A panorama of Singular Sub-Laplacians and their spectra

Marcello Seri
University of Groningen

(joint works w. U. Boscain, D. Prandi, L. Rizzi)

30 August 2023
U. Angers

MOTIVATION

GEOMETRY OF RIEMANNIAN MANIFOLD $\leftrightsquigarrow$ SPECTRAL PROPERTIES OF LAPLACE-BELTRAMI OPERATOR

Such as

BOUNDS ON LOW LYING SPECTRUM

$$\lambda_1(M) \overset{*}{\geq} \frac{h^2(M)}{4}$$

$$h(M) = \inf_{A \subset\subset M} \frac{\mathrm{Vol}_{d-1}(\partial A)}{\mathrm{Vol}_d(A)} \quad \text{Cheeger constant}$$

WEYL'S LAW

$$N(\lambda) \overset{*}{=} C_d \, \mathrm{Vol}_d(M) \, \lambda^{d/2} - C_{d-1} \, \mathrm{Vol}_{d-1}(\partial M) \, \lambda^{\frac{d-1}{2}} + o\left(\lambda^{\frac{d-1}{2}}\right)$$

QUANTUM - CLASSICAL CORRESPONDENCE ...

MOTIVATION

GEOMETRY OF RIEMANNIAN MANIFOLD $\leftrightsquigarrow$ SPECTRAL PROPERTIES OF LAPLACE-BELTRAMI OPERATOR

Such as

BOUNDS ON LOW LYING SPECTRUM, WEYL'S LAW, QUANTUM - CLASSICAL CORRESPONDENCE

HOW MUCH OF THE CORRESPONDENCE IS RETAINED, AND IN WHICH FORM, IF WE ALLOW IMPORTANT OBJECTS TO VANISH OR BLOW UP?

PROPERTY ZERO: SELF-ADJOINTNESS (A BRIEF DETOUR)

Tension between

- BOUNDARY $\partial M \neq \emptyset \Rightarrow$ BOUNDARY CONDITIONS
- SINGULARITIES, NON COMPACTNESS $\Rightarrow$ DECAY OF EIGENFUNCTIONS

No brainer for "nice operators"

- (M,g) complete Riemannian manifold $\Rightarrow$ $\Delta_g = \mathrm{div}_{\mu_g} \nabla_g$ essentially S.A.
- M closed manifold, μ smooth density, P symmetric op on $L^2(M,\mu)$.

  P elliptic $\Rightarrow$ P essentially S.A

PROPERTY ZERO: SELF-ADJOINTNESS

Tension between

- BOUNDARY $\partial M \neq \emptyset \Rightarrow$ BOUNDARY CONDITIONS
- SINGULARITIES, NON COMPACTNESS $\Rightarrow$ DECAY

classical confinement of the geodesic flow

quantum confinement

No brainer for "nice operators"

- (M, g) $\boxed{\text{complete}}$ Riemannian manifold $\Rightarrow \Delta_g = \mathrm{div}_{\mu_g} \nabla_g$ $\boxed{\text{essentially S.A.}}$
- M closed manifold, μ smooth density, P symmetric op on $L^2(M, \mu)$.

  P elliptic $\Rightarrow P$ essentially S.A

SELF-ADJOINTNESS & QUANTUM-CLASSICAL CORR.

closed smooth manifold
w. density μ

CONJECTURE (Y. COLIN DE VERDIÈRE & C. LE BIHAN, 2020)

P formally self-adjoint differential operator on $C^\infty(M)$.

Hamiltonian flow of symbol p of P complete (classical confinement) $\iff$ P essentially self-adjoint (quantum confinement)

SELF-ADJOINTNESS & QUANTUM-CLASSICAL CORR.

closed smooth manifold w. density μ

CONJECTURE (Y. COLIN DE VERDIÈRE & C. LE BIHAN, 2020)

P formally self-adjoint differential operator on $C^\infty(M)$.

Hamiltonian flow of symbol p of P complete (classical confinement) $\iff$ P essentially self-adjoint (quantum confinement)

VERIFIED FOR

- P elliptic on complete Riemannian manifold M, not necessarily compact
- P of deg 1 (e.g. $P = i(X + \frac{1}{2}\mathrm{div}_\mu(X))$, $X \in \Gamma(TM)$)
- P Sturm-Liouville on $\mathbb{S}^1$ [Colin de Verdière, Bihan 2020, Taire 2020]

$\exists$ examples in which NOT COMPLETE $\Rightarrow$ NOT E.S.A.

SELF-ADJOINTNESS & QUANTUM-CLASSICAL CORR.

closed smooth manifold w. density M

CONJECTURE (Y. COLIN DE VERDIÈRE & C. LE BIHAN, 2020)

P formally self-adjoint differential operator on $C^\infty(M)$.

Hamiltonian flow of symbol p of P complete (classical confinement) $\iff$ P essentially self-adjoint (quantum confinement)

WHAT ABOUT

NOT GEODESICALLY COMPLETE BUT ESSENTIALLY SELF-ADJOINT?

PROPERTY ZERO: SELF-ADJOINTNESS

Tension between

- BOUNDARY $\partial M \neq \emptyset$ $\Rightarrow$ BOUNDARY CONDITIONS
- SINGULARITIES, NON COMPACTNESS $\Rightarrow$ DECAY

results in

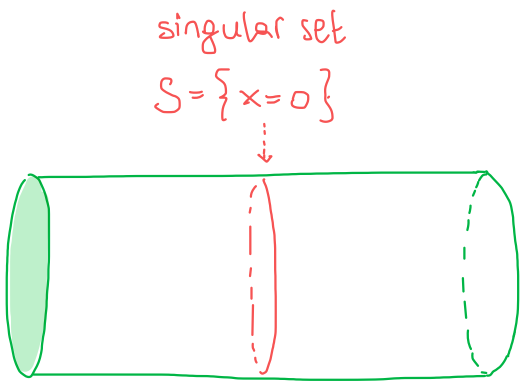
$-\Delta + V(x)$

- on $C_0^\infty(0,1)$ is
  - ESA if $V(x) \geq \frac{3}{4} \frac{1}{\mathrm{dist}(x,\{0,1\})^2}$
  - not ESA if $V(x) < \frac{3}{4}(1-\varepsilon) \frac{1}{\mathrm{dist}(x,\{0,1\})^2}$
- on $\Omega \subset \mathbb{R}^d$ is "as above" w.r.t. $\mathrm{dist}(x, \partial\Omega)$

