Hypoellipticity, pseudodifferential operators, tangent groupoids, and the Helffer-Nourrigat conjecture

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Hypoelliptic differential operators

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M — smooth manifold. Recall…

Definition (Hypoellipticity)

A differential operator P on M is **hypoelliptic** if the differential equation

$$Pu = f$$

has the smoothness of solutions property :

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Remark.
P is **maximally hypoelliptic** if moreover we have Sobolev estimates.

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Q. What governs hypoellipticity?

A. [Rockland, **Helffer-Nourrigat**]: Representation theory of osculating nilpotent Lie groups.

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A polynomial P in V_i is maximally hypoelliptic iff $\pi([P]_p)$ is left-invertible on $\mathcal{H}_\pi^\infty$ for every $\pi \in \bigsqcup_p \mathrm{HN}(\mathcal{F})_p$.

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Remarks.

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2. We can prove the conjecture using the groupoid approach to ΨD calculus (pioneered by Connes and Debord-Skandalis).

The groupoid approach to pseudodifferential operators

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Theorem

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Proof.

Given $Pu = f$, write $u = \underbrace{(I-QP)u}_{\substack{\text{smooth}\\ \text{by regularity}}} \quad + \quad \underbrace{Q(Pu)}_{\substack{\text{same sing. supp. as } f = Pu\\ \text{by pseudolocality}}}$ □

Groupoids and ΨDOs

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Connes (1980s)

A pseudodifferential operator P and its principal symbol $\sigma^m(P)$ are the restrictions at $t = 1$ and $t = 0$, respectively, of a larger deformation family $(\mathbb{P}_t)$ over $t \in \mathbb{R}$.

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Debord-Skandalis (2010s)

Can characterize these deformation families $(\mathbb{P}_t)$ using a canonical $\mathbb{R}_+^\times$-action on $\mathbb{T}M$.

A basic fact about homogeneous polynomials

Proposition

There is a 1–1 correspondence

$$\left\{ \begin{array}{c} \textit{polynomials on } \mathbb{R}^n \textit{ of} \\ \textit{order} \leqslant m \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homog. polynomials on } \mathbb{R}^{n+1} \\ \textit{of degree} = m \end{array} \right\}$$

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$$a(\xi_1, \ldots, \xi_n) = \underline{a}(\xi_1, \ldots, \xi_n, 1) \quad \longleftarrow \quad \underline{a}(\xi_1, \ldots, \xi_n, t)$$

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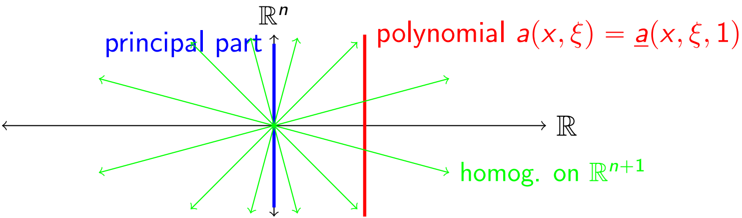
Remark.

- The restriction $\underline{a}(\xi_1, \ldots, \xi_n, 0)$ at $t = 0$ is the **principal part** of a.

A basic fact about polynomials



A basic fact about polyhomogeneous symbols

Recall: A classical ΨDO on $\mathbb{R}^n$ is an operator of the form

$$Pf(x) = \int_{\mathbb{R}^n} a(x,\xi)\hat{f}(\xi)e^{i\langle \xi,x\rangle}\,d\xi,$$

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Homog. mod Schwartz means

$$\underline{a}(x,s\xi,st) - s^m \underline{a}(x,\xi,t) \in C^\infty(\mathbb{R}^n, \mathcal{S}(\mathbb{R}^{n+1})) \qquad \forall s > 0.$$

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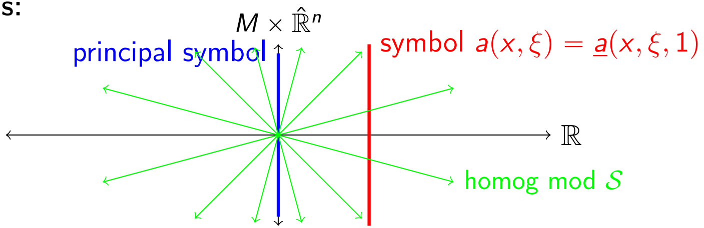

