Magnetic Hardy inequalities in the Heisenberg group

David KREJČIŘÍK
Czech Technical University in Prague

Based on: Communications in Partial Differential Equations (2023)
Collaboration with: Biagio Cassano, Valentina Franceschi & Dario Prandi

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Motivation

Classical Hardy inequality

[Hardy 1920]

$$\boxed{-\Delta \geq \frac{c_n}{\varrho^2} \quad \text{in} \quad L^2(\mathbb{R}^n)}$$

$$c_n := \left(\frac{n-2}{2}\right)^2 \qquad n \geq 2$$

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Magnetic improvement

[Laptev, Weidl 1999]

⋮

[Cazacu, K 2016]

$$\boxed{-\Delta_A - \frac{c_n}{\varrho^2} \geq \frac{c_{n,B}}{1+\varrho^2 \log^2 \varrho} \quad \text{in} \quad L^2(\mathbb{R}^n)}$$

$$-\Delta_A := (-i\nabla + A)^2 \ \text{ with } \ A : \mathbb{R}^n \to \mathbb{R}^n$$

$$c_{n,B} > 0$$

$$\Updownarrow$$

$$B := \mathrm{d}A \neq 0$$

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(Euclidean) (Heisenberg)

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Riemannian (Euclidean) → sub-Riemannian (Heisenberg)

elliptic → sub-elliptic

Euclid (300 BC)	*versus*	**Heisenberg** (AD 1925)

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		strategy: [Rumin 1994] complex

Horizontal magnetic fields

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NB $\boxed{\mathbb{H}^1 \quad \leftsquigarrow \quad \mathrm{span}\{X, Y\}}$

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basis $T\mathbb{H}^1$ dual basis $T^*\mathbb{H}^1$

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$A = A_x\ \mathrm{d}x + A_y\ \mathrm{d}y + A_\omega\ \omega$ with $A_x, A_y, A_\omega : \mathbb{R}^3 \to \mathbb{R}$

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$$\underset{XA_y - YA_x - A_\omega}{\overset{b_\omega}{\|}}$$

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↖ irrelevant (can be chosen arbitrarily)

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$$b_\omega \;\|\; XA_y - YA_x - A_\omega = 0$$

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$\|$

$XA_y - YA_x - A_\omega = 0$

$\mathrm{d}B = 0$ (physics) $\Longrightarrow$ freedom in choice of A_x, A_y (gauge invariance)

Uniform magnetic fields

$:\Longleftrightarrow \quad B(x,y,z) = b_1 \, \mathrm{d}x \wedge \omega + b_2 \, \mathrm{d}y \wedge \omega$ with $b_1, b_2 \in \mathbb{R}$

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Theorem. $\exists c > 0, \ \forall b_1, b_2 \in \mathbb{R},$ $\boxed{\inf \sigma(-\Delta_A) = c\,|B|^{2/3}}$

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Remark. Linear growth in the Euclidean case: $\boxed{\inf \sigma(-\Delta_A) = |B|}$ in $L^2(\mathbb{R}^2)$.

However, non-linearity known for fields vanishing on curves: [Montgomery 1995]

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Corollary. $\boxed{-\Delta_A \geq c \, |B|^{2/3}}$ (Poincaré inequality)

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Theorem. $\exists c>0,\ \forall b_1, b_2 \in \mathbb{R},\ \boxed{\inf\sigma(-\Delta_A) = c\,|B|^{2/3}}$

Proof. $-\Delta_A = -X^2 - \left(Y + i|B|\dfrac{x^2}{2}\right)^2$ $\left(\text{gauge choice } A := |B|\,\dfrac{x^2}{2}\mathrm{d}y\right)$

$$\cong -\partial_x^2 - \left(\partial_y + x\partial_w + i|B|\frac{x^2}{2}\right)^2 \qquad \left(w := z + \frac{xy}{2}\right)$$

$$\cong \int_{\eta,\nu\in\mathbb{R}}^{\oplus}\left[-\partial_x^2 + \left(\nu + x\eta - |B|\frac{x^2}{2}\right)^2\right] \qquad \text{(partial Fourier transforms)}$$