[Nenciu, Nenciu 2009]

MOTIVATING EXAMPLE (BAOUENDI – GRUSHIN CYLINDER)

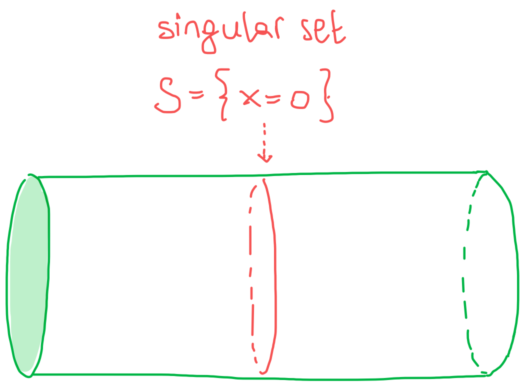
$$M = \mathbb{R}_x \times \mathbb{S}^1_\theta, \qquad g = dx^2 + \frac{d\theta^2}{x^2}, \qquad \mu_g = \frac{dx\,d\theta}{|x|}$$



$$\Rightarrow \quad \Delta = \mathrm{div}_\mu \circ \mathrm{grad}_g = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial \theta^2} - \frac{1}{x}\frac{\partial}{\partial x} \quad \text{on} \quad L^2(M, \mu_g)$$

Motivating Example (Baouendi – Grushin cylinder)

$M = \mathbb{R}_x \times \mathbb{S}^1_\theta$, $\quad g = dx^2 + \dfrac{d\theta^2}{x^2}$, $\quad \mu_g = \dfrac{dx\, d\theta}{|x|}$



BOSCAIN, LAURENT (2011)

Δ essentially self-adjoint on each connected component of $M \setminus S$, $\sigma_c(H) = [0, \infty)$ with embedded eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

$$\Rightarrow \quad \Delta = \mathrm{div}_\mu \circ \mathrm{grad}_g = \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial \theta^2} - \frac{1}{x}\frac{\partial}{\partial x} \quad \text{on} \quad L^2(M, \mu_g)$$

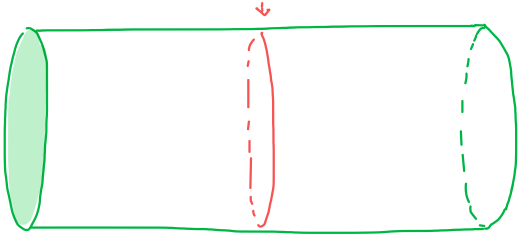
Motivating Example (Baouendi – Grushin cylinder)

$$M = \mathbb{R}_x \times \mathbb{S}^1_\theta, \qquad g = dx^2 + \frac{d\theta^2}{x^2}, \qquad \mu_g = \frac{dx\,d\theta}{|x|}$$

Quantum confinement, what about classical confinement? (1)

singular set

$S = \{x = 0\}$



Boscain, Laurent (2011)

Δ essentially self-adjoint on each connected component of $M \setminus S$, $\sigma_c(H) = [0, \infty)$ with embedded eigenvalues

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

We have eigenvalues to count, what about Weyl's law? (2)

$$\Rightarrow \quad \Delta = \mathrm{div}_\mu \circ \mathrm{grad}_g = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial \theta^2} - \frac{1}{x}\frac{\partial}{\partial x} \quad \text{on } L^2(M, \mu_g)$$

Motivating Example (Baouendi – Grushin cylinder)

$$M = \mathbb{R}_x \times \mathbb{S}^1_\theta, \quad g = dx^2 + \frac{d\theta^2}{x^2}, \quad \mu_g = \frac{dx\,d\theta}{|x|}$$

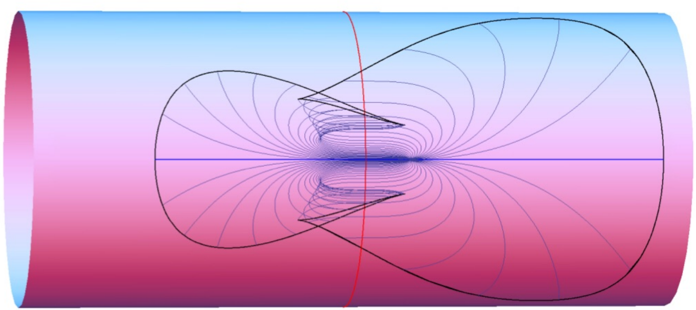
(1)

- symbol of Δ

$$p(x,\xi) = \xi_x^2 + x^2 \xi_\theta^2$$

- flow of p is not complete on connected components of $M \setminus S$

$\Rightarrow$ QUANTUM CONFINEMENT WITHOUT CLASSICAL CONFINEM.



Geodesic front of length L from (1,0)

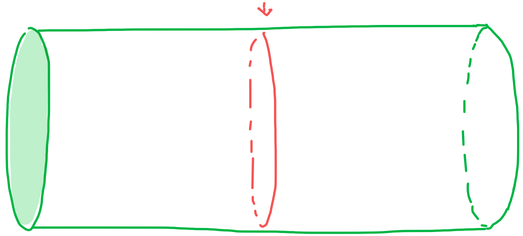
Motivating Example (Baouendi – Grushin cylinder)

$$M = \mathbb{R}_x \times \mathbb{S}^1_\theta, \qquad g = dx^2 + \frac{d\theta^2}{x^2}, \qquad \mu_g = \frac{dx\, d\theta}{|x|}$$

2

singular set

$S = \{x = 0\}$



Boscain, Prandi, S. (2015)

Δ essentially self-adjoint on each connected component of $M \setminus S$, $\sigma_c(H) = [0, \infty)$ with embedded eigenvalues

$$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \quad \text{satisfying}$$

$$N(\lambda) = \frac{\lambda}{2} \log \lambda + \left(\gamma - \log 2 - \frac{1}{2}\right)\lambda + O(\sqrt{\lambda})^*$$

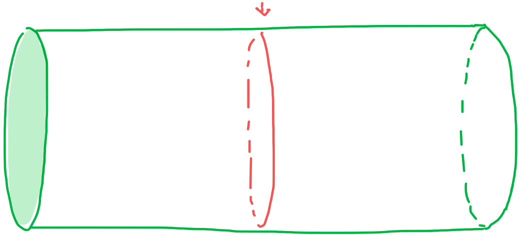
* $O\left(\lambda^{\frac{131}{416}+\varepsilon}\right)$ since remainder is from a Dirichlet counting problem...