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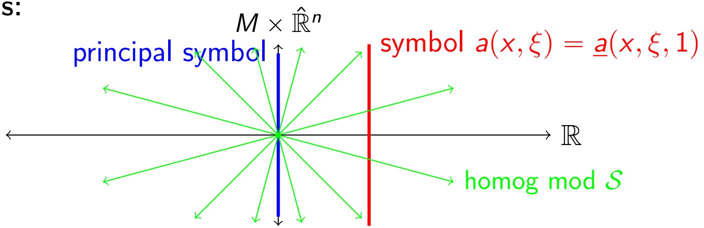

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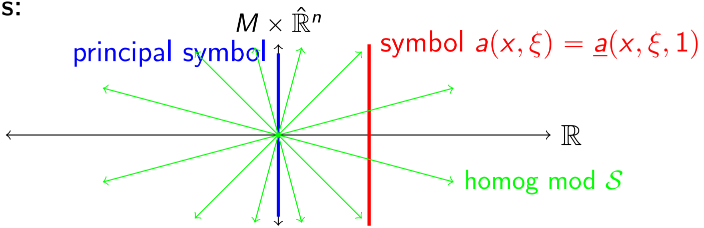

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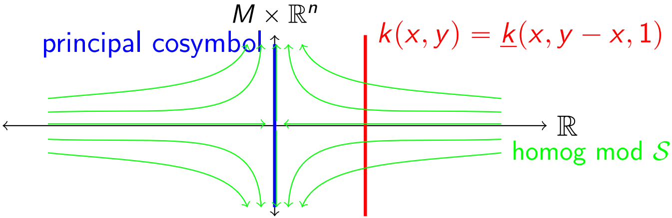

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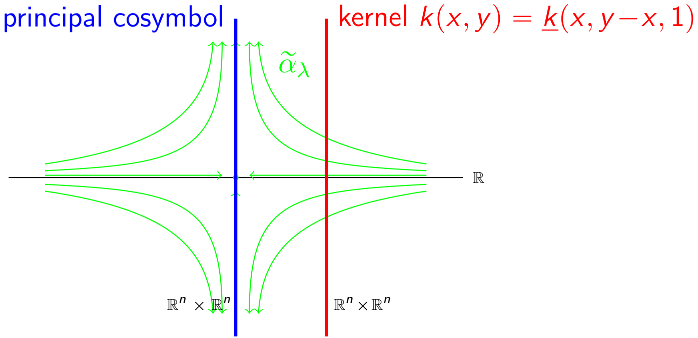

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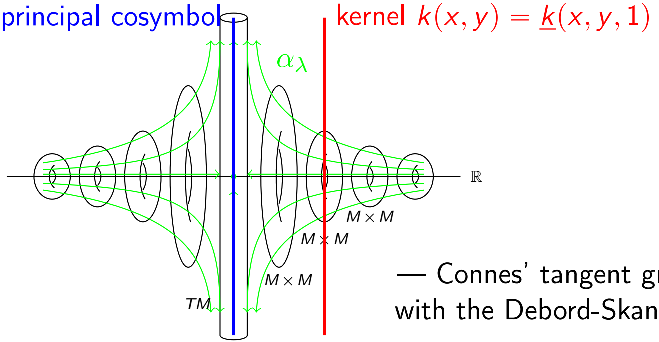

— Connes' tangent groupoid $\mathbb{T}M$ with the Debord-Skandalis action.

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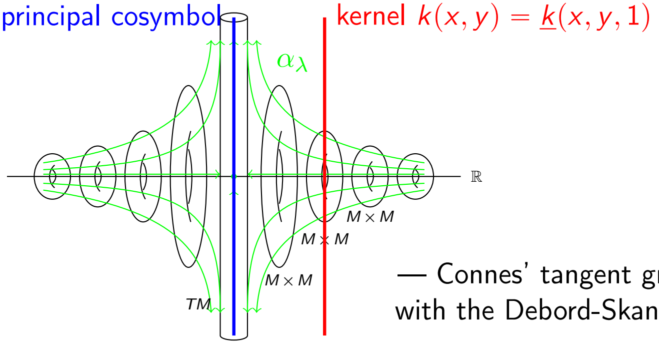

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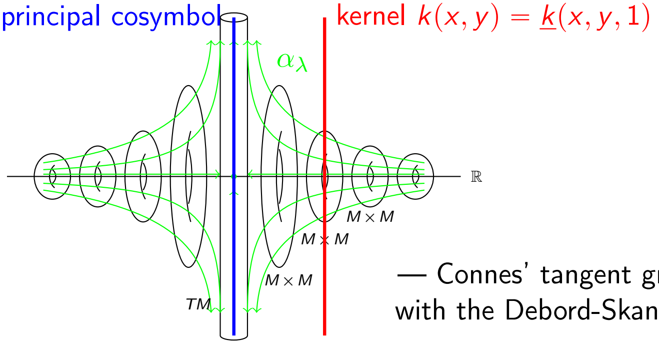

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Topology: $(x_i, y_i, t_i) \to (V_x, 0)$ iff $t_i \to 0$ and $\frac{x_i - y_i}{t_i} \to V_x$.

