive estimate writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|t|^{\frac{n-1}{2}}}\big(\|u_0\|_{L^1(\mathbb{R}^n)} + \|u_1\|_{L^1(\mathbb{R}^n)}\big)\,.$$

$\rightarrow$ oscillatory integrals and stationary phase theorem.

Strichartz estimate for wave equation

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C(p,q)\big(\|\nabla u_0\|_{L^2(\mathbb{R}^n)} + \|u_1\|_{L^2(\mathbb{R}^n)}\big)\,,$$

where (p,q) satisfies the scaling admissibility condition

$$\frac{1}{q} + \frac{n}{p} = \frac{n}{2} - 1, \qquad p, q \geq 2, \quad q < \infty\,.$$

Previous Stricharts estimate for wave on $\mathbb{H}$

On $\mathbb{H}^d$ one can prove a dimension-independent dispersive estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{H}^d)} \leq \frac{C}{|t|^{\frac{1}{2}}}(\|u_0\|_{L^1(\mathbb{H}^d)} + \|u_1\|_{L^1(\mathbb{H}^d)}),$$

- only the center is involved in the dispersive effect.
- this estimate is optimal.

This dispersive estimate gives rise to a Strichartz estimate

[Bahouri, Gérard, Xu, '00]

$$\|u\|_{L^q_t L^p_{z,s}} \leq C_{p,q,p_1,q_1}\left(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^{q_1'}_t L^{p_1'}_{z,s}}\right)$$

with $\frac{1}{q} + \frac{Q}{p} = \frac{Q}{2} - 1$ and $q \geq 2Q - 1$.

Our result for wave

In the case of the wave equation on $\mathbb{H}$ we obtain the following Strichartz estimate.

Theorem (Bahouri, DB, Gallagher, '19)

With the above notation, given (p, q) and (p_1, q_1) belonging to the admissible set

$$\mathcal{A}^W = \left\{ (p, q) \in [2, \infty]^2 \,/\, q \leq p \quad \textit{and} \quad \frac{1}{q} + \frac{2d}{p} = \frac{Q}{2} - 1 \right\},$$

there is a constant C_{p,q,p_1,q_1} such that the solution to the wave equation $(W_{\mathbb{H}})$ associated with radial data satisfies the following Strichartz estimate:

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C_{p,q,p_1,q_1} \left(\|\nabla_{\mathbb{H}^d} u_0\|_{L^2(\mathbb{H}^d)} + \|u_1\|_{L^2(\mathbb{H}^d)} + \|f\|_{L^1_s L^{q_1'}_t L^{p_1'}_z} \right).$$

- q can be small. In the previous $q \geq 2Q - 1$.
- we pay a price in the s variable.

Theorem [Bahouri-Gallagher 2022]
If Supp $u_0 \subset B(x_0, R)$ then for all $\kappa < 1$ and $|t|^{\frac{1}{2}} \geq R/(1-\kappa)$

$$\|u(t)\|_{L^\infty(B(x_0,\kappa|t|^{\frac{1}{2}}))} \leq \frac{C}{|t|^{\frac{Q}{2}}} \|u_0\|_{L^1(\mathbb{H})} .$$

The Schrödinger equation on $\mathbb{E}$

Theorem [Bahouri-DB-Gallagher-Léautaud 2023] The following holds

$$\exists u_0 \in \mathcal{S}(\mathbb{E}), \quad \liminf_{t\to\infty} |t| \|u(t)\|_{L^\infty(\mathbb{E})} \geq C$$

Moreover for any $u_0 \in L^1(\mathbb{E})$,

$$\sup_{x\in\mathbb{E}} \left| \langle x \rangle^{-1} u(t,x) \right| \leq \frac{C}{t} \|u_0\|_{L^1(\mathbb{E})} .$$