Motivating Example (Baouendi – Grushin cylinder)

$M = \mathbb{R}_x \times \mathbb{S}^1_\theta$, $\quad g = dx^2 + \dfrac{d\theta^2}{x^2}$, $\quad \mu_g = \dfrac{dx\,d\theta}{|x|}$

singular set

$S = \{x = 0\}$



$\mathrm{Vol}(M) = +\infty$

$K = -\dfrac{2}{x^2} \xrightarrow[x \to 0]{} -\infty$

Boscain, Prandi, S. (2015)

Δ essentially self-adjoint on each connected component of $M \setminus S$, $\sigma_c(H) = [0, \infty)$ with embedded eigenvalues

$\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ satisfying

$$N(\lambda) = \frac{\lambda}{2} \log \lambda + \left(\gamma - \log 2 - \frac{1}{2}\right)\lambda + O(\sqrt{\lambda})^{*}$$

so what is Weyl's law picking up now?

A SINGULAR SUB-LAPLACIAN

Almost-Riemannian: Riemannian outside singular set; D changes dimension at $\{x=0\}$

BAOUENDI-GRUSHIN EXAMPLE is sub-Riemannian

$M = \mathbb{R} \times S^1$, $\mu = \dfrac{dx\,d\theta}{|x|}$, $g = dx^2 + \dfrac{dy^2}{x^2}$ and $D = \mathrm{span}\left\{ \dfrac{\partial}{\partial x}, x\dfrac{\partial}{\partial \theta} \right\}$

$$\Rightarrow \Delta_\mu = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial \theta^2} - \frac{1}{x}\frac{\partial}{\partial x}$$

A singular Sub-Laplacian

Baouendi-Grushin example is sub-Riemannian

$M = \mathbb{R} \times \mathbb{S}^1$, $\mu = \frac{dx\,d\theta}{|x|}$, $g = dx^2 + \frac{dy^2}{x^2}$ and $D = \mathrm{span}\left\{ \frac{\partial}{\partial x}, x \frac{\partial}{\partial \theta} \right\}$

$$\Rightarrow \Delta_\mu = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial \theta^2} - \frac{1}{x}\frac{\partial}{\partial x}$$

"Simple" but interesting example, e.g.

[Boscain, Prandi 2013] Heat evolution & stochastic completeness

[Boscain, Prandi, S. 2015] Weyl's law, Aharonov-Bohm effect & spectral degeneration

[Gallone, Michelangeli, Pozzoli 2019, 2021] Fully non-compact case & scattering

[Bauer, Furutani, Iwasaki 2013]
[Caserini, Ciatti, Martini 2020] Spectral analysis of d-dimensional Grushin spheres

OBJECTIVES

1) How general is the quantum confinement with classical tunnelling for singular sub-Riemannian manifolds?

2) A brief panoramic of sub-Riemannian Weyl's laws in the regular & singular setting

SUB-RIEMANNIAN MANIFOLDS

SUB-RIEMANNIAN MANIFOLD (M, D, g)

Smooth manifold

$D \subset TM$ tangent distribution

section of positive definite quadratic forms on D

$M = \mathbb{R}^3$

$D = \mathrm{span}\{\partial_x - \frac{y}{2}\partial_z, \partial_y + \frac{x}{2}\partial_z\}$

$g = dx^2 + dy^2$

$\oplus$ $\mathrm{Lie}(D) = TM$ (HÖRMANDER CONDITION / BRACKET GENERATING)

SUB-RIEMANNIAN MANIFOLDS

SUB-RIEMANNIAN MANIFOLD (M, D, g)

Smooth manifold

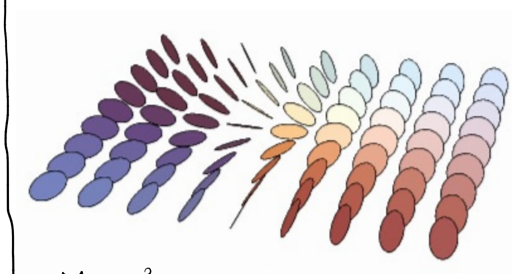
$D \subset TM$ tangent distribution

section of positive definite quadratic forms on D

⊕ Lie $(D) = TM$ (HÖRMANDER CONDITION BRACKET GENERATING)

Chow-Rashewski ~39 — Any two points in M are joined by admissible curves $\gamma : \mathbb{R} \to M$ s.t. $\dot{\gamma}(t) \in D_{\gamma(t)}$

d_{SR} sub Riemannian distance & geodesic flow



$M = \mathbb{R}^3$

$D = \mathrm{span}\{\partial_x - \frac{y}{2}\partial_z, \partial_y + \frac{x}{2}\partial_z\}$

$g = dx^2 + dy^2$

SUB-LAPLACIANS

SUB-RIEMANNIAN MANIFOLD (M, D, g) $\oplus$ BRACKET GENERATING D

(!)

$\oplus$ $\mu \in \Omega^n(M)$ SMOOTH VOLUME ON M

$\Rightarrow$
- HORIZONTAL GRADIENT $\langle \nabla_H f, X \rangle_g = X(f)$ for $X \in \Gamma(D)$, $f \in C^\infty(M)$
- DIVERGENCE $\mathcal{L}_X \mu = \mathrm{div}_\mu(X)\,\mu$ for $X \in \Gamma(TM)$

Sub-Laplacians

Sub-Riemannian Manifold $(M, \mathcal{D}, g)$ $\oplus$ **bracket generating** $\mathcal{D}$

(!)

$\oplus$ $\mu \in \Omega^n(M)$ **smooth volume on** M

$\Rightarrow$ • Horizontal Gradient $\quad \langle \nabla_H f, X \rangle_g = X(f) \quad$ for $X \in \Gamma(\mathcal{D})$, $f \in C^\infty(M)$

• Divergence $\quad \mathcal{L}_X \mu = \operatorname{div}_\mu(X)\,\mu \quad$ for $X \in \Gamma(TM)$

$\Rightarrow$ sub-Laplacian (associated to μ)

$$\Delta_\mu f = \operatorname{div}_\mu(\nabla_H f)$$

$$= \sum \left(X_i^2(f) + \operatorname{div}_\mu(X_i)\, X_i(f) \right)$$

on a frame $\{X_i\}$ of $\mathcal{D}$

$M = \mathbb{R}^3$

$\mathcal{D} = \operatorname{span}\{ \underbrace{\partial_x - \tfrac{y}{2}\partial_z}_{X_1}, \underbrace{\partial_y + \tfrac{x}{2}\partial_z}_{X_2} \}$

$g = dx^2 + dy^2 \qquad \mu = dx\,dy\,dz$

$\Delta_\mu = X_1^2 + X_2^2$

SUB-LAPLACIANS

SUB-RIEMANNIAN MANIFOLD (M, D, g) ⊕ BRACKET GENERATING D

(!)

