Semiclassical normal forms for the magnetic Laplacian

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The operator

Consider the magnetic Laplacian,

$$\mathcal{H}_h = (-ih\nabla - \mathrm{A})^2 = \sum_{j=1}^{d} (-ih\partial_{q_j} - A_j)^2.$$

- Self-adjoint realization on $\mathbb{R}^d$.
- Potential vector field $\mathrm{A} : \mathbb{R}^d \to \mathbb{R}^d$ (smooth).
  In fact A is a 1-form, $A = \sum_{j=1} A_j \mathrm{d}q_j$.
- The magnetic field is a $d \times d$ matrix $\mathbb{B} = (\partial_i A_j - \partial_j A_i)_{i<j}$.
  In fact B is a 2-form, $B = \mathrm{d}A$.
- Assumption $|B(q)| \to \infty$ as $q \to \infty$.

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Goal

→ Study the spectrum when $h \to 0$ depending on the variations of B.

→ Construct a normal form for $\mathcal{H}_h$.

The symbol

$$\mathcal{H}_h = (-ih\nabla - A)^2$$

- Note that $\mathbb{B}_{jk} = \frac{i}{h}\big[-ih\nabla_j - A_j, -ih\nabla_k - A_k\big]$. To be related with hypoelliptic sum of squares ?
- The semiclassical symbol is,

$$H(q,p) = \sum_{j=1}^{d} (p_j - A_j(q))^2 = |p - A(q)|^2.$$

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For example, in $d = 2$, you can compare H with the $d+1$ dimensional symbol

$$H_{\mathrm{hyp}} = (p_1 - A_1 p_3)^2 + (p_2 - A_2 p_3)^2 \quad ?$$

Construct a normal form for H microlocally near the surface

$$\Sigma = H^{-1}(0) = \Big\{(q,p) \in \mathbb{R}^{2d};\ p = A(q)\Big\}.$$

Of importance is the symplectic structure $\mathrm{d}p \wedge \mathrm{d}q$ on the phase space.

Proposition

The restriction of $\mathrm{d}p \wedge \mathrm{d}q$ to Σ is $\pi^* B$.

Here π is the canonical projection $\pi(q,p) = q$.

- The rank of B will have strong influence. We use

$$d = 2s + k, \quad k = \dim\big(\mathrm{Ker}\mathbb{B}\big).$$

We assume s, k are constant i.e. independent of q.

1. Symplectic magnetic fields $d = 2s$, $k = 0$.
   - Normal form
   - Applications

2. Constant rank magnetic fields $d = 2s + k$, $k > 0$.

When B is symplectic

Assume $d = 2s$. Then (Σ, B) is a symplectic submanifold of $\mathbb{R}^{2d}$ and

$$T\mathbb{R}^{2d} = T\Sigma \oplus T\Sigma^{\perp}.$$

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$$T\mathbb{R}^{2d} = T\Sigma \oplus T\Sigma^{\perp}.$$

Near Σ, we can approximate H by its Hessian.

Proposition

The Hessian of H satisfies

$$\nabla^2 H(\mathcal{V}, \mathcal{V}) = 2|\mathbb{B}(q)\pi_* \mathcal{V}|^2 \quad \textit{for } \mathcal{V} \in T\Sigma^{\perp}.$$

Hence B appears at two different places :

- The curvature of Σ,
- The Hessian of H.

Normal form I

We obtain the following normal form for the symbol. We denote by $(\pm i\beta_j(q))_{1\leq j\leq d}$ the eigenvalues of $\mathbb{B}$ and assume they are simple.

Proposition

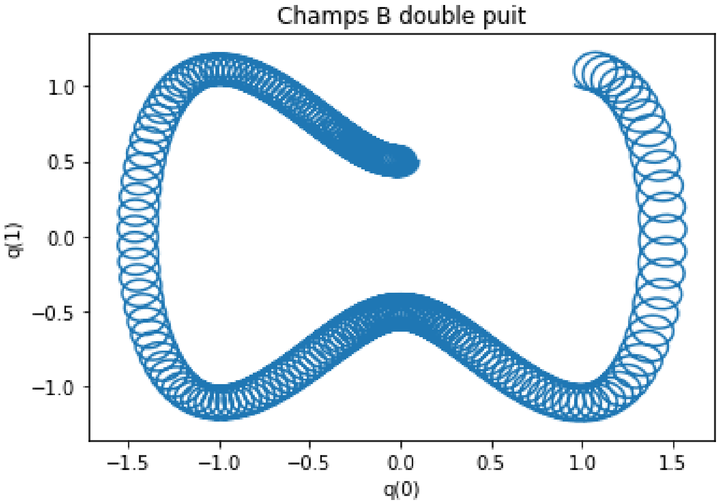
Assume B is symplectic, i.e. $d = 2s$. Then there exist local variables $(x, \xi, y, \eta) = \Phi(q, p)$ near Σ such that

