Welcome to the workshop High Frequency Analysis, from Operator Algebras to PDEs

Analysis in subelliptic and subRiemannian settings, Challenges and recent approaches

Véronique Fischer (University of Bath, UK)

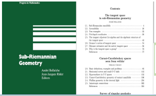
Université d'Angers, August 28th - September 1st, 2023

Historical perspectives - XXth century

Caratheodory '09

Cartan '28

Chow-Rashevskii '38



'92

Agrachev '94

Carnot dynamics

CR, contact, non-holonomic geometries
Control theory

SubRiemannian geometry

Hörmander '67

Folland '75
Rotschild+Stein '76

Helffer+Nourrigat '79

Christ+Geller+Glowacki+Polin '92

Analysis of subelliptic operators

Harmonic analysis on nilpotent Lie groups

With microlocal analysis

Fefferman+Phong '81, then Sanchez-Calle, Parmeggiani etc

Maliavin '81

Ben Arous '88

Probability

Hypoelliptic stochastic PDEs

Definitions?

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A pseudodifferential operator T is elliptic when

its principal symbol is invertible, or equivalently

$$u \in \mathcal{D}', \ Tu \in H^s_{loc} \Longrightarrow u \in H^{s+m}_{loc}, \qquad \text{where } m = \text{order } T, \ H^s \text{ Sobolev.}$$

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A pseudodifferential operator T is

- hypoelliptic when Tu smooth $\Longrightarrow$ u smooth.
- subelliptic with loss of $\delta \in (0,1)$ derivatives when $Tu \in H^s_{loc} \Longrightarrow u \in H^{s+m-\delta}_{loc}$.

Subelliptic operators?

Hörmander '67

Let $X_0, X_1, \dots, X_r$ be real vector fields on M. If X_i, $[X_{i_1}, X_{i_2}]$, …, $[X_{i_1}, [X_{i_2}, X_{i_3}, \dots, X_{j_k}]]], \dots$ generate $\mathbb{T}M$, then $X_0 + \sum_{i=1}^r X_i^2$ is hypoelliptic.

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Examples of $X_0 + \sum_{i=1}^r X_i^2$

- subLaplacians on sR M.
- The magnetic Laplacian in $\mathbb{R}^2$ (Montgomery '95)

$$H(\lambda) = (\partial_x - i\lambda A_x(x,y))^2 + (\partial_y - i\lambda A_y(x,y))^2 \rightsquigarrow H = (\partial_x - A_x\partial_z)^2 + (\partial_y - A_y\partial_z)^2.$$

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Rothschild+Stein '76, with $\mathscr{L} = -\sum_i X_i^2$, (X_i) Hörmander

- $\mathscr{L}u \in H^s_{loc} \Longrightarrow u \in H^{s+2/r}_{loc}$, where $r =$ rank of X_i's (optimal).
- $\mathscr{L}u \in L^2_{s,loc} \Longrightarrow u \in L^2_{s+2,loc}$, where $L^2_{s,loc}$ Sobolev spaces adapted to X_i's.

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$\rightsquigarrow$ analysis of e.g. polynomials in (X_i)

Challenges (still in XXth century)

1. (X_i) noncommutative.
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2. Replace the notions of local coordinates & tangent space.
   - Rothschild & Stein '76 $\rightsquigarrow$ osculating nilpotent Lie group.
   - Mitchell '85 the metric tangent space above of point of a regular sR M is a nilpotent Lie group 'nilpotentisation'.
   - Bellaïche '92 singular sR M and priviledged coordinates.

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3. Replace the notion of ellipticity?
   - With microlocal analysis? but even Fefferman+Phong '81, 2nd order diff. op. with semidefinite princ. symbol $\rightsquigarrow$ nilpotentisation.
   - Rockland condition on nilpotent Lie groups.
     Hellfer+Nourrigat '79, CGGP '92.

Challenges

1. Singular case?

   (self-adjointness of subLaplacian, heat kernel expansion, Rumin complex etc. are known on regular sR M)

2. Geometric spectral theory for sR M?

   - limits of $|\varphi_j(x)|^2 dx$ where φ_j eigenfunctions of a subLaplacian $\mathscr{L}$?
   - semiclassical or microlocal analysis for $\mathscr{L}$?
   - problems with $e^{it\mathscr{L}}$, Egorov...
   - recent progress on pseudodifferential calculi on sR M or from (X_i), many links with operators algebras.

3. Sections of vector bundles adapted to sR structures?

   - on forms: sR cohomology / Hodge theory?
   - sR tensors should appear in geometric spectral theory but also in examples from gauge theory (theoretical physics)...