⊕ $\mu \in \Omega^n(M)$ SMOOTH VOLUME ON M

$\Rightarrow$ • HORIZONTAL GRADIENT $\langle \nabla_H f, X \rangle_g = X(f)$ for $X \in \Gamma(D)$, $f \in C^\infty(M)$

• DIVERGENCE $\mathcal{L}_X \mu = \mathrm{div}_\mu(X)\mu$ for $X \in \Gamma(TM)$

$\Rightarrow$ sub-Laplacian (associated to μ)

$$\Delta_\mu f = \mathrm{div}_\mu(\nabla_H f)$$

$$= \sum \Big(\underbrace{X_i^2(f)}_{\mu \text{ independent}} + \mathrm{div}_\mu(X_i) X_i(f) \Big)$$

on a frame $\{X_i\}$ of D

• Δ_μ SYMMETRIC, NEGATIVE operator on $C_0^\infty(M)$

• (M, d_{SR}) complete

$\Rightarrow$ Δ_μ essentially self-adjoint on $L^2(M, \mu)$ with domain $C_0^\infty(M)$

⊕ $e^{-t\Delta_\mu}$ has positive C^∞ kernel & defines contractive semigroup

Quantum Confinement with Classical Tunnelling for sub-Laplacians

THEOREM [Prandi, Rizzi, S. 2019; Franceschi, Prandi, Rizzi 2019]

$(M, \mathcal{D}, g)$ complete sR manifold, $S \subset M$ smooth embedded compact hypersurface with no characteristic points, μ smooth measure on $M \setminus S$.

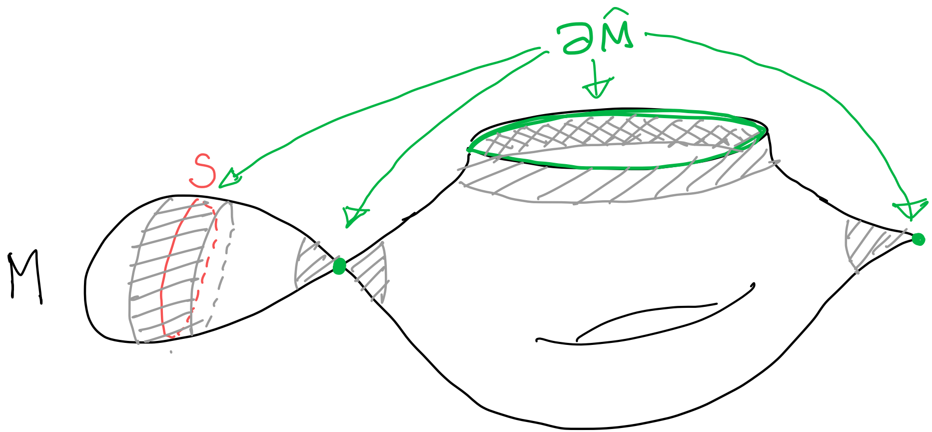
Denote $\delta := d(S, \cdot)$, if $\delta \in C^2$ and for some $\varepsilon > 0$ $\exists \kappa \geq 0$ such that in the region $0 < \delta \leq \varepsilon$ it holds

$$\left(\frac{\Delta_\mu \delta}{2}\right)^2 + \left(\frac{\Delta_\mu \delta}{2}\right)' \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta}$$

$\Rightarrow$ Δ_μ with domain $C_0^\infty(M \setminus S)$ is essentially self-adjoint on $L^2(M \setminus S, \mu)$ or any of the connected components of $M \setminus S$.

$\oplus$ If M relatively compact $\Rightarrow$ Δ_μ has compact resolvent

A SKETCH OF THE PROOF



In the spirit of

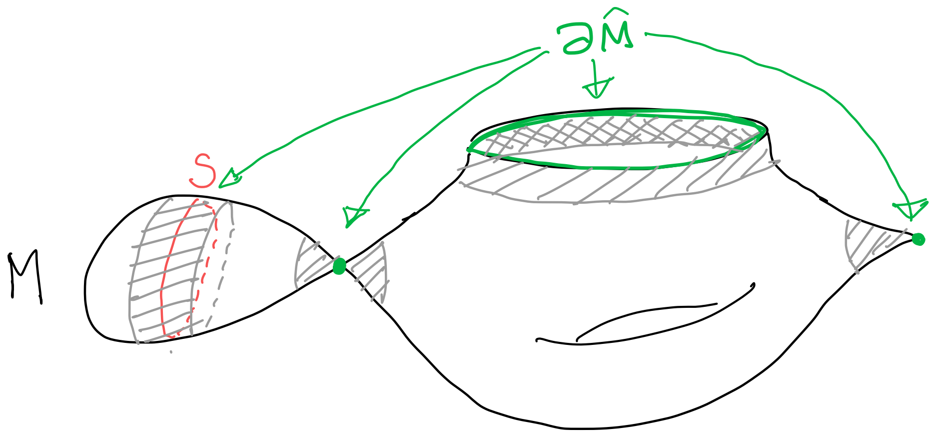
[Colin de Verdière, Truc 2010]
[Nenciu, Nenciu 2009]

$$\left(H = -\Delta_\mu + V \text{ on } C_c^\infty(M), \quad V_{eff} + V \geq \frac{3}{4\delta^2} - \frac{\overset{\geq 0}{K}}{\delta} - \nu^2 \text{ on } \delta \leq \varepsilon \right)$$