$$H \circ \Phi^{-1}(x, \xi, y, \eta) = \sum_{j=1}^{s} \beta_j(y, \eta)(\xi_j^2 + x_j^2) + \mathcal{O}((x, \xi)^3),$$

and $\Phi_*(\mathrm{d}p \wedge \mathrm{d}q) = \mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y$.

Remarks

- We can use a Birkhoff normal form to get a higher order precision.
- In dimension $d = 2$, you have only one oscillator $\beta(y, \eta)(\xi^2 + x^2)$.
- The oscillator $\xi^2 + x^2$ is the cyclotron motion.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q(0)

Normal form II

We come back to the operator...

$$\mathcal{H}_h = (-ih\nabla - A)^2 = \mathrm{Op}_h^w H,$$

where Op_h^w is the semiclassical Weyl quantization.

- We recall the normal form on the symbol

$$H \circ \Phi^{-1} = \sum_{j=1}^{s} \beta_j(y,\eta)(\xi_j^2 + x_j^2) + \dots$$

- We quantize this result. To main order, $\mathcal{H}_h$ will be described by

$$\mathcal{N}_h = \sum_{j=1}^{s} \mathrm{Op}_h^w(\beta_j)(-h^2\partial_{x_j}^2 + x_j^2),$$

  and the spectrum is given by a familly of operators,

$$\mathcal{N}_h^{[n]} = \sum_{j=1}^{s} h(2n_j+1)\mathrm{Op}_h^w(\beta_j).$$

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Application 1

We have the following Weyl law.

Theorem

Assume $(\beta_j(q))_j$ are pairwise distinct for $\{b_0(q) \leq b_1\}$. The number of eigenvalues of $\mathcal{H}_h$ below $b_1 h$ is given by

$$N(\mathcal{H}_h, b_1 h) \sim \frac{1}{(2\pi h)^s} \sum_{n \in \mathbb{N}^s} \int_{\{b_n(q) \leq b_1\}} \frac{B^s}{s!}.$$

with $b_n(q) = \sum_{j=1}^{s} (2n_j + 1)\beta_j(q)$.

References.

[1] J.P. Demailly, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*. CMP, 1986.

[2] L. Morin, *A semiclassical Birkhoff normal form for symplectic magnetic fields*. Journal of spectral theory. 2022.

Application 2

We can also deduce eigenvalue asymptotics.

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Theorem

Assume $d = 2$. Assume $|B|$ admits a unique and non-degenerate minimum b_0. Then for p, $n \in \mathbb{N}$, you can find eigenvalues of $\mathcal{H}_h$ such that

$$\lambda_{n,p}(h) = (2n+1)b_0 h\big(1 + \big((2p+1)c_0 + c_1\big)h\big) + o(h^2).$$

There is also a full expansion in powers of h. Moreover, the first eigenvalues of $\mathcal{H}_h$ are

$$\lambda_{0,p}(h) = b_0 h\big(1 + \big((2p+1)c_0 + c_1\big)h\big) + o(h^2)).$$

Here c_0, c_1 are explicit constants depending on the Hessian of β at the minimum.

Application 2

Also in higher dimension.

Theorem

Assume $d = 2s$. Assume $b = \sum_j \beta_j$ admits a unique and non-degenerate minimum b_0. Then the first eigenvalues of $\mathcal{H}_h$ are of the form

$$\lambda_j(q) = hb_0\Big(1 + \big(E_j + c\big)h\Big) + o(h^2),$$

where hE_j are the eigenvalues of a s-dim. oscillator with symbol $\nabla^2 b(q_0)$.

References.

[1] B. Helffer, Y. Kordyukov, *Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator*. 2011.

[2] N. Raymond, S. Vu Ngoc, *Geometry and spectrum in 2d magnetic wells*. 2015.

[3] L. Morin, *A semiclassical Birkhoff normal form for symplectic magnetic fields*. 2022.

Application 3

In $d = 2$, we used the same ideas to describe the spectrum of non-selfadjoint operators of the form

$$(-ih\nabla - A)^2 + hV,$$

with V complex valued. Note that V acts at same order as B. Under suitable assumptions on B, V, we prove existence of discrete eigenvalues,

$$\lambda_n(h) = \mu_0 h + \big((2n+1)c_0 + c_1\big)h^2 + o(h^2),$$

where c_0, $c_1 \in \mathbb{C}$.

