Field C^*-algebra and spectral analysis of quantum many channel Hamiltonians.

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The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy.

Steven Weinberg, *The First Three Minutes*

The purpose of this lecture is to highlight the utility of C^*-algebras in the spectral theory of "complicated" quantum systems: N-body systems and beyond, quantum fields, etc.

To sum up the key idea: *instead of studying by ad hoc means the properties of a Hamiltonian H, study the structure of the C^*-algebra generated by a class of Hamiltonians that are similar (in some sense) to H.*

Main references: VG+A.Iftimovici, *On the structure of the C^*-algebra generated by the field operators*, J. Func. Analysis 284(8), 2023; V.G. *On the structure of the essential spectrum of elliptic operators on metric spaces*, J. Func. Analysis, 260:1734-1765, 2011.

1. Notations

(1) If $\mathscr{C}$ is a C^*-algebra and $\mathcal{A}, \mathcal{B} \subset \mathscr{C}$ then

$\mathcal{A}\mathcal{B} =$ linear span of $\{AB \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A}\cdot\mathcal{B} =$ closure of $\mathcal{A}\mathcal{B}$.

$\sum_{i\in I}^{\mathrm{c}} \mathcal{A}_i =$ norm closure of the sum $\sum_{i\in I} \mathcal{A}_i$ of the subspaces $\mathcal{A}_i$ of $\mathscr{C}$.

(2) Some algebras associated to a finite dimensional real vector space X:

$$C_{\mathrm{c}}(X) \subset C_0(X) \subset C_\infty(X) \subset C_{\mathrm{b}}^{\mathrm{u}}(X) \subset C_{\mathrm{b}}(X) \subset C(X).$$

$L^2(X)$ and its norm are defined by a translation invariant Radon measure on X *but the norm in* $B(L^2(X))$ *is independent of this choice.* We set

$$\mathscr{B}(X) = B(L^2(X)) \quad \text{and} \quad \mathscr{K}(X) = K(L^2(X)).$$

Position and momentum observables q, p:

$\varphi : X \to \mathbb{C}$ Borel $\Rightarrow \varphi(q) =$ multiplication by φ on $L^2(X)$,
$\psi : X^* \to \mathbb{C}$ Borel $\Rightarrow F\psi(p)F^{-1} =$ multiplication by ψ on $L^2(X^*)$,
(where $F : L^2(X) \to L^2(X^*)$ is a Fourier transform).

(1) $C_{\mathrm{b}}(X) \subset \mathscr{B}(X)$ via $\varphi \mapsto \varphi(q)$.
(2) $C_b(X^*) \subset \mathscr{B}(X)$ via $\psi \mapsto \psi(p)$. (Also set $C^*(p) = C_0(X^*)$.)

2. OBSERVABLES AFFILIATED TO C^*-ALGEBRAS

Self-adjoint operators affiliated to a C*-algebra

Let $\mathcal{H}$ a Hilbert space, $\mathscr{C} \subset B(\mathcal{H})$ a C^*-subalgebra, and A a self-adjoint operator with spectrum $\mathrm{Sp}(A)$.

C^*-algebra generated by A: $\quad C^*(A) \equiv C_0(A) \doteq \{\theta(A) \mid \theta \in C_0(\mathbb{R})\}$.

A *is affiliated to* $\mathscr{C}$ if the next equivalent conditions are satisfied:

$C^*(A) \subset \mathscr{C} \Leftrightarrow \theta(A) \in \mathscr{C}\ \forall\theta \in C_0(\mathbb{R}) \Leftrightarrow (A-z)^{-1} \in \mathscr{C}$ for some $z \notin \sigma(A)$.

A is *strictly affiliated* to $\mathscr{C}$ if it is affiliated to $\mathscr{C}$ and $C^*(A) \cdot \mathscr{C} = \mathscr{C}$.

Example. Let $\mathcal{H} = L^2(\mathbb{R})$ and q the operator defined by $(qu)(x) = xu(x)$. Clearly $C^*(q) \equiv C_0(\mathbb{R})$. *Then* $q+q^{-1}$ *is affiliated but not strictly to* $C_0(\mathbb{R})$.

Affiliation criterion

1) H_0 = self-adjoint operator on $\mathcal{H}$ and $\mathcal{G} = D(|H_0|^{\frac{1}{2}})$. Then

$$\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^* \quad \text{continuous dense embeddings}$$

and H_0 extends to a continuous map $\mathcal{G} \to \mathcal{G}^*$.

2) $V : \mathcal{G} \to \mathcal{G}^*$ symmetric such that for some numbers $\mu, \nu \geq 0$ with $\mu < 1$

$\pm V \leq \mu |H_0| + \nu$ or H_0 is bounded from below and $V \geq -\mu H_0 - \nu$.

3) Then the restriction of $H = H_0 + V : \mathcal{G} \to \mathcal{G}^*$ to $D(H) \doteq \{g \in \mathcal{G} \mid Hg \in \mathcal{H}\}$ is a self-adjoint operator on $\mathcal{H}$ still denoted H.

Theorem. If H_0 is strictly affiliated to $\mathscr{C}$ and for some $s \geq 1/2$

$$(|H_0| + 1)^{-s} V (|H_0| + 1)^{-1/2} \in \mathscr{C}$$

then H is strictly affiliated to $\mathscr{C}$.

Observables affiliated to C*-algebras

Let $\mathscr{C}$ be an arbitrary C^*-algebra. An *observable affiliated to* $\mathscr{C}$ is just a morphism $A : C_0(\mathbb{R}) \to \mathscr{C}$. We often use the notation $\theta(A) = A(\theta)$ and

$C^*(A) \equiv C_0(A) \doteq \{\theta(A) \mid \theta \in C_0(\mathbb{R})\}$ = C^*-subalgebra of $\mathscr{C}$.

The zero morphism is an observable affiliated to $\mathscr{C}$ *denoted* ∞; this is natural because $\theta(\infty) = 0$ for any $\theta \in C_0(\mathbb{R})$.

$\mathcal{A} =$ set of observables affiliated to $\mathscr{C}$. The C^**-algebra generated by* $\mathcal{A}$ is $C^*(\mathcal{A})$ = smallest C^*-subalgebra which contains $\theta(A)$ if $A \in \mathcal{A}, \theta \in C_0(\mathbb{R})$.

A is *strictly affiliated* to $\mathscr{C}$ if it is affiliated to $\mathscr{C}$ and $C^*(A) \cdot \mathscr{C} = \mathscr{C}$.

$\mathcal{P} : \mathscr{C} \to \mathscr{D}$ = morphism $\Rightarrow \mathcal{P}(A) \doteq \mathcal{P} \circ A$ observable affiliated to $\mathscr{D}$.

Notation: $A \,\widetilde{\in}\, \mathscr{C} \Leftrightarrow A$ belongs to $\mathscr{C}$ or is an observable affiliated to $\mathscr{C}$.

Fix a Hilbert space $\mathcal{H}$. Then a self-adjoint operator is identified with the observable defined by its C_0-functional calculus.

Let $\mathscr{C} \subset B(\mathcal{H})$. *The observables affiliated to $\mathscr{C}$ can be identified with self-adjoint operators acting in closed subspaces of $\mathcal{H}$.* The observable ∞ is the only operator with domain $\{0\}$.