1. Localize around the boundaries

2. Agmon-type estimates $\Rightarrow$ $\exists E < 0$ s.t. $H^* \psi = E\psi$ iff $\psi \equiv 0$

A SKETCH OF THE PROOF



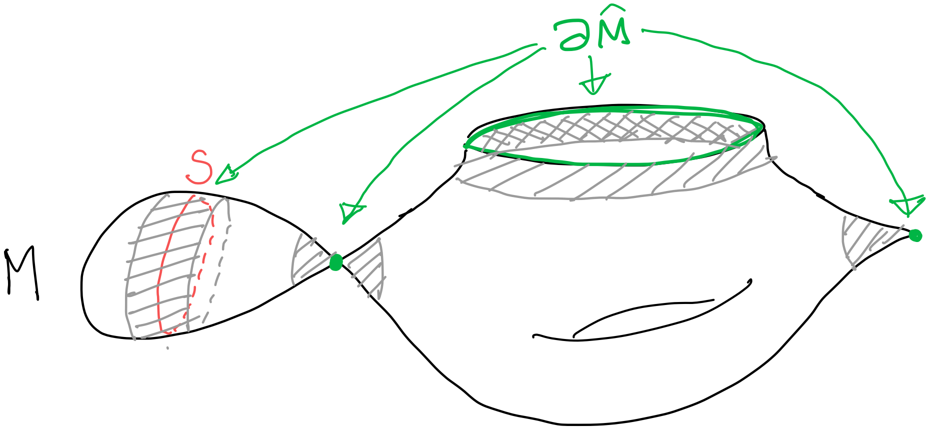
In the spirit of

[Colin de Verdière, Truc 2010]
[Nenciu, Nenciu 2009]

$\left(H = -\Delta_\mu + V \text{ on } C_c^\infty(M), \quad V_{eff} + V \geq \frac{3}{4\delta^2} - \frac{\kappa}{\delta} - \nu^2 \text{ on } \delta \leq \varepsilon \right)$

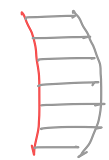
1. Localize around the boundaries
2. Agmon-type estimates $\Rightarrow$ $\nexists E<0$ s.t. $H^*\psi = E\psi$ iff $\psi \equiv 0$
   via sR Hardy inequality
   and sR Rellich-Kondrakov
   [Franceschi, Prandi, Rizzi 2019]

A SKETCH OF THE PROOF



- $(M, d_{SR}) \longrightarrow (\hat{M}, \hat{d}_{SR})$ metric completion
- δ distance to $\partial\hat{M} = \hat{M} \setminus M$
- $\delta \in C^2(M_\varepsilon)$

$\Rightarrow M_\varepsilon$ has tubular nbd



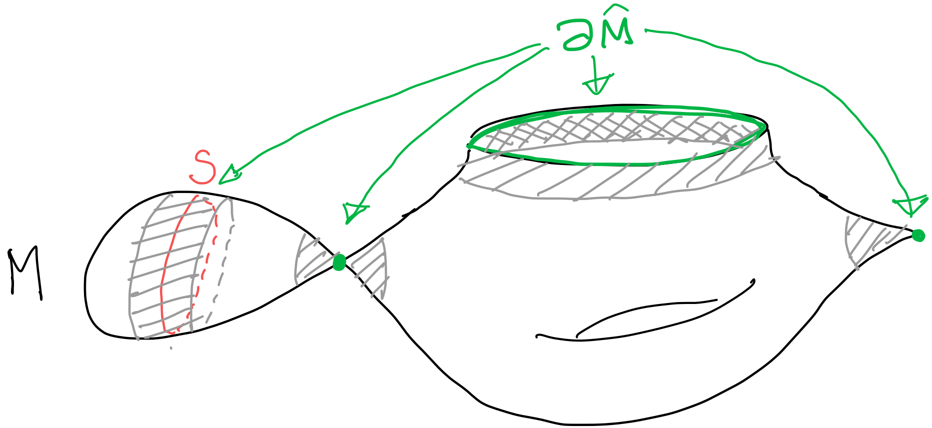
$M_\varepsilon \simeq (0,\varepsilon]_t \times X_\varepsilon$

$X_\varepsilon = \{ t = \varepsilon \}$

where

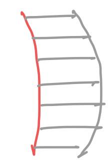
$d\mu = e^{2(t,x)}\, dt\, d\omega(x)$

A SKETCH OF THE PROOF



- $(M, d_{SR}) \longrightarrow (\hat{M}, \hat{d}_{SR})$ metric completion
- δ distance to $\partial\hat{M} = \hat{M} \setminus M$
- $\delta \in C^2(M_\varepsilon)$

$\Rightarrow M_\varepsilon$ has tubular nbd



$M_\varepsilon \simeq (0,\varepsilon]_t \times X_\varepsilon$

$X_\varepsilon = \{ t = \varepsilon \}$

where

$$d\mu = e^{2\vartheta(t,x)} dt\, d\omega(x)$$

$$V_{eff} = (\partial_t \vartheta)^2 + \partial_t^2 \vartheta \quad \Leftarrow$$

does not depend on the choice of $d\omega$

Intrinsic quantity!

COROLLARY I

If $\mu = t^a \, dt \, d\omega(x)$ on $M_\varepsilon \simeq (0,\varepsilon]_t \times X_\varepsilon$

$\Rightarrow \Delta_\mu$ e.s.a. for $\underbrace{a \leq -1}$ and $a \geq 3$

$a = -1$ includes

Baouendi - Grushin

cylinder & sphere

A remark on V_{eff}

$$V_{eff} = \frac{1}{4}\left((\Delta_\mu \delta)^2 + \|\mathrm{Hess}(\delta)\|_{HS}^2 + \mathrm{Ric}_\mu(\nabla\delta, \nabla\delta) \right)$$

a function of mean curvature $\Delta_\mu \delta$ of levelsets of δ

Corollary II

Let $\mu = \mathrm{vol}_g$.

If $\exists\, c_1 \geq c_2 \geq 0$ and $r \geq 2$ s.t

$$-\frac{c_1}{\delta^r} \leq \mathrm{Sec}(\sigma) \leq -\frac{c_2}{\delta^r}$$ for all σ planes containing $\nabla\delta$

and $\mathrm{Hess}(\delta)\big|_{X_\varepsilon} < h^*_\varepsilon(c_2, r)$

$\Longrightarrow$ Δ with domain $C_c^\infty(M)$ is ess. s.a. in $L^2(M)$.

Both sectional curvature and principal curvature of levelsets of δ contribute to self-adjointness!