Pseudodifferential operators on sR M or from (X_i)

Main examples: CR or contact manifold

More generally, Heisenberg manifold, i.e. nilpotentisation Heis $\times \mathbb{R}^n$.

- Beals+Greiner '88 *Calculus on Heisenberg manifolds*
- Ponge '08 *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*

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Two recent breakthroughs:

1. Groupoid techniques (see next slide)
2. Symbolic calculus on nilpotent Lie groups (Fischer+Ruzhansky) with applications to spectral theory (with Fermanian). Rest of the talk.

A word about the groupoid approach

- Connes '94 tangent groupoid of a compact Riemannian manifold (equivalent description of Hörmander's $\Psi_{cl}(M)$ for index theory)
- van Erp '10 index formula on contact manifold.
  idea: groupoid but tangent space replaced with Heisenberg group.
- Debord+Skandalis '14 deeper understanding & description of Lie groupoids with dilations.
- van Erp+Yuncken '17 Pseudodifferential calculus on regular sR (more generally filtered) M via osculating groupoids
- Androulidakis+Mohsen+Yuncken '22 *A pseudodifferential calculus for maximally hypoelliptic operators and the Helffer-Nourrigat conjecture*

The symbolic approach

Abstract Fourier analysis on a (reasonable) group G, Dixmier 60's

The *Fourier transform* of $f \in L^1(G)$ is $\widehat{f}(\pi) = \int_{x\in G} f(x)\pi(x)^*\, dx \in \mathscr{L}(\mathcal{H}_\pi)$,

for π rep. of G, thus $\pi \in \widehat{G} = \{\text{unirrep's}\}/\sim$.

Plancherel formula: $\|f\|^2_{L^2(G)} = \int_{\pi\in\widehat{G}} \|\widehat{f}(\pi)\|^2_{HS(\mathcal{H}_\pi)}\, d\mu(\pi)$.

$\rightsquigarrow$ *Kohn-Nirenberg quantization, Taylor '84*

for a symbol $\sigma(x,\pi)$, $x\in G, \pi\in\widehat{G}$,

$$\mathrm{Op}(\sigma) f(x) = \int_{\pi\in\widehat{G}} \mathrm{tr}\left(\pi(x)\sigma(x,\pi)\widehat{f}(\pi)\right) d\mu(\pi), \quad x\in G.$$

Formally, $\sigma(x,\pi) = \widehat{\kappa}_x(\pi)$ where $\mathrm{Op}(\sigma) f(x) = f * \kappa_x(x)$.

Pseudo-differential calculus on G graded nilpotent Lie groups (Fischer+Ruzhansky '16)

Symbol classes S^m à la Hörmander

$$\forall \alpha, \beta \quad \forall (x,\pi) \in G \times \widehat{G} \quad \|\pi(\mathrm{Id}+\mathscr{R})^{\frac{[\alpha]-m}{\nu}} X_x^\beta \Delta^\alpha \sigma(x,\pi)\|_{\mathscr{L}(\mathscr{H}_\pi)} \le C_{\alpha,\beta},$$

where $\mathscr{R}$ is a positive Rockland operator of degree ν. $(m \in \mathbb{R}.)$
Difference operator $\Delta^\alpha \sigma\ (\pi) = \pi(x^\alpha \kappa)$ where $\sigma(\pi) = \widehat{\kappa}(\pi) = \pi(\kappa)$.

Symbolic calculus

The resulting classes of operators $\Psi^m := \mathrm{Op}(S^m)$, $m \in \mathbb{R}$, form a pseudo-differential calculus, i.e. algebra of operators, stable by $*$,

+ symbolic asymptotics,

containing the left-invariant differential calculus and
acting continuously on the adapted Sobolev spaces $L^p_s(G) \to L^p_{s-m}(G)$.
Rockland condition $\rightsquigarrow$ parametrices.

Case of the Heisenberg group $\leftrightarrow$ Bahouri+Fermanian+Gallagher '12.

With Clotilde: what about phase-space analysis like MDM?

In the Riemannian / Euclidean setting,

Microlocal Defect Measures, Gérard '91, Tartar '90

Let $(f_j)_{j\in\mathbb{N}} \subset L^2(M)$ with $\|f_j\|_{L^2} = 1$ and $f_j \to 0$ weakly.
$\exists\ (j_k)_{k\in\mathbb{N}}$ and μ positive Radon measure on $\mathbb{S}M^*$ such that

$$(Af_j, f_j)_{L^2} \longrightarrow_{j=j_k,\ k\to\infty} \int_{\mathbb{S}M^*} a_0(q,p)\, d\mu(q,p),$$

for all classical pseudodifferential operator A (principal symbol a_0).
Applications: first PDE's but then Quantum Ergodicity.