Example. The Hamiltonians of N-body systems with hard core interactions are observables affiliated to the C^*-algebra generated by the usual N-body Hamiltonians but are not self-adjoint operators on $\mathcal{H}$.

If $\mathcal{P} : \mathscr{C} \to \mathscr{D} \subset B(\mathcal{K})$ and A is a self-adjoint operator on $\mathcal{H}$ affiliated to $\mathscr{C}$, then $\mathcal{P}(A)$ in general is not associated to a (densely defined) self-adjoint operator on $\mathcal{K}$. But:

if A is a self-adjoint operator strictly affiliated to $\mathscr{C}$ then $\mathcal{P}(A)$ is a (densely defined) self-adjoint operator in any non-degenerate representation $\mathcal{P}$ of $\mathscr{C}$.

3. C^*-ALGEBRAS GRADED BY SEMILATTICES

$\mathcal{S}$ = *semilattice* = ordered set s.t. the upper bound $a \vee b$ exists $\forall a, b \in \mathcal{S}$.

Definition. A C^*-algebra $\mathscr{C}$ is $\mathcal{S}$-*graded* if a linearly independent family $\{\mathscr{C}(a)\}_{a \in \mathcal{S}}$ of C^*-subalgebras of $\mathscr{C}$ (called *components*) is given such that:

(i) $\mathscr{C}(a) \cdot \mathscr{C}(b) \subset \mathscr{C}(a \vee b)$ for all $a, b \in \mathcal{S}$;

(ii) $\sum_{a \in \mathcal{S}} \mathscr{C}(a)$ is dense in $\mathscr{C}$, i.e. $\sum^{\mathrm{c}}_{a \in \mathcal{S}} \mathscr{C}(a) = \mathscr{C}$.

$\mathscr{C}(\mathcal{E}) \doteq \sum^{\mathrm{c}}_{a \in \mathcal{E}} \mathscr{C}(a)$ if $\mathcal{E} \subset \mathcal{S}$. If $\mathcal{E} \subset \mathcal{S}$ is $\vee$-stable then $\mathscr{C}(\mathcal{E})$ is $\mathcal{E}$-graded.

Remark. $\mathcal{E}$ finite $\Rightarrow \sum_{a \in \mathcal{E}} \mathscr{C}(a)$ is a closed subspace.

Remark. We will also come across situations where $\mathcal{S}$ is a $\wedge$-semilattice, i.e. the lower bound $a \wedge b$ exists $\forall a, b \in \mathcal{S}$, and the condition (i) is replaced by $\mathscr{C}(a) \cdot \mathscr{C}(b) \subset \mathscr{C}(a \wedge b)$. If needed to avoid confusion we then say that $\mathscr{C}$ *is* $\wedge$*-graded, or inf-graded, by* $\mathcal{S}$. Above, $\mathscr{C}$ is $\vee$-graded, or sup-graded.

Remark. Only sub-semilattices of Grassmannians will be of interest.

$\mathbb{G}(X)$ **=** ***Grassmannian*** **of** $X \doteq$ set of finite dimensional subspaces of the real vector space X with inclusion as order relation. This is a lattice: $Y \wedge Z = Y \cap Z$ and $Y \vee Z = Y + Z$.

Exercise

The simplest nontrivial example of graded C^*-algebra.

Let $\mathcal{H} = L^2(\mathbb{R})$. If $(\alpha, \beta) \in \mathbb{R}^2$ then $\alpha p + \beta q$ is a self-adjoint operator so we my consider the C^*-algebra generated by such operators:

$$\mathscr{F} \doteq C^*(\alpha p + \beta q \mid (\alpha, \beta) \in \mathbb{R}^2).$$

If $(\alpha', \beta') \in \mathbb{R}^2$ then

$$[\alpha p + \beta q, \alpha' p + \beta' q] = \mathrm{i}(\alpha' \beta - \alpha \beta').$$

This is zero if and only if the vectors (a, b) and (a', b') are collinear and then $\alpha' p + \beta' q = \lambda(\alpha p + \beta q)$ for some real λ if $\alpha p + \beta q \neq 0$.

For each line $L \subset \mathbb{R}^2$ choose a nonzero $(a, b) \in L$ and denote $\mathscr{F}(L) = C^*(\alpha p + \beta q)$. Then set $\mathscr{F}(0) = \mathbb{C}$, $\mathscr{F}(\mathbb{R}^2) = K(\mathcal{H})$. Show that

$$\mathscr{F} = \text{ norm closure of } \mathbb{C} + \textstyle\sum_{L \in \mathbb{P}} \mathscr{F}(L) + K(\mathcal{H}).$$

$\mathscr{F}$ is $\mathbb{G}(\mathbb{R}^2)$-graded by the family of C^*-subalgebras $\{\mathscr{F}(E)\}_{E \in \mathbb{G}(\mathbb{R}^2)}$.

Proposition. $a \in \mathcal{S} \Rightarrow$
$\mathcal{S}_a = \{b \in \mathcal{S} | b \leq a\}$ and $\mathcal{S}'_a = \{b \in \mathcal{S} | b \not\leq a\}$ are sub-semilattices. $\mathscr{C}_a = \mathscr{C}(\mathcal{S}_a)$ is a C^*-subalgebra and $\mathscr{C}'_a = \mathscr{C}(\mathcal{S}'_a)$ is an ideal of $\mathscr{C}$ such that $\mathscr{C} = \mathscr{C}_a + \mathscr{C}'_a$ direct sum. The projection $\mathcal{P}_a : \mathscr{C} \to \mathscr{C}_a$ is a morphism.

HVZ Theorem. Assume $\mathcal{S}$ has a greatest element e and is co-atomic. Let $\mathcal{S}_{\max}$ = set of maximal elements of $\mathcal{S} \setminus \{e\}$. Then $\mathcal{P} : S \mapsto (\mathcal{P}_a S)_{a \in \mathcal{S}_{\max}}$ is a morphism $\mathscr{C} \to \bigoplus_{a \in \mathcal{S}_{\max}} \mathscr{C}_a$ with kernel $\mathscr{C}(e)$, hence

$$\mathscr{C}/\mathscr{C}(e) \hookrightarrow \bigoplus_{a \in \mathcal{S}_{\max}} \mathscr{C}_a.$$

(i) If $T \,\widetilde{\in}\, \mathscr{C}$ then:

$$\mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T) \doteq \text{ spectrum of } \mathcal{P}(T) \equiv \mathrm{Sp}\,\mathcal{P}(T) = \overline{\cup}_{a \in \mathcal{S}_{\max}} \mathrm{Sp}\,\mathcal{P}_a(T).$$

(ii) If $\mathscr{C} \subset B(\mathcal{H})$ and $\mathscr{C}(e) = \mathscr{C} \cap K(\mathcal{H})$ then $\mathrm{Sp}_{\mathrm{ess}}(T) = \mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T)$.

(iii) If $\mathcal{S} = \mathbb{G}(X)$, so $\mathcal{S}_{\max} = \mathbb{H}$ = set of hyperplanes of X, then $\{\mathcal{P}_a(S) \mid a \in \mathbb{H}\}$ is a compact in $\mathscr{C}$ and $\mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T) = \cup_{a \in \mathbb{H}} \mathrm{Sp}\,\mathcal{P}_a(T)$.

Remark. Everything is very easy. Only the assertion concerning the compactness requires a little bit of thinking! $\{\mathcal{P}_a(S) \mid a \in \mathcal{S}_{\max}\}$ is always a relatively compact subset of $\mathscr{C}$.