Corollary III (Riemannian Kalf-Walter-Schmincke-Simon theorem)

(N, g) n-dim complete Riemannian manifold

$Z \subset N$ embedded compact C^2 submanifold, $\delta(\cdot) := d_g(\cdot, Z)$

If $V \in L^2_{loc}(N \setminus Z)$ such that

$$V(q) \geq -\frac{(n-k)(n-k-4)}{4\,\delta^2(q)} - \frac{k}{\delta(q)} - \nu(q) \quad \text{on } \{\delta \leq \varepsilon\}$$

For points, $k=0$, weaker than previous conditions

submanifolds of co-dimension $n-k \geq 4$ are "invisible" to solutions free laplacians $(V \equiv 0)$

and $V(q) \geq -\nu(q)$ on $\{\delta > \varepsilon\}$

$\Rightarrow$ $H = -\Delta + V$ w. domain $C_c^\infty(M)$ is ess. s.a. on $L^2(M)$ or any of its connected components

RECENT UPDATES

- $-\Delta_\omega$ ess.s.a. on $M\setminus\{q\}$ iff Flag $Q(q) \geq 4$
  [Colin de Verdière, Hillairet, Trélat 2023]

- Sharp ess.s.a. on regular α-Grushin manifolds via adapted 0-calculus
  [Beschastnyi, Quan 2023]

- Ess.s.a. via Pseudodifferential calculus on Lie groupoid for generic 2 ARS, allows unique treatment of some tangency points [Beschastnyi 2021]

- Sharp ess.s.a. on conic and anticonic manifolds
  [Gallone, Michelangeli, Pozzoli 2019]

- Confinement for curvature Laplacian on 2-ARS [Beschastnyi, Boscain, Pozzoli 2021]

- Point interactions for 3D sub-Laplacians [Adami, Boscain, Franceschi, Prandi 2019]

A Landscape of Open Problems: Two Explicit Open Questions

(1) $M=\mathbb{R}^2$, $\mathcal{D}=\operatorname{Span}\{\partial_x, x(x^{2\ell}+y^2)\frac{\partial}{\partial y}\}$

$\Rightarrow S=\{x=0\}$, $\delta(x,y)=|x|$ and $\mu_g=\frac{1}{|x|(x^{2\ell}+y^2)}\,dx\,dy$

- $\ell=1$: Δ_μ is ESA
- $\ell\geq 2$: Δ_μ is ESA (conjecture)

(2) $M=\mathbb{R}^2$, $\mathcal{D}=\operatorname{Span}\{\partial_x, \phi(x,y)\,\partial_y\}$ where $\phi(x,y)=y-x^2$

$\Delta_\mu-\frac{k}{\phi^2}$ is ess. s.a. [Beschastnyi 2021]

$\Rightarrow S=\{\phi(x,y)=0\}$, 0 is a tangency point ($\mathcal{D}_0 \,/\!/\, T_0S$)

$\delta(x,y)\approx|\sqrt{y}-|x||$ and $\mu_g=\frac{1}{|\phi(x,y)|}\,dx\,dy$

$\Delta_\mu=\partial_x^2+\phi^2\partial_y^2+\phi\,\partial_y+\frac{\partial_x\phi}{\phi}\partial_x$ symmetric, very explicit

but its self-adjointness is unknown

SPECTRUM OF SUB-LAPLACIANS: THE REGULAR CASE

3D CONTACT E.G. $(\mathbb{R}^3, dx^2+dy^2)$, $D = \mathrm{span}\{\partial_x - \frac{y}{2}\partial_z, \partial_y + \frac{x}{2}\partial_z\}$

$\Rightarrow \underline{\Delta}$ is related to magnetic laplacian for constant field $(0,0,1)$

MANY ADVANCES IN PAST FEW YEARS, E.G., (SORRY FOR THE MANY PAPERS I COULD NOT FIT)

- 3D contact:

  [Colin de Verdière, Hillairet, Trélat 2017-2023, Colin de Verdière 2021] Weyl's law, Semiclassics, Quantum ergodicity

  [Flynn 2020] X-Ray transform, Santaló formula

  [Fermanian-Kammerer, Fischer 2019-2022] Semiclassical analysis

- 4D quasicontact: [Savale 2019] Microlocalization, Weyl's law w. contribution by abnormals
- Smooth compact sub laplacians, equiregular D:

  [Hassannezhad, Kokarev 2014] upper bounds on spectrum & Weyl's law

  [Chen, Chen 2019] lower bounds on spectrum

  [Colin de Verdière, Hillairet, Trélat 2020-2023] small time asymptotics of heat kernel & Weyl's law

  [Savale, Marinescu 2018] Heat trace & Weyl's law for sub-laplacians on unit circle bundles

  [Prandi, Rizzi, S. 2018] Santaló formula, functional & isoperimetric-type inequalities

  [Letrouit 2020-2021] Quantum limits

SPECTRUM OF SUB-LAPLACIANS: THE REGULAR CASE

FUNCTIONAL INEQUALITIES [Prandi, Rizzi, Seri 2018]

HARDY $\displaystyle\int_M |\nabla_H f|^p \omega \geq C_{p,\mathrm{rank} D} \int_M \frac{f^p}{r^p}\,\omega$, r harmonic mean distance from boundary

see also [Prandi, Franceschi 2020] for sharp versions

BOTTOM OF THE SPECTRUM (DIRICHLET) $\displaystyle\lambda_1(M) \geq \frac{(\mathrm{rank}\, D)\, \pi^2}{L^2}$, L^2 longest <u>reduced</u> geodesic in M

M compact n-dim sub-Riemannian mfld w. boundary, ω smooth volume form

SPECTRUM OF SUB-LAPLACIANS: THE REGULAR CASE

WEYL'S LAW

$$N(\lambda) = \begin{cases} \dfrac{\mathcal{P}_{opp}(M)}{32}\lambda^{2} + o(\lambda^2) \\[2ex] \dfrac{\int \hat{e}^{q}(1,0,0)\,d\mu(q)}{\Gamma(Q/2+1)}\lambda^{Q/2} + o(\lambda^{Q/2}) \end{cases}$$

(compact 3D contact)

[Colin de Verdière, Hillairet, Trélat 2018]

Q, Hausdorff dimension, replaces d
(Flag of the distribution)

Depends on q
but not on μ!