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Definition (Van Erp-Y., inspired by Debord-Skandalis)

A (properly supported, classical polyhomogeneous) ΨDO of order $\leqslant m$ on M is an operator with Schwartz kernel $k(x,y) = \underline{k}(x,y,1)$, where

$$\underline{k}(x,y,t) \in \mathcal{E}'_r(\mathbb{T}M)$$ [Lescure-Manchon-Vassout]

is homogeneous of degree m modulo C^∞_{p} for the Debord-Skandalis action:

$$\alpha_s : \begin{cases} (x,y,t) = (x,y,s^{-1}t), & t \neq 0, \\ (x,v,0) = (x,sv,0), & \text{t=0}. \end{cases}$$

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Theorem (Van Erp-Y.)

1. *This is equivalent to the usual definition of classical ΨDOs [Kohn-Nirenberg].*
2. *The restriction $\underline{k}|_{t=0}$ is the principal cosymbol.*

Groupoid convolution

Theorem (Lescure-Manchon-Vassout)

The product of the Lie groupoid $\mathbb{T}M$ integrates to a ***convolution product*** *of r-fibred distributions on $\mathbb{T}M$,*

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$\Rightarrow$ Groupoid convolution in $\mathcal{E}'_r(\mathbb{T}M)$ describes both composition of ΨDOs (at $t = 1$) and product of symbols (at $t = 0$). [Connes]

Elliptic ⇒ Hypoelliptic

Definition

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2. Pseudolocality
3. Parametrices of elliptic elements.

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2. Pseudolocality
3. Parametrices of elliptic elements.

Corollary

P elliptic $\Rightarrow$ P hypoelliptic.

A calculus for Helffer-Nourrigat operators

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Corollary (The Helffer-Nourrigat Conjecture)

Helffer-Nourrigat operators are (maximally) hypoelliptic.

The singular tangent groupoid

Families of vector fields

M — manifold

$V_1, \ldots, V_n$ — vector fields on M.

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NB. More general frameworks are possible, eg, by ascribing an *order* to each V_i.

The osculating groups

Not'n. $I_p := \{f \in C^\infty(M) \mid f(p) = 0\}$ — vanishing ideal.

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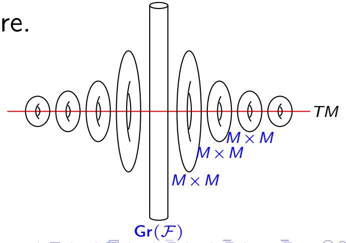
... but this doesn't explain the smooth structure.



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$$(p,t) \mapsto \begin{cases} t^k X(p) & \in TM_p, \quad t \neq 0, \\ [X]_p & \in \mathfrak{gr}_k \mathcal{F}_p, \quad t = 0. \end{cases}$$

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- This induces a smooth structure on the tangent groupoid $\mathbb{T}\mathcal{F}$ by a tubular neighbourhood construction (exponential charts).

ΨDOs & the Helffer-Nourrigat Conjecture

The $\mathbb{R}_+^\times$-action

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Definition

A distribution $\underline{k} \in \mathcal{E}'_r(\mathbb{T}\mathcal{F})$ is **essentially homogeneous of degree** m if

$$\alpha_{s*}\underline{k} - s^m\underline{k} \in C^\infty_{\mathrm{p}}(\mathbb{T}\mathcal{F}) \qquad \forall s > 0.$$

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It is easy to show that $\Psi^m(\mathcal{F})$ is a $\mathbb{Z}$-filtered algebra satisfying

1. Regularity
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The question of parametrices of "elliptic" elements is far more subtle.

[Helffer-Nourrigat]

Topology of the dual bundle $\mathfrak{t}^*\mathcal{F}$

Definition (Local generating family of vector fields)

A family of vector fields $(X_1, \ldots, X_d)$ is a **local generating family** for $\mathcal{F}$ on $U \subseteq M$ if there are $d_1, \ldots, d_N$ s.t. $(X_1, \ldots, X_{d_k})$ spans $\mathcal{F}^k$ on U.

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- [!] Not all of $\mathfrak{gr}\mathcal{F}_p^* = \mathfrak{t}^*\mathcal{F}|_{t=0}$ will be a limit of $\mathfrak{t}^*\mathcal{F}|_{t>0}$ as $t \rightarrow 0$.