4. Field C^*-algebra of a symplectic space

Symplectic space = real vector space Ξ equipped with a symplectic form (bilinear anti-symmetric non-degenerate map $\sigma : \Xi^2 \to \mathbb{R}$). Set
$E \subset \Xi \Rightarrow E^\sigma \doteq \{\xi \in \Xi \mid \sigma(\xi, \eta) = 0 \; \forall \eta \in E\}$.
E is *isotropic* if $E \subset E^\sigma$, *Lagrangian* if $E = E^\sigma$, *symplectic* if σ is non-degenerate on it.

Remark. $\mathbb{G}_{\mathrm{s}}(\Xi) \doteq$ set of symplectic finite dimensional subspaces. Then:

$\forall E \in \mathbb{G}(\Xi) \; \exists F \in \mathbb{G}_{\mathrm{s}}(\Xi)$ such that $E \subset F$.

Example. X = finite dimensional real vector space; $T^*X = X \oplus X^*$ with the symplectic form $\sigma(\xi, \eta) = \langle y, k \rangle - \langle x, l \rangle$ if $\xi = x + k, \eta = y + l$ with $x, y \in X$ and $k, l \in X^*$.

Representation of Ξ on a Hilbert space $\mathcal{H}$: a map $W : \Xi \to U(\mathcal{H})$ with
$W(\xi + \eta) = \mathrm{e}^{\frac{\mathrm{i}}{2}\sigma(\xi,\eta)} W(\xi) W(\eta) \; \forall \xi, \eta \in \Xi$ and $\text{w-}\lim_{t \to 0} W(t\xi) = 1$.

Then $\forall \xi \in \Xi$ the *field operator* $\phi(\xi) \equiv \phi_W(\xi)$ is the self-adjoint operator such that $W(t\xi) = \mathrm{e}^{\mathrm{i}t\phi(\xi)} \; \forall t \in \mathbb{R}$. We set $R_\xi(z) = (\phi(\xi) - z)^{-1}$.

Short history . . .

D. Kastler. *The C^*-algebras of a free boson field. I. Discussion of the basic facts,* Commun. Math. Phys. 1:14-48, **1965**.

A. Boutet de Monvel and V. Georgescu. *Graded C^*-algebras associated to symplectic spaces and spectral analysis of many channel Hamiltonians,* in Bielefeld Encount. Math. Phys. VIII, World Sci. Publishing, **1993**.

V. Georgescu and A. Iftimovici. *C^*-algebras of energy observables. II. Graded Symplectic Algebras and Magnetic Hamiltonians,* in *Math.Phys.Arch.* 01-99, **2001**.

D. Buchholz, H. Grundling. *The resolvent algebra: a new approach to canonical quantum systems,* J. Func. Anal. 254:2725-2779, **2008**.

V. Georgescu and A. Iftimovici. *On the structure of the C^*-algebra generated by the field operators,* J. Func. Anal. 284(8), art. 109867, 73 pp, **2023**.

$E \in \mathbb{G}(\Xi) \Rightarrow M(E) =$ set of bounded Borel measures on E and $L^1(E) =$ subset of absolutely continuous measures (with $M(0) = L^1(0) = \mathbb{C}\delta_0$).

Definitions. The Kastler algebra was introduced by *Kastler in 1965*.

(1) The *Kastler C^*-algebra $\mathscr{K}$ of* Ξ is the norm closure in $B(\mathcal{H})$ of the set of operators $W(\mu) = \int_E W(\xi)\mu(d\xi)$ with $E \in \mathbb{G}(\Xi)$, $\mu \in M(E)$.

(2) The *field C^*-algebra $\mathscr{F}$ of* Ξ is the norm closure of the set of operators $W(\mu)$ with $E \in \mathbb{G}(\Xi)$ and $\mu \in L^1(E)$ *(AI+VG 2001)*.

(3) The *resolvent algebra* $\mathcal{R} \doteq C^*(\phi(\xi)|\xi \in \Xi)$ introduced by *Buchholz and Grundling in 2008* coincides with the field algebra.

Proposition. The Kastler C^*-algebras associated to different representations W are canonically isomorphic. Similarly for the field algebras.

Thus we may think of $\mathscr{K}, \mathscr{F}$ as some abstractly given objects independent of W. In fact they were so constructed by Kastler (1965) by using the unital $*$-algebra structure on $\cup_{E\in\mathbb{G}(\Xi)} M(E)$ defined by the natural involution and the twisted convolution

$$\int f(\xi)(\mu \circledast \nu)(\mathrm{d}\xi) = \iint \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(\xi,\eta)} f(\xi+\eta)\mu(\mathrm{d}\xi)\nu(\mathrm{d}\eta) \quad \forall f \in C_0(E).$$

$\mathbb{G}$-grading of $\mathscr{F}$

Definition. $E \in \mathbb{G}(\Xi)$ and $\xi_1, \ldots, \xi_n$ is a generating set for $E \in \mathbb{G}(\Xi)$

$$\begin{aligned}\mathscr{F}(E) &\doteq \text{ norm closure of the set of operators } W(\mu) \text{ with } \mu \in L^1(E) \\ &= C^*(\phi(\xi_1)) \cdot C^*(\phi(\xi_2)) \cdot \ldots \cdot C^*(\phi(\xi_n)).\end{aligned}$$

Theorem. The set of C^*-subalgebras $\mathscr{F}(E)$ of $\mathscr{F}$ has the properties:

$$E, F \in \mathbb{G}(\Xi) \Rightarrow \mathscr{F}(E) \cdot \mathscr{F}(F) = \mathscr{F}(E + F),$$
$$\mathring{\mathscr{F}} \doteq \textstyle\sum_{E \in \mathbb{G}(\Xi)} \mathscr{F}(E) \text{ is a linear direct sum and is dense in } \mathscr{F},$$
$$\text{if } \mathcal{S} \subset \mathbb{G}(\Xi) \text{ is finite then } \mathscr{F}(\mathcal{S}) \doteq \textstyle\sum_{E \in \mathcal{S}} \mathscr{F}(E) \text{ is norm closed.}$$

In other terms: $\mathscr{F}$ **is a** $\mathbb{G}(\Xi)$**-graded** C^***-algebra with components** $\mathscr{F}(E)$.

$$\mathring{\mathscr{F}} \doteq \textstyle\sum_{E \in \mathbb{G}(\Xi)} \mathscr{F}(E), \quad \mathscr{F} = \text{ closure of } \mathring{\mathscr{F}} = \textstyle\sum^{\mathrm{c}}_{E \in \mathbb{G}(\Xi)} \mathscr{F}(E).$$

Remark. If $\mathcal{S} \subset \mathbb{G}(\Xi)$ is finite and $E \notin \mathcal{S}$ then $\mathscr{F}(E) \cap \mathscr{F}(\mathcal{S}) = 0$.