$\Rightarrow$ new canonical measure:
Local Weyl Measure

The complete expansion with a geometric interpretation of all terms is in

[Colin de Verdière, Hillairet, Trélat 2023]

SPECTRUM OF SINGULAR SUB-LAPLACIANS

WEYL'S LAW (I)

- [Boscain, Prandi, S 2015] Grushin sphere & cylinder, first two terms of Weyl's law but no geometric description

- [Chitour, Prandi, Rizzi 2019] Under rather general conditions on the growth and oscillation of volume(nbd of S at distance $> \lambda^{-1/2}$ from S) $= V(\lambda)$:

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{n/2} V(\lambda)} = \frac{\omega_n}{(2\pi)^n}$$

For Grushin-type 2-ARS $\quad N(\lambda) \sim \dfrac{\text{Regularized Vol}(S)}{8\pi} \lambda \log \lambda$

Spectrum of singular sub-Laplacians

Weyl's law (II) [Colin de Verdière, Hillairet, Trélat 2023]

- Full asymptotic expansion of heat kernel for a very large class of nilpotentizable singular sub-laplacians with Whitney stratifiable singular set
- Detailed 2-terms asymptotics for Baouendi-Grushin and Martinet
- Quantum Ergodicity for Baouendi-Grushin with singular set containing at most one tangency point

In these cases the Weyl's law is led by

$$N(\lambda) \underset{\lambda\to+\infty}{\sim} \frac{\nu(N_s)}{\Gamma(Q^s/2+1)} \lambda^{Q^s/2} (\ln \lambda)^{m_s-1}$$

SPECTRUM OF SINGULAR SUB-LAPLACIANS

WEYL'S LAW (III) [Colin de Verdière, Hillairet, Trélat 2023]

- Weyl's law for some examples with non-stratifiable singular sets

	Case	$\mathrm{Tr}(\mathcal{M}_f\, e^{t\triangle})$	$N(\lambda)$	$\mathcal{N}$	
1	$X_1 = \partial_1, \quad X_2 = (x_1^k - x_2)\,\partial_2$ $k \geqslant 2$	$\frac{\lvert\ln t\rvert}{t}$	$\lambda \ln \lambda$	$\{x_2 = x_1^k\}$	
2	$X_1 = \partial_1, \quad X_2 = (x_1^{2p} + x_1 x_2^k)\,\partial_2$ $p, k \in \mathbb{N}^*$	$\frac{\lvert\ln t\rvert^2}{t}$ $\frac{1}{t^{p+\frac{1}{2}-\frac{2p-1}{2k}}}$	$\lambda \ln^2 \lambda$ $\lambda^{p+\frac{1}{2}-\frac{2p-1}{2k}}$	$\{(0,0)\}$ $\{(0,0)\}$	if $k = 1$ if $k \geqslant 2$
3	$X_1 = \partial_1, \quad X_2 = (x_1^2 + x_2^2)^k\,\partial_2$ $k \in \mathbb{N}^*$	$\frac{\lvert\ln t\rvert}{t}$ $\frac{1}{t^k}$	$\lambda \ln \lambda$ λ^k	$\{(0,0)\}$ $\{(0,0)\}$	if $k = 1$ if $k \geqslant 2$
4	$X_1 = \partial_1, \quad X_2 = x_1^m (x_1^{2p} + x_2^{2k})\,\partial_2$ $m \in \mathbb{N}, \quad p, k \in \mathbb{N}^*$	$\frac{\lvert\ln t\rvert}{t^{1+\frac{m}{2}}}$ $\frac{1}{t^{\frac{m+1}{2}+p-\frac{p}{2k}}}$	$\lambda^{1+\frac{m}{2}} \ln \lambda$ $\lambda^{\frac{m+1}{2}+p-\frac{p}{2k}}$	$\{(0,0)\}$ $\{(0,0)\}$	if $p = k = 1$ $\{(0,0)\}$
5	$X_1 = \partial_1, \quad X_2 = (x_1^2 - x_2^3)\,\partial_2$	$\frac{1}{t^{7/6}}$	$\lambda^{7/6}$	$\{(0,0)\}$	
6	$X_1 = \partial_1, \quad X_2 = (x_1^4 + x_1^2 x_2^2 + x_2^{2k})\,\partial_2$	$\frac{1}{t^2}$	λ^2	$\{(0,0)\}$	if $k \geqslant 3$
7	$X_1 = \partial_1, \quad X_2 = \partial_2 + x_1\,\partial_3 + x_1^2\,\partial_5,$ $X_3 = \partial_4 + (x_1^k + x_2^k)\,\partial_5$ $k \geqslant 2$	$\frac{1}{t^{7/2}}$ $\frac{\lvert\ln t\rvert}{t^{\frac{7}{2}}}$ $\frac{1}{t^{4-\frac{1}{k-1}}}$	$\lambda^{7/2}$ $\lambda^{7/2} \ln \lambda$ $\lambda^{4-\frac{1}{k-1}}$	$\mathbb{R}^5$ $\{x_1 = x_2 = 0\}$ $\{x_1 = x_2 = 0\}$	if $k = 2$ if $k = 3$ if $k \geqslant 4$

$$N(\lambda) \sim \frac{\nu(\mathcal{N})}{\Gamma(\gamma+1)} \lambda^{\gamma} \ln^k \lambda$$

$\mathbb{Q} \ni \gamma \geqslant \frac{\mathcal{Q}^s}{2}$ depends only on $\mathcal{D}$

$k \in \mathbb{N}$

SPECTRUM OF SINGULAR SUB-LAPLACIANS

IS THERE ANYTHING LEFT TO DO?

[Savale 2019]

- Role of obnormal minimizers in the Weyl's law [Colin de Verdière, Letrouit 2021]
- Self-adjointness and Weyl's law in presence of tangency pts
- Inverse problems:
  - given $\gamma \geq \frac{Q}{2}$ rational and $k \in \mathbb{N}$ is there a singular sR with those values as its leading Weyl order
  - reconstruction of singularities from asymptotic data
- Localization of Weyl's measure
- Magnetic sR Laplacians
- Still few methods: more techiques & alternative proofs are needed!
- Structure of nodal domains [Eswarathasan, Letrouit 2023]
- Bounds on low lying eigenvalues
- Quantum ergodicity


"That's all Folks!"
Thanks!