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- Chart $\mathbb{R}^3 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a, b, c) \mapsto \begin{cases} ta\partial_x + (t^2xb + t^3c)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0, \ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p, & t = 0, \ x = 0. \end{cases}$$

Example: A hypoellipitic Laplacian

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- $X = \partial_x$, $Y = x\partial_y$, $Z = [X, Y] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \quad \subseteq \quad \mathcal{F}^2 = \langle X, Y \rangle \quad \subseteq \quad \mathcal{F}^3 = \langle X, Y, Z \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle & \cong \mathbb{R}^2, \quad \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle & \cong \mathfrak{h}^3, \quad \text{if } x = 0, \end{cases}$
- Chart $\mathbb{R}^3 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a, b, c) \mapsto \begin{cases} ta\partial_x + (t^2xb + t^3c)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0, \ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p, & t = 0, \ x = 0. \end{cases}$$

- Image of $\mathfrak{t}^*\mathcal{F}|_{(x,y,t)} \hookrightarrow \mathbb{R}^{3*}$ is:

$$\mathrm{span}\{(1,0,0), (0, t^2x, t^3)\} = (0, t, x)^\perp.$$

Example: Another hypoellipitic Laplacian

$P = \partial_x^2 + \frac{x^2}{2}\partial_y$ on $M = \mathbb{R}^2$.

- $X = \partial_x$, $Y = \frac{x^2}{2}\partial_y$, $Z = [X, Y] = x\partial_y$, $W = [X, Z] = \partial_y$.
- $\mathcal{F}^1 = \langle X \rangle \subseteq \mathcal{F}^2 = \langle X, Y \rangle \subseteq \mathcal{F}^3 = \langle X, Y, Z \rangle \subseteq \mathcal{F}^4 = \langle X, Y, Z, W \rangle = \Gamma^\infty(M)$
- $\mathfrak{gr}\mathcal{F}_p = \begin{cases} \langle [X]_p \rangle \oplus \langle [Y]_p \rangle & \cong \mathbb{R}^2, \quad \text{if } x \neq 0, \\ \langle [X]_p \rangle \oplus \langle [Y]_p \rangle \oplus \langle [Z]_p \rangle \oplus \langle [W]_p \rangle & \cong \mathfrak{n}^4, \quad \text{if } x = 0, \end{cases}$
- Chart $\mathbb{R}^4 \twoheadrightarrow \mathfrak{t}\mathcal{F}_{(x,y,t)}$ is

$$(a, b, c, d) \mapsto \begin{cases} ta\partial_x + (\frac{1}{2}t^2x^2b + t^3xc + t^4d)\partial_y, & t \neq 0, \\ a[X]_p + b[Y]_p, & t = 0, \ x \neq 0, \\ a[X]_p + b[Y]_p + c[Z]_p + d[W]_p, & t = 0, \ x = 0. \end{cases}$$

- Image of dual $\mathfrak{t}^*\mathcal{F}|_{(x,y,t)}$ is:

$$\mathrm{span}\{(1, 0, 0), (0, \tfrac{1}{2}t^2x^2, t^3x, t^4)\}.$$

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Definition (Helffer-Nourrigat cone)

$$\mathrm{HN}(\mathcal{F}) := \overline{\mathfrak{t}^*\mathcal{F}|_{t>0}} \cap \mathfrak{t}^*\mathcal{F}|_{t=0}$$

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Lemma (Helffer-Nourrigat)

The Helffer-Nourrigat cone $\mathrm{HN}(\mathcal{F})_p \subseteq \mathfrak{gr}\mathcal{F}_p^*$ is invariant under the coadjoint action of $\mathrm{Gr}\mathcal{F}_p$.

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Kirillov $\Rightarrow$ $\mathrm{HN}(\mathcal{F})_p$ corresponds to a family of unitary irreps of $\mathrm{Gr}\mathcal{F}_p$.

Parametrices

Theorem (Androulidakis-Mohsen-Y.)

- $P \in DO(M)$ — *polynomial in vector fields of* $\mathcal{F}$ *with total order* $\leqslant m$,
- $k \in \mathcal{E}'_r(M \times M)$ *its Schwartz kernel,*
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The statement suggests that the appropriate osculating groupoid is **not** in fact $\mathrm{Gr}\mathcal{F}$, but a new groupoid whose reps are only $\mathrm{HN}(\mathcal{F})$:

$$\mathrm{Moh}(\mathfrak{a}\mathcal{F}) = \text{Mohsen's blow-up groupoid.}$$

Thank you