Any subspace $E \subset \Xi$ determines three C^*-subalgebras of $\mathscr{F}$:

$$\mathscr{F}_E \doteq \sum\nolimits^{\mathrm{c}}_{F\subset E} \mathscr{F}(F), \quad \mathscr{F}'_E \doteq \sum\nolimits^{\mathrm{c}}_{F\not\subset E} \mathscr{F}(F), \quad \mathscr{F}_{\supset E} \doteq \sum\nolimits^{\mathrm{c}}_{F\supset E} \mathscr{F}(F).$$

$\mathscr{F}_E =$ unital C^*-subalgebra; $\mathscr{F}'_E$ and $\mathscr{F}_{\supset E}$ are ideals. In terms of fields

$$\mathscr{F}_E = C^*(\phi(\xi) \mid \xi \in E).$$

Theorem. *The C^*-algebra $\mathscr{F}_E$ and the ideal $\mathscr{F}'_E$ satisfy*

$$\mathscr{F} = \mathscr{F}_E + \mathscr{F}'_E \quad \textit{and} \quad \mathscr{F}_E \cap \mathscr{F}'_E = 0.$$

The projection $\mathcal{P}_E \colon \mathscr{F} \to \mathscr{F}_E$ determined by this direct sum decomposition is a morphism and it is the unique continuous linear map $\mathcal{P}_E \colon \mathscr{F} \to \mathscr{F}$ such that $T = \sum_F T(F) \in \mathring{\mathscr{F}} \Rightarrow \mathcal{P}_E T = \sum_{F\subset E} T(F)$.
For any subspaces E, F we have

$$\mathscr{F}_{E\cap F} = \mathscr{F}_E \cap \mathscr{F}_F \quad \textit{and} \quad \mathcal{P}_{E\cap F} = \mathcal{P}_E\mathcal{P}_F = \mathcal{P}_F\mathcal{P}_E.$$

If Ξ is finite dimensional we may describe $\mathscr{F}_E$ and its commutant in $\mathscr{F}$ independently of the graded structure of $\mathscr{F}$.

Theorem. *If Ξ is finite dimensional, for any subspace $E \subset \Xi$ we have*
(1) $\mathscr{F}_E = \{T \in \mathscr{F} \mid [T, W(\xi)] = 0 \,\forall \xi \in E^\sigma\}$,
(2) $\mathscr{F}_{E^\sigma} = \{T \in \mathscr{F} \mid [S, T] = 0 \,\forall S \in \mathscr{F}_E\}$.

Corollary. *If $X \in \mathbb{G}(\Xi)$ is Lagrangian then $\mathscr{F}_X$ is a maximal abelian subalgebra of $\mathscr{F}$, i.e. if $T \in \mathscr{F}$ then:* $[S, T] = 0 \,\forall S \in \mathscr{F}_X \Leftrightarrow T \in \mathscr{F}_X$.

The next one is a much more subtle result and it requires a real proof!

Theorem. *If Ξ is finite dimensional and W irreducible then $\mathscr{F}(E)$ is the set of $T \in B(\mathcal{H})$ such that:*

$$\lim_{\xi \to 0} \|[W(\xi), T]\| = 0, \quad \lim_{\xi \in E, \xi \to 0} \|(W(\xi) - 1)T\| = 0, \quad [W(\xi), T] = 0 \,\forall \xi \in E^\sigma.$$

The first condition is equivalent to:

$\xi \mapsto W(\xi)^* T W(\xi)$ *is norm continuous on finite dimensional subspaces*.

We now consider the HVZ theorem for a finite dimensional Ξ. Then $\mathscr{F}(\Xi) \subset K(\mathcal{H})$ if W is of finite multiplicity and $\mathscr{F}(\Xi) = K(\mathcal{H})$ if W is irreducible. So if W is of finite multiplicity

$$\mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T) = \mathrm{Sp}_{\mathrm{ess}}(T)\ \forall T \mathrel{\widetilde{\in}} \mathscr{F}.$$

Clearly $\mathbb{G}(\Xi)_{\max} = \mathbb{H}(\Xi)$ is the set of hyperplanes of Ξ.

Theorem. If $T \mathrel{\widetilde{\in}} \mathscr{F}$ then $\mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T) = \bigcup_{H \in \mathbb{H}(\Xi)} \mathrm{Sp}(\mathcal{P}_H T)$. Moreover, for any $H \in \mathbb{H}(\Xi)$ and any nonzero $\xi \in H^\sigma$ we have

$$\mathcal{P}_H T = \text{s-}\lim_{r \to \infty} W(r\xi)^* T W(r\xi).$$

If $\mathcal{S} \subset \mathbb{G}(\Xi)$ is a subsemilattice with $\Xi \in \mathcal{S}$ and $T \mathrel{\widetilde{\in}} \mathscr{F}(\mathcal{S})$ then

$$\mathscr{C}\text{-}\mathrm{Sp}_{\mathrm{ess}}(T) = \bigcup_{E \in \mathcal{S}_{\max}} \mathrm{Sp}(\mathcal{P}_E T).$$

5. Phase space of a finite dimensional real space

Let X be a finite dimensional real vector space.

If $Y \subset X$ linear subspace, $\pi_Y : X \to X/Y$ the natural surjection, embed

$$C_0(X/Y) \subset C_{\mathrm{b}}^{\mathrm{u}}(X) \text{ via } \varphi \mapsto \varphi \circ \pi_Y$$

$C_0(X/Y)$ = set of continuous $\varphi : X \to \mathbb{C}$ with $\varphi(x+y) = \varphi(x)$ if $x \in X, y \in Y$ and such that $\varphi(x) \to 0$ when $\operatorname{dist}(x, Y) \to \infty$.

Lemma. *The family of C^*-subalgebras $C_0(X/Y)$ with $Y \in \mathbb{G}(X)$ is*

(1) *linearly independent,*

(2) $C_0(X/Y) \cdot C_0(X/Z) = C_0(X/(Y \cap Z)) \quad \forall Y, Z \in \mathbb{G}$.

Definition. $\mathcal{G} \equiv \mathcal{G}^X \doteq \sum_{Y \in \mathbb{G}}^{\mathrm{c}} C_0(X/Y) \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ *is the (abelian) Grassmann C^*-algebra of X*; it is $\wedge$-graded by $\mathbb{G}(X)$.

Lemma. If $\mathcal{S} \subset \mathbb{G}$ is finite and $\wedge$-stable then $\mathcal{G}(\mathcal{S}) \doteq \sum_{Y \in \mathcal{S}} C_0(X/Y)$ is a C^*-subalgebra of $\mathcal{G}$.

(Quantum) Grassmann C*-algebra

$\mathscr{G} \doteq \mathcal{G} \cdot C^*(p) =$ closed linear span of $\{\varphi(q)\psi(p) \mid \varphi \in \mathcal{G}, \psi \in C_0(X^*)\}$.

For each $Y \in \mathbb{G}(X)$ set

$\mathscr{G}(Y) \doteq$ closed linear span of $\{\varphi(q)\psi(p) \mid \varphi \in C_0(X/Y), \psi \in C_0(X^*)\}$

Proposition. $\mathscr{G}(Y)$ is a nondegenerate C^*-algebra of operators on $\mathcal{H}$ and
(1) the family $\{\mathscr{G}(Y)$ is linearly independent,
(2) $\mathscr{G}(Y) \cdot \mathscr{G}(Z) = \mathscr{G}(Y \cap Z) \quad \forall Y, Z \in \mathbb{G}$,
(3) $\mathscr{G} = \sum_{Y \in \mathbb{G}}^{\mathrm{c}} \mathscr{G}(Y)$.

This means: the C^*-algebra $\mathscr{G}$ is $\wedge$-graded by $\mathbb{G}$ with components $\mathscr{G}(Y)$.