Abnormal extremals (EXAMPLE)

$$M = \mathbb{R}^3, \quad D = \ker\left(dz - \frac{y^2}{2}dx\right) = \operatorname{span}\Big\{ \overbrace{\partial_y}^{X_1}, \overbrace{\partial_x + \frac{y^2}{2}\partial_z}^{X_2} \Big\}$$

$\Rightarrow$ singular set $Z = \{y = 0\}$ — $\det(X_1, X_2, [X_1, X_2]) = 0$

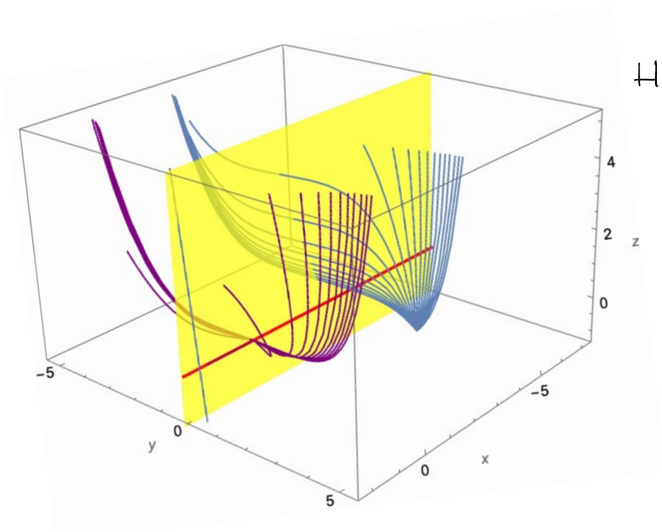
MARTINET SURFACE

$\Rightarrow$ $\gamma(t) = (\pm t, 0, 0)$ ABNORMAL EXTREMAL

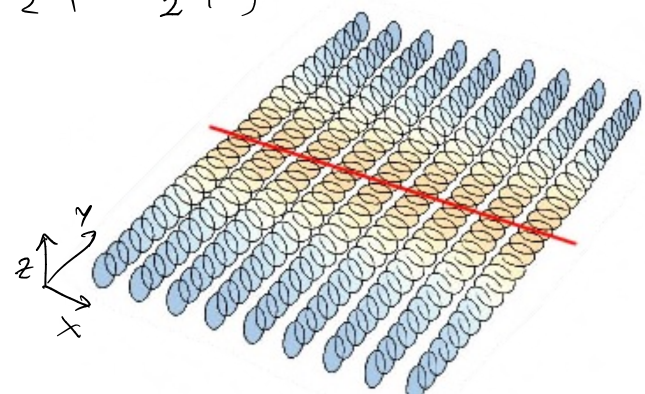
Simplest example.

No abnormal in step 2 Carnot

Plenty of abnormals in quasi-contact and Engels



$$H = \frac{1}{2}p_y^2 + \frac{1}{2}\left(p_x + \frac{y^2}{2}p_z\right)^2$$



Flags and dimensions

Canonical flag of a distribution D of step s is filtration

$$\{0\} = D_q^0 \subset \underset{\underset{D_q}{\shortparallel}}{D_q^1} \subset \cdots \subset D_q^j \subset \cdots \subset D_q^{s(q)} = T_qM$$

$D_q^j = D_q^{j-1} + [D_q, D_q^{j-1}]$
span of applications of j Lie brackets

$\Rightarrow$ Growth vector $k^D(q) = (\dim D_q^1, \ldots, \dim D_q^{s(q)})$

If $k^D(q) \equiv k^D$ constant $\Rightarrow$ D is called EQUIREGULAR

DIDO: $D^0 = \{0\}$, $D^1 = \mathrm{span}\{\partial_x, \partial_y + x\partial_z\}$, $D^2 = TM$

$k^D(q) = (2, 3)$ CONSTANT $\Rightarrow$ EQUIREGULAR

Hausdorff dimension

(X,d) metric space

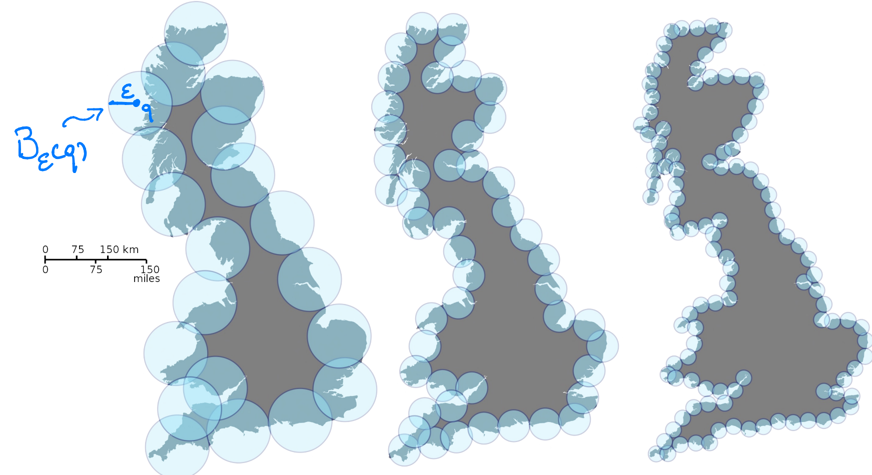
$\Rightarrow$ Hausdorff dimension

$$Q(q) = \lim_{\varepsilon \to 0} \frac{\ln(\operatorname{vol} B_\varepsilon(q))}{\ln \varepsilon}$$

BALL-BOX THEOREM

In the case of an equiregular subRiemannian structure (M^n, D, g)

$$= \sum_{j=1}^{s} j \left(\dim D_q^j - \dim D_q^{j-1}\right) \begin{cases} > n & \text{in sub-Riemannian case} \\ = n & \text{in Riemannian case} \end{cases}$$



[Wikimedia Foundation]

Examples

① Contact structures

M dim $2n+1$

$\mathcal{D} = \ker(dz - y\,dx)$

$= \operatorname{span}\{\partial_{y_i}, \partial_{x_i} + y_i \partial_z\}$

STEP 2 EQUIREGULAR

② Quasi-contact structures

M dim $2n+2$

$\mathcal{D} = \operatorname{span}\{\partial_{y_i}, \partial_{x_i} + y_i\partial_z, \partial_w\}$

STEP 2 EQUIREGULAR

③ Grushin plane

M (dim 2)

$\mathcal{D} = \operatorname{span}\{\partial_x, x\partial_y\}$

STEP 2 NOT EQUIREGULAR

④ Engel structure

M dim 4

$\mathcal{D} = \operatorname{span}\{\partial_x, \partial_y + x\partial_z + z\partial_w\}$

STEP 3 EQUIREGULAR

⑤ Martinet structure

M dim 3

$\mathcal{D} = \ker(dz - y^2 dx)$

$= \operatorname{span}\{\partial_y, \partial_x + \frac{y^2}{2}\partial_z\}$

STEP 3 EQUIREGULAR

⑥ (???)

M dim 3

$\mathcal{D} = \operatorname{span}\{\partial_x, \partial_y + xz\partial_z\}$

$z=0 \Rightarrow$ NOT BRACKET GENERATING