$$\cong \int_{\eta,\nu\in\mathbb{R}}^{\oplus}\left[-\partial_t^2 + \left(\frac{|B|}{2}t^2 + g\right)^2\right] \qquad \left(t := x - \frac{\eta}{|B|},\quad g := -\frac{\eta^2}{2|B|} - \nu\right)$$

$$\cong |B|^{2/3}\int_{\eta,\nu\in\mathbb{R}}^{\oplus}\left[-\partial_s^2 + \left(\frac{1}{2}s^2 + \tilde{g}\right)^2\right] \qquad \left(s := |B|^{1/3}t,\quad \tilde{g} := \frac{g}{|B|^{1/3}}\right)$$

$$c := \min_{\tilde{g}\in\mathbb{R}} \inf\sigma\left(-\partial_s^2 + \left(\frac{1}{2}s^2 + \tilde{g}\right)^2\right) > 0$$

q.e.d.

Uniform magnetic fields

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Theorem. $\exists c > 0, \;\; \forall b_1, b_2 \in \mathbb{R}, \;\; \boxed{\inf \sigma(-\Delta_A) = c\,|B|^{2/3}}$

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$$
\begin{aligned}
-\Delta_A &= -X^2 - \left(Y + i|B|\frac{x^2}{2}\right)^2 && \left(\text{gauge choice } A := |B|\,\frac{x^2}{2}\mathrm{d}y\right)\\
&\cong -\partial_x^2 - \left(\partial_y + x\partial_w + i|B|\frac{x^2}{2}\right)^2 && \left(w := z + \frac{xy}{2}\right)\\
&\cong \int_{\eta,\nu\in\mathbb{R}}^{\oplus} \left[-\partial_x^2 + \left(\nu + x\eta - |B|\frac{x^2}{2}\right)^2\right] && (\text{partial Fourier transforms})\\
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\end{aligned}
$$

$$c := \min_{\tilde{g}\in\mathbb{R}} \inf \sigma\left(-\partial_s^2 + \left(\frac{1}{2}s^2 + \tilde{g}\right)^2\right) \approx 0.57 > 0$$

[**Helffer** , Persson 2010]

q.e.d.

Aharonov–Bohm potentials

$:\Longleftrightarrow \quad A = \alpha\, \mathrm{d}\varphi$ with $\alpha \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

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Remark. Analogue of [Laptev, Weidl 1999]: $\boxed{-\Delta_A \geq \frac{\operatorname{dist}(\alpha,\mathbb{Z})^2}{r^2}}$ in $L^2(\mathbb{R}^2)$.

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However, $-\Delta_0$ is not critical in $L^2(\mathbb{H}^1)$. [Ruzhansky, Suragan 2017]

Nevertheless, no inequality $-\Delta_0 \geq \frac{c}{r^2}$ with $c > 0$ possible in $L^2(\mathbb{H}^1)$.

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q.e.d.

Mild magnetic fields

$:\Longleftrightarrow$ A is smooth with $\mathrm{d}A =: B \neq 0$ (typically, $B \xrightarrow{\infty} 0$)

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NB $r(x,y,z) = \sqrt{x^2+y^2}$, $\rho(x,y,z) = \sqrt[4]{(x^2+y^2)^2 + 16z^2}$

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Remark 2. The same result holds for $A = \alpha\,\mathrm{d}\varphi$ with $\alpha \notin \mathbb{Z}$ $(c = c(\alpha,\Omega))$.

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q.e.d.

Conclusions

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Moral: magnetic field $\Longrightarrow$ (improved) Hardy-type inequalities in $\mathbb{H}^1$

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- H_+ is subcritical ($\exists$ Hardy)
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→ pointwise variant: [Fraas, K, Pinchover 2010]

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In arte veritas, 16 Rue des Lices, Angers

CIRM conference on

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3 – 7 June 2024
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Cristina Câmara (Lisbon)
Isabelle Chalendar (Paris)
Kirill Cherednichenko (Bath)
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Jean-Claude Cuenin (Loughborough)
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Marcel Hansmann (Chemnitz)
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Cécilia Lancien (Grenoble)
Giorgio Metafune (Lecce)
Sandra Pott (Lund)
Birgit Schörkhuber (Innsbruck)
Johannes Sjöstrand (Dijon)
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Calanque (six heures du soir)
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