[4] L. Morin, N. Raymond, S. Vu Ngoc, *Eigenvalue asymptotics for confining magnetic Schrödinger operators with complex potentials.* 2022.

Application 4

These ideas are also behind other works :

- Propagation of coherent states in 2D magnetic fields
  [5] G. Boil, S. Vu Ngoc, *Long-time dynamics of coherent states in strong magnetic fields*. Amer. J. Math. 2021.

- Results on the decay of the eigenfunctions in 2D magnetic wells
  [6] Y. G. Bonthonneau, N. Raymond, S. Vu Ngoc, *Exponential localization in 2D pure magnetic wells*. Arkiv for Mat. 2021.

1 Symplectic magnetic fields $d = 2s$, $k = 0$.
- Normal form
- Applications

2 Constant rank magnetic fields $d = 2s + k$, $k > 0$.

When B has non-zero kernel...

We come back to the caracteristic surface

$$\Sigma = H^{-1}(0) = \Big\{(q,p) \in \mathbb{R}^{2d};\ p = A(q)\Big\}.$$

If the 2-form B has constant rank, we have another splitting of the tangent phase-space,

$$T\mathbb{R}^{2d} = \underbrace{E \oplus K}_{T\Sigma} \overbrace{\oplus F}^{T\Sigma^\perp} \oplus L$$

where

- $K = \operatorname{Ker} B$,
- E, F are symplectic with dimension $2s$,
- L is a Lagrangian complement of K in $(E \oplus F)^\perp$.

We use three sets of variables $(x,\xi) \in \mathbb{R}^{2s}$, $(y,\eta) \in \mathbb{R}^{2s}$, $(t,\tau) \in \mathbb{R}^{2k}$.

Normal form

Theorem

Assume B has constant rank and non-zero kernel ($k > 0$). Then there exist local variables $(x, \xi, y, \eta, t, \tau) = \Phi(q, p)$ such that

$$H \circ \Phi^{-1} = \langle M(y, \eta, t)\tau, \tau \rangle + \sum_{j=1}^{s} \beta_j(y, \eta, t)(\xi_j^2 + x_j^2) + \mathcal{O}((x, \xi, \tau)^3),$$

and $\Phi^*(\mathrm{d}\xi \wedge \mathrm{d}x + \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\tau \wedge \mathrm{d}t) = \mathrm{d}p \wedge \mathrm{d}q$, for some $k \times k$ positive matrix $M(y, \eta, t)$.

Here $(\pm i\beta_j)_{1 \leq j \leq s}$ are the non-zero eigenvalues of $\mathbb{B}$.

- Again, we can use a Birkhoff normal form to improve the precision order.
- We deduce eigenvalue asymptotics as $h \to 0$.

Applications

- We deduce eigenvalue asymptotics. For instance in $d = 3$ if $|B|$ admits a non-degenerate minimum, you can find eigenvalues of the form

$$\lambda_{n,p,j}(h) = (2n+1)b_0 h\Big(1+\frac{(2p+1)}{(2n+1)^{1/2}}\nu_0 h^{1/2}+((2j+1)\alpha+c_{n,p})h\Big)+o(h^2$$

- In any dimension the first eigenvalues of $\mathcal{H}_h$ are of the form

$$\lambda_j(h) = b_0 h\Big(1 + \nu_0 h^{1/2} + (E_j + c)h\Big) + o(h^2).$$

References.

[1] B. Helffer, Y. Kordyukov, *Eigenvalue estimates for a three-dimensional magnetic Schrödinger operator*, Asymptotic Analysis. 2013.

[2] B. Helffer, Y. Kordyukov, N. Raymond, S. Vu Ngoc, *Magnetic wells in dimension three*, Analysis and PDE. 2016.

[3] L. Morin, *A semiclassical Birkhoff normal form for constant-rank magnetic fields*, Analysis and PDE. 2022.

Discussions.

1. Do these results have non-semiclassical analogues ? How much is known ?
2. Can we describe semiclassical operators with more general symbols

$$H(q,p) = \sum_{j=1}^{\ell} X_j(q,p)^2,$$

   depending on the symplectic structure of $\Sigma = H^{-1}(0)$? Applications ?
3. When B has eigenvalue crossings ?
4. Effect of resonances between the $(\beta_j(q))$?
5. When the rank of B is not constant ?

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Thank you !