Semi-classical (aka Wigner) measures

Let $(f_\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(X=\mathbb{R}^n)$. Then $\exists\ (\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to_{k\to\infty} 0$ and a positive Radon measure μ on $X\times X^*$ s.t.

$$\forall a\in C_c^\infty(X\times X^*) \qquad (\mathrm{Op}^\varepsilon(a) f^\varepsilon, f^\varepsilon)_{L^2} \longrightarrow_{\varepsilon=\varepsilon_k\to 0} \int_{\mathbb{R}^n\times\mathbb{R}^n} a(q,p)\, d\mu(q,p).$$

Application to e.g. the semi-classical Schrödinger equation.

⇝ Quantum Limits

Our definition

An accumulation point of functionals $A \longmapsto (Af_j, f_j)_{\mathcal{H}}$ for a given sequence (f_j) of unit vectors in a Hilbert $\mathcal{H}$.

This object is often a state on a space of symbols, hence a measure in the commutative case, as for the microlocal defect and semi-classical measures…

Micro-local Defect Measure as quantum limits

Microlocal Defect Measures, Gérard '91, Tartar '90

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for all classical pseudodifferential operator A (principal symbol a_0).

Alternative pf of $\exists$ - thanks to V. Georgescu explaining works by Cordes:

- $\ell_j : A \mapsto (Af_j, f_j)$ states on a subC^*algebra $\mathscr{A}$ of $\mathscr{L}(L^2(X))$ s.t. $\lim_{j\to\infty} \ell_j(A) = 0$ for all $A \in \mathscr{K} = \{\text{compact op. in } \mathscr{A}\}$.
- Let $\mathscr{A}$ = the subC^*algebra of $\mathscr{L}(L^2(X))$ generated by the classical PDO's. The symbol space $\mathscr{A}/\mathscr{K}$ is abelian. Pass to the limit.

Alternative pf of $\exists$ of semi-classical measures even simpler!!!

Quantum limits on graded Lie groups G

Microlocal Defect Measures on G, Fermanian+F., '20

Let $(f_j)_{j\in\mathbb{N}} \subset L^2(G)$ with $\|f_j\|_{L^2} = 1$ and $f_j \to 0$ weakly. $\exists$ $(j_k)_{k\in\mathbb{N}}$ and an operator valued measure $\Gamma(x,\dot\pi)d\gamma(x,\dot\pi)$, $x \in$, $\dot\pi \in \Sigma_{\widehat{G}} := \widehat{G} \setminus \{1_{\widehat{G}}\}/\mathbb{R}^+(x,\dot\pi)$ such that

$$(Af_j, f_j)_{L^2} \longrightarrow_{j=j_k,\ k\to\infty} \int_{G\times\Sigma_{\widehat{G}}} \mathrm{tr}\left(\sigma_0(x,\dot\pi)\Gamma(x,\dot\pi)\right) d\gamma(x,\dot\pi),$$

for all 'classical' FR-pseudodiff. op. A on G (principal symbol σ_0).

- Main idea of the proof: determining the states of C^*-algebra generated by the 'classical' pseudo-differential calculus on G modulo {compact operators} $\sim$ {operators of < 0 order}.
- Fermanian+F. '19 Semiclassical measures on G.
- Proof without Gårding inequality although we can prove it Benedetto+Fermanian+F. '19.

Typical application of quantum limits

MDM on compact manifolds helps with

$$\mathscr{L}\psi_j = \lambda_j\psi_j, \quad j = 0, 1, 2, \ldots, \qquad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \|\psi_j\|_{L^2(M)} = 1,$$

and describe (properties of) the weak limits of $|\psi_j(x)|^2 dx$, $j \to \infty$.

Semiclassical measure helps with

Schrödinger equation $i\varepsilon\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\mathscr{L}\psi^\varepsilon$, $(\psi^\varepsilon)_{|t=0} = \psi_0^\varepsilon$,

and describe (properties of) the weak limits of $|\psi^\varepsilon(x)|^2 dx$, $\varepsilon \to 0$.

On sR settings, works with more standard analysis

- QE by Zelditch '97 in a few cases in complex analysis, by Colin de Verdière+Hillairet+Trélat '18 on 3D contact M with ergodic Reeb flow, see also Savale on 4D quasi-contact, Letrouit on products of 3-dim. nilHeisenberg manifolds.
- on Grushin operators, Burq+Sun and Arnaiz et al.

Fermanian+F., JST 2021

Setting

Let G be a H-type group, $\mathscr{L}$ its subLaplacian.
Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$ satisfying

$$\