Proposition. Let Z be a subspace supplementary to Y, so $X = Y \oplus Z$ and $X/Y \cong Z$, and let us identify $\mathcal{H} = L^2(X) = L^2(Y) \otimes L^2(Z)$. Then

$$\mathscr{G}(Y) = C_0(Y^*) \otimes K(L^2(Z)) \cong C_0(Y^*, K(L^2(Z))).$$

In other terms: $T \in B(\mathcal{H})$ belongs to $\mathscr{G}(Y)$ if and only if FTF^{-1} is the operator of multiplication by a function $\widehat{T} : Y^* \to K(L^2(Z))$ of class C_0.

Remarks.

The symplectic space is $\Xi = T^*X = X \oplus X^*$ with symplectic form

$\sigma(\xi,\eta) = \langle y,k\rangle - \langle x,l\rangle$ if $\xi = x+k, \eta = y+l$ with $x,y \in X$ and $k,l \in X^*$.

Note that X and X^* are Lagrangian subspaces of Ξ. We have:

$$\mathscr{G}^X = \mathscr{F}_{\supset X} = \sum\nolimits^{\mathrm{c}}_{E \supset X} \mathscr{F}(E) \quad \text{and} \quad \mathscr{G}(Y) = \mathscr{F}(Y^\sigma)\,\forall Y \subset X.$$

There are very few functions of the position observable in $\mathscr{F}$. Indeed:

$$\varphi(q) \in \mathscr{F} \Leftrightarrow \varphi \in \mathcal{G}^X.$$

Half-Lagrangian decompositions of Ξ

A simple modification of the symplectic form allows one to introduce constant magnetic fields into the formalism and so to treat N-body systems which interact with an external asymptotically constant magnetic field. The constant magnetic field may be interpreted as a bilinear anti-symmetric form $\beta : X \times X \to \mathbb{R}$. The new symplectic form on T^*X

$$\sigma(\xi,\eta) = \beta(x,y) + \langle y,k\rangle - \langle x,l\rangle.$$

HVZ theorem

$\mathbb{G} \equiv \mathbb{G}(X)$ has a least element $0 = \{0\}$ and $\mathbb{G}_{\min} = \mathbb{P}$ = set of one dimensional subspaces L of X (projective space of X).

Lemma. If $T \in \mathscr{G}$ and $a \in L \setminus \{0\}$ then $\mathcal{P}_L(T) =$ s-$\lim_{r\to\infty} \mathrm{e}^{\mathrm{i}rap}\, T\, \mathrm{e}^{-\mathrm{i}rap}$.

Lemma. H = self-adjoint operator affiliated to $\mathscr{G}_Y$ with $Y \neq 0 \Rightarrow$ $\sigma(H)$ is an interval.

Proposition. $\{\mathcal{P}_L(T) \mid L \in \mathbb{P}\}$ is a compact subset of $\mathscr{G}$ for any $T \in \mathscr{G}$.

Theorem. $T \,\widetilde{\in}\, \mathscr{G} \Longrightarrow \mathrm{Sp}_{\mathrm{ess}}(T) = \bigcup_{L\in\mathbb{P}} \mathrm{Sp}(\mathcal{P}_L T)$.

Proof. If $Y \in \mathbb{G}$ and $\mathscr{G}_Y = \sum^{\mathrm{c}}_{Z\supset Y} \mathscr{G}(Z)$ and $\mathscr{G}'_Y = \sum^{\mathrm{c}}_{Z\not\supset Y} \mathscr{G}(Z)$ then $\mathscr{G} = \mathscr{G}_Y + \mathscr{G}'_Y$ direct sum and $\mathcal{P}_Y : \mathscr{G} \to \mathscr{G}_Y$ is the associated projection. Then $T \mapsto (\mathcal{P}_L T)_{L\in\mathbb{P}}$ is a morphism $\mathscr{G} \to \bigoplus_{L\in\mathbb{P}} \mathscr{G}_L$ with kernel $\mathscr{G}(0) = C_0(q)\cdot C_0(p) = K(\mathcal{H})$. Hence $\mathscr{G}/K(\mathcal{H}) \hookrightarrow \bigoplus_{L\in\mathbb{P}} \mathscr{G}_L$.

Intrinsic description of $\mathscr{G}(Y)$

Notations: if $f \in \mathcal{H}$, $x \in X$, $k \in X^*$ then

$$(\mathrm{e}^{\mathrm{i}xp} f)(y) = f(x+y) \quad \text{and} \quad (\mathrm{e}^{\mathrm{i}kq} f)(y) = \mathrm{e}^{\mathrm{i}ky} f(y) \quad \forall y \in X.$$

Simplest cases: $\mathscr{G}(0) = K(\mathcal{H})$ and $\mathscr{G}(X) = C^*(p) = C_0(p)$.

Theorem (really!). $\mathscr{G}(Y)$ is the set of $T \in B(\mathcal{H})$ such that

(i) $[\mathrm{e}^{\mathrm{i}xp}, T] = 0$ for all $x \in Y$,

(ii) $\lim_{x\to 0} \|[\mathrm{e}^{\mathrm{i}xp}, T]\| = 0$ and $\lim_{x\to 0} \|(\mathrm{e}^{\mathrm{i}xp} - 1)T\| = 0$,

(iii) $\lim_{k\to 0} \|[\mathrm{e}^{\mathrm{i}kq}, T]\| = 0$ and $\lim_{k\to 0, k\in Y^\perp} \|(\mathrm{e}^{\mathrm{i}kq} - 1)T\| = 0$.

$\mathscr{G}$ is generated by elementary N-body type Hamiltonians

First, un exercise:

Lemma. $\mathcal{G} = \{\varphi(q) \mid \varphi \in \mathcal{G}\} \subset B(\mathcal{H})$ *is the* C^**-algebra generated by the self-adjoint operators* $\alpha(q)$ *with* $\alpha \in X^*$.

$\mathcal{S}_0 \subset \mathbb{G}$; $\mathcal{S}$ = set of finite intersections of subspaces from $\mathcal{S}_0$ (so $X \in \mathcal{S}$).

$$\mathscr{G}(\mathcal{S}) = \sum\nolimits_{Y \in \mathcal{S}}^{\mathrm{c}} \mathscr{G}(Y)$$

Theorem. Let $h : X^* \to \mathbb{R}$ continuous with $\lim_{k\to\infty} h(k) = \infty$.
Then $\mathscr{G}(\mathcal{S})$ = C^*-algebra generated by the operators $H = h(p+k) + v(q)$ with $k \in X^*$ and $v \in \sum_{Y \in \mathcal{S}_0} C_0(X/Y)$ real.

Proposition. If h is a real elliptic polynomial of order m on X then $\mathscr{G}(\mathcal{S})$ is the C^*-algebra generated by the self-adjoint operators $h(p) + S$, where S runs over the set of symmetric differential operators of order $< m$ with coefficients in $\sum_{Y \in \mathcal{S}_0} C_0^\infty(X/Y)$.

Hamiltonians affiliated to $\mathscr{G}$

$\mathcal{S} \subset \mathbb{G}$ sub-semilattice with $X \in \mathcal{S}$
$h : X^* \to \mathbb{R}$ continuous positive and with $\lim_{k\to\infty} h(k) = \infty$.

Then $h(p)$ is a kinetic energy operator strictly affiliated to $\mathscr{G}(X) = C_0(X^*)$, hence to $\mathscr{G}(\mathcal{S})$.

Let $\mathcal{H}_h = D(h(p)^{1/2})$ be the form domain of $h(p)$ and $\mathcal{H}_h \subset \mathcal{H} \subset \mathcal{H}_h^*$ the associated scale.

Theorem. $\forall Y \in \mathcal{S}$ let $V(Y) \in B(\mathcal{H}_h, \mathcal{H}_h^*)$ a symmetric operator s.t.

(1) $V(X) = 0$ and $(h(p)+1)^{-s} V(Y) (h(p)+1)^{-1/2} \in \mathscr{G}(Y)$ with $s > 1/2$;

(2) the family $\{V(Y)\}_{E \in \mathcal{S}}$ is norm summable in $B(\mathcal{H}_h, \mathcal{H}_h^*)$;

(3) $V(Y) \geq -\mu_{Y} h(p) - \nu_{Y}$ with $\mu_{Y}, \nu_{Y} \geq 0$, $\sum_{Y\in\mathcal{S}} \mu_{Y} < 1$, $\sum_{Y\in\mathcal{S}} \nu_{Y} < \infty$.

Let $V = \sum_{Y\in\mathcal{S}} V(Y)$ and $V_{Y} = \sum_{Z\supset Y} V(Z)$ for any $Y \in \mathcal{S}$. Then $H = h(p) + V$ and $H_{Y} = h(p) + V_{Y}$ are bounded from below self-adjoint operators, with form domain $\mathcal{H}_h$, strictly affiliated to $\mathscr{G}(\mathcal{S})$, and $\mathcal{P}_{Y} H = H_{Y}$ $\forall Y \in \mathcal{S}$. If $0 \in \mathcal{S}$ then

$$\mathrm{Sp}_{\mathrm{ess}}(H) = \cup_{Y\in\mathcal{S}_{\min}} \mathrm{Sp}(\mathcal{P}_{Y} H).$$

Assume that for some $s > 0$

$$c'|k|^{2s} \leq h(k) \leq c''|k|^{2s} \quad \text{for some constants } c', c'' \text{ and all large } k.$$

Sobolev spaces $\mathcal{H}^r$, $r \in \mathbb{R}$.

Theorem. Let $V : \mathcal{H}^s \to \mathcal{H}^{-s}$ symmetric such that $V \geq -\mu h(p) - \nu$ with $\mu < 1, \nu \geq 0$ and $\langle p \rangle^{-t} V \langle p \rangle^{-s} \in \mathscr{G}(\mathcal{S})$ for some $t > s$. Then $H = h(p) + V$ is a self-adjoint operator strictly affiliated to $\mathscr{G}(\mathcal{S})$.

Remark. The condition $t > s$ allows perturbations of a differential operator by operators of the same order and locally irregular. For example, $\Delta + V$ with $V = \sum_{jk} \partial_j g_{jk} \partial_k$ and g_{jk} locally bounded or nonlocal operators with some conditions at infinity is allowed.

6. A remark on the infinite dimensional case

Let Ξ be a complex infinite dimensional Hilbert space and $\mathcal{H} = \Gamma(\Xi)$ the symmetric Fock space associated to it. We keep the notation Ξ for the underlying real vector space of Ξ equipped with the symplectic structure defined by $\sigma(\xi, \eta) = \Im\langle\xi|\eta\rangle$.

Then $\mathscr{F}$ is a C^*-algebra on $\mathcal{H}$ which does not contain compact operators and the usual quantum field Hamiltonians are not affiliated to it. The problem comes from the fact that $\Gamma(A) \notin \mathscr{F}$ if A is a bounded operator on the one particle Hilbert space Ξ.

A solution is to extend $\mathscr{F}$ by adding the necessary *free kinetic energies*. More precisely, if $\mathcal{O}$ is an abelian C^*-algebra on the Hilbert space Ξ whose strong closure does not contain compact operators then

$$\Phi(\mathcal{O}) \doteq C^*(\phi(\xi)\Gamma(A) \mid \xi \in \Xi, A \in \mathcal{O}, \|A\| < 1)$$

is a C^*-algebra of operators on $\mathcal{H}$ which contains the compacts and whose quotient with respect to the ideal of compact operators is canonically embedded in $\mathcal{O} \otimes \Phi(\mathcal{O})$ which allows one to describe the essential spectrum of the operators affiliated to $\Phi(\mathcal{O})$. The Hamiltonians of the $P(\varphi)_2$ models with a spatial cutoff are affiliated to such algebras. The algebra $\mathscr{A} \doteq \Phi(\mathbb{C})$ has a remarkable property: $K(\mathcal{H}) \subset \mathscr{A}$ and $\mathscr{A}/K(\mathcal{H}) \cong \mathscr{A}$.

[V.G. 2007 [9]]

7. Many-body systems

Many-body systems are obtained by coupling a certain number (finite or infinite) of N-body systems. An N-body system consists of a fixed number N of particles which interact through k-body forces which preserve N. The many-body type interactions include forces which allow the system to make transitions between states with different numbers of particles. These transitions are realized by creation-annihilation processes as in quantum field theory.

The main difficulty in the present algebraic approach is to isolate the correct C^*-algebra. *This is especially problematic in the present situations since it is not a priori clear how to define the couplings between the various N-body systems but in very special situations. It is rather remarkable that the C^*-algebra generated by a small class of elementary and natural Hamiltonians will finally prove to be a fruitful choice. These elementary Hamiltonians are analogs of the Pauli-Fierz Hamiltonians.*

[M. Damak and V.G. 2010 [6]]

$\mathcal{X}$ = real prehilbert space; $\mathbb{G}(\mathcal{X})$ = set of finite dimensional subspaces.

$X \in \mathbb{G}$ is an Euclidean space. Set (change of notations: $L(\mathcal{H}) \equiv B(\mathcal{H})$):
$\mathcal{H}_X = L^2(X) \quad \mathscr{L}_X = L(\mathcal{H}_X) \quad \mathscr{K}_X = K(\mathcal{H}_X) \quad \mathscr{T}_X = C^*(p_x)$

If $X, Y \in \mathbb{G}$ set $\mathscr{L}_{XY} = L(\mathcal{H}_Y, \mathcal{H}_X)$ and $\mathscr{K}_{XY} = K(\mathcal{H}_Y, \mathcal{H}_X)$.

If $\varphi \in \mathcal{C}_{\mathrm{c}}(X+Y)$ then $(T_{XY}(\varphi)f)(x) = \int_Y \varphi(x-y)f(y)\mathrm{d}y$ defines a continuous operator $\mathcal{H}_Y \to \mathcal{H}_X$. Define $\mathscr{T}_{XY}$ by (clearly $\mathscr{T}_{XX} = \mathscr{T}_X$)

$\mathscr{T}_{XY}$= norm closure of the set of operators $T_{XY}(\varphi)$ with $\varphi \in \mathcal{C}_{\mathrm{c}}(X+Y)$.

Fix a sub-semilattice $\mathcal{S} \subset \mathbb{G}$. For each $X \in \mathcal{S}$ the Hilbert space $\mathcal{H}_X$ is thought as the state space of an N-body system with X as configuration space. The state space of the many-body system is

$$\mathcal{H} \equiv \mathcal{H}_{\mathcal{S}} = \oplus_{X \in \mathcal{S}} \mathcal{H}_X.$$

We have a natural embedding $\mathscr{L}_{XY} \subset L(\mathcal{H})$ and define

$$\mathscr{L} \equiv \mathscr{L}_{\mathcal{S}} = \text{ closed linear span of the subspaces } \mathscr{L}_{XY}.$$

This is a C^*-subalgebra of $L(\mathcal{H})$ equal to $L(\mathcal{H})$ if and only if $\mathcal{S}$ is finite.

We will be interested in subspaces $\mathscr{R}$ of $\mathscr{L}$ constructed as follows: for each couple X, Y we are given a closed subspace $\mathscr{R}_{XY} \subset \mathscr{L}_{XY}$ and $\mathscr{R} \equiv (\mathscr{R}_{XY})_{X,Y\in\mathcal{S}} = \sum^{\mathrm{c}}_{X,Y\in\mathcal{S}} \mathscr{R}_{XY}$.

Note that $\mathscr{K} \equiv \mathscr{K}_{\mathcal{S}} = (\mathscr{K}_{XY})_{X,Y\in\mathcal{S}} = K(\mathcal{H})$.

Theorem. $\mathscr{T} \equiv \mathscr{T}_{\mathcal{S}} = (\mathscr{T}_{XY})_{X,Y\in\mathcal{S}}$ is a closed self-adjoint subspace of $\mathscr{L}$ and $\mathscr{C} \equiv \mathscr{C}_{\mathcal{S}} = \mathscr{T}^2$ is a non-degenerate C^*-algebra of operators on $\mathcal{H}$.

We say that $\mathscr{C}$ *is the Hamiltonian algebra of the many-body system* $\mathcal{S}$.
We equip $\mathscr{C}$ with an $\mathcal{S}$-graded C^*-algebras structure. Let

$$\mathcal{C}_X(Y) \cong \mathcal{C}_{\mathrm{o}}(X/Y) \text{ if } Y \subset X \quad \text{and} \quad \mathcal{C}_X(Y) = \{0\} \text{ if } Y \not\subset X.$$

Then $\mathcal{C}_X \equiv \mathcal{C}_X(\mathcal{S}) := \sum^{\mathrm{c}}_{Y\in\mathcal{S}} \mathcal{C}_X(Y) \subset \mathscr{L}_X$ by $\varphi = \varphi(q_X)$. Then

$$\mathcal{C} \equiv \mathcal{C}_{\mathcal{S}} = \oplus_{X\in\mathcal{S}} \mathcal{C}_X$$

is a C^*-algebra of operators on $\mathcal{H}$ included in $\mathscr{L}$. If $Z \in \mathcal{S}$ let

$$\mathcal{C}(Z) \equiv \mathcal{C}_{\mathcal{S}}(Z) = \oplus_X \mathcal{C}_X(Z) = \oplus_{X\supset Z} \mathcal{C}_X(Z) = C^*\text{-subalgebra of } \mathcal{C}.$$

Theorem. We have $\mathscr{C} = \mathscr{T} \cdot \mathcal{C} = \mathcal{C} \cdot \mathscr{T}$. For each $Z \in \mathcal{S}$ the space $\mathscr{C}(Z) = \mathscr{T} \cdot \mathcal{C}(Z) = \mathcal{C}(Z) \cdot \mathscr{T}$ is a C^*-subalgebra of $\mathscr{C}$ and the family $\{\mathscr{C}(Z)\}_{Z\in\mathcal{S}}$ defines a graded C^*-algebra structure on $\mathscr{C}$.

8. ELLIPTIC C*-ALGEBRA

$X =$ **finite dimensional real vector space.**

Elliptic algebra $\mathscr{E}(X)$: defined by the following equivalent conditions:

(i) $\mathscr{E} = C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$;

(ii) $\mathscr{E} = \{T \in \mathscr{B} \mid \lim_{a\to 0} \|(\mathrm{e}^{\mathrm{i}ap} - 1)T^{(*)}\| = 0,\ \lim_{a\to 0} \|[T, \mathrm{e}^{\mathrm{i}aq}]\| = 0\}$;

(iii) $\mathscr{E} = C_{\mathrm{b}}^{\mathrm{u}}(X) \cdot C_0(X^*)$;

(iv) $\mathscr{E}$ = C^*-algebra generated by the operators $h(p)+S$, where h is a fixed real elliptic polynomial of order m on X and S runs over the set of symmetric differential operators of order $< m$ with coefficients in $C_{\mathrm{b}}^{\infty}(X)$.

Let $X^\dagger$ = set of all ultrafilters finer than the Fréchet filter on X.

Theorem. If $T \,\widetilde{\in}\, \mathscr{E}$ then s-$\lim_{x\to\varkappa} \mathrm{e}^{\mathrm{i}xp} A \mathrm{e}^{-\mathrm{i}xp} \doteq A_\varkappa$ exists $\forall \varkappa \in X^\dagger$ and

$$\mathrm{Sp}_{\mathrm{ess}}(A) = \cup_{\varkappa \in X^\dagger} \mathrm{Sp}(A_\varkappa).$$

Generalization

Let $C_\infty(X) \subset \mathcal{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ a translation invariant C^*-subalgebra.
Crossed product: $\mathscr{A} = \mathcal{A} \rtimes X = \mathcal{A} \cdot C_0(X^*)$.

Let $h(p)$ be a real elliptic polynomial of order m on X. Then $\mathscr{A}$ is the C^*-algebra generated by the self-adjoint operators of the form $h(p) + S$, where S runs over the set of symmetric differential operators of order $< m$ with coefficients in $\mathcal{A}^\infty = \{\varphi \in C^\infty(X) \mid \varphi^{(\alpha)} \in \mathcal{A}\, \forall \alpha\}$.

The *character space*, or *spectrum*, of $\mathcal{A}$ is the compact space $\sigma(\mathcal{A})$ consisting of nonzero morphisms $\mathcal{A} \to \mathbb{C}$ equipped with the weak* topology inherited from the embedding $\sigma(\mathcal{A}) \subset$ dual of $\mathcal{A}$.

Each $x \in X$ defines a character $\chi_x : \varphi \mapsto \varphi(x)$ and the map $x \mapsto \chi_x$ is a homeomorphism of X onto an open dense subset of $\sigma(\mathcal{A})$ that we identify with X. *The boundary of X in $\sigma(\mathcal{A})$* is the compact set

$$\mathcal{A}^\dagger = \sigma(\mathcal{A}) \setminus X = \{\varkappa \in \sigma(\mathcal{A}) \mid \varkappa(\varphi) = 0\, \forall \varphi \in C_0(X)\}.$$

Theorem. For any $A \in \mathscr{A}$ the map $x \mapsto A_x \doteq \mathrm{e}^{\mathrm{i}xp} A \mathrm{e}^{-\mathrm{i}xp}$ is norm continuous and extends to a continuous map $\sigma(\mathcal{A}) \ni \chi \mapsto A_\chi \in \mathscr{E}_{\mathrm{loc}}$ and

$$\tau_\varkappa(A) = 0 \,\forall \varkappa \in \mathcal{A}^\dagger \Longleftrightarrow A \in \mathscr{K}(X).$$

Thus the map $\Phi(A) = (A_\varkappa)_{\varkappa \in \mathcal{A}^\dagger}$ defines a morphism

$$\Phi : \mathscr{A} \to \bigoplus\nolimits_{\varkappa \in \mathcal{A}^\dagger} \mathscr{E}$$

whose kernel is $K(\mathcal{H})$ hence it induces an embedding

$$\widehat{\Phi} : \mathscr{A}/\mathscr{K} \hookrightarrow \bigoplus\nolimits_{\varkappa \in \mathcal{A}^\dagger} \mathscr{E}.$$

Theorem. For any $A \,\widetilde{\in}\, \mathscr{A}$ we have

$$\mathrm{Sp}_{\mathrm{ess}}(A) = \cup_{\varkappa \in \mathcal{A}^\dagger} \mathrm{Sp}(A_\varkappa).$$

$X =$ **locally compact non-compact abelian group.**

Let U_x be the operator of translation by $x \in X$ and V_k the operator of multiplication by the character $k \in X^*$. Then

$$\mathscr{C}(X) = \{T \in \mathscr{B}(X) \mid \lim_{k \to 0} \|[T, V_k]\| = 0 \text{ and } \lim_{x \to 0} \|(U_x - 1)T^{(*)}\| = 0\}$$

is a C^*-algebra. $\mathscr{C}_s(X)$ is $\mathscr{C}(X)$ equipped with the topology defined by the seminorms $\|S\|_\theta = \|S\theta(q)\| + \|\theta(q)S\|$ with $\theta \in C_0(X)$.

Theorem Let H be an observable affiliated to $\mathscr{C}(X)$. For each $\varkappa \in X^\dagger$ the limit $\varkappa.H := \lim_{x \to \varkappa} x.H$ exists in the following sense: there is an observable $\varkappa.H$ affiliated to $\mathscr{C}(X)$ such that $\lim_{x \to \varkappa} U_x \varphi(H) U_x^* = \varphi(\varkappa.H)$ in $\mathscr{C}_s(X)$ for all $\varphi \in C_0(\mathbb{R})$. We have

$$\mathrm{Sp}_{\mathrm{ess}}(H) = \overline{\bigcup}_{\varkappa} \mathrm{Sp}(\varkappa.H).$$

General metric spaces

V.G. *On the structure of the essential spectrum of elliptic operators on metric spaces*; J. Func. Analysis 260:1734-1765 (2011).

Description of the essential spectrum of a large class of operators on metric measure spaces, analogues of the elliptic operators on Euclidean spaces, in terms of their *localizations at infinity*. The main result concerns the ideal structure of the C^*-algebra generated by them.

9. Mourre estimate

A, H = self-adjoint operators on a Hilbert space $\mathcal{H}$.

H is of class $C^1(A)$ or $C^1_{\rm u}(A)$ if $\tau \mapsto e^{-{\rm i}\tau A}(H+{\rm i})^{-1}e^{{\rm i}\tau A}$ is strongly C^1 or norm C^1_u respectively. Then $D(A) \cap D(H)$ is a core for H and $[H, {\rm i}A]$ extends to a continuous sesquilinear form on $D(H)$.

H satisfies *Mourre estimate* at $\lambda \in \mathbb{R}$ if there are: a number $c > 0$, a real function $\varphi \in C_{\rm c}(\mathbb{R})$ with $\varphi(\lambda) \neq 0$, and a compact operator K, such that

$$\varphi(H)[H, {\rm i}A]\varphi(H) \geq c\varphi(H)^2 + K.$$

If this holds with $K = 0$ say that H satisfies a *strict Mourre estimate* at λ

Remark. We have $\varphi(H)[H, {\rm i}A]\varphi(H) = \varphi(H)[\psi(H), {\rm i}A]\varphi(H)$ if $\psi \in C^\infty_{\rm c}(\mathbb{R})$ and $\psi(x) = x$ on $\operatorname{supp}\varphi$, hence the Mourre estimate can be expressed in terms of the observable H.

The set of A*-thresholds of* H is the closed set

$$\tau_A(H) = \{\lambda \in \mathbb{R} \mid H \text{does not satisfy a Mourre estimate at } \lambda\}$$

and set of A*-critical points of* H is the closed set

$$\kappa_A(H) = \{\lambda \in \mathbb{R} \mid H \text{ does not satisfy a strict Mourre estimate at } \lambda \}.$$

Define $\tilde{\rho}_H$ and ρ_H as the functions $\mathbb{R} \to (-\infty, +\infty]$ defined as follows:

$\widetilde{\rho}_H(\lambda)$ = upper bound of the numbers c s.t. the Mourre estimate holds for some φ, K;

$\rho_H(\lambda)$ = upper bound of the numbers c s.t. the strict Mourre estimate holds for some φ.

Then $\tau_A(H) = \{\lambda \in \mathbb{R} \mid \tilde{\rho}_H(\lambda) \leq 0\}$ and $\kappa_A(H) = \{\lambda \in \mathbb{R} \mid \rho_H(\lambda) \leq 0\}$.

Proposition. $\rho_H(\lambda) = \widetilde{\rho}_H(\lambda)$ *with the exception of the points* λ *which are eigenvalues of* H *and* $\tilde{\rho}_H(\lambda) > 0$*; at these points* $\rho_H(\lambda) = 0$.

In particular, $\rho_H(\lambda) > 0$ if and only if $\tilde{\rho}_H(\lambda) > 0$ and $\lambda \notin \sigma_p(H)$. In other terms

$$\kappa_A(H) = \tau_A(H) \cup \sigma_p(H).$$

Theorem. Let $\mathcal{S}$ a $\wedge$-semilattice with a least element o and atomic and $\mathscr{C} \subset B(\mathcal{H})$ an $\mathcal{S}$-graded C^*-algebra such that:

(i) each $\mathscr{C}(a)$ is nondegenerate on $\mathcal{H}$;

(ii) $\mathscr{C}(o) = K(\mathcal{H})$;

(iii) $\forall S \in \mathscr{C}$ the set $\{\mathcal{P}_a(S) \mid a \in \mathcal{S}_{\min}\}$ is compact in $\mathscr{C}$;

(iv) $\forall a \in \mathcal{S}$ and $\tau \in \mathbb{R}$ we have $e^{-\mathrm{i}\tau A}\mathscr{C}(a)\mathrm{e}^{\mathrm{i}\tau A} \subset \mathscr{C}(a)$.

Let H be a self-adjoint operator strictly affiliated to $\mathscr{C}$ and of class $C^1_{\mathrm{u}}(A)$ and let $H_a = \mathcal{P}_a(H)$. Then H_a is a self-adjoint operator strictly affiliated to $\mathscr{C}_a$, of class $C^1_{\mathrm{u}}(A)$ and

$$\tilde{\rho}_H(\lambda) = \min_{a \in \mathcal{S}_{\min}} \rho_{H_a}(\lambda)\ \forall \lambda \quad \text{and} \quad \boxed{\tau_A(H) = \cup_{a \in \mathcal{S}_{\min}} \kappa_A(H_a).}$$

Idea. Need to think in terms of observables. Let $\tilde{H} = \mathcal{P}(H)$ where $\mathcal{P} : \mathscr{C} \to \mathscr{C}/K(\mathcal{H})$. The $\mathrm{e}^{\mathrm{i}\tau A}$ induce a group automorphisms of $\mathscr{C}/K(\mathcal{H})$, the observable $\tilde{H}$ is of class C^1_{u} with respect to this group, and the $\tilde{\rho}$ of H is the ρ of $\tilde{H}$. Finally, the ρ for a direct sum of H_a is $\inf_a \rho_a$ (due to (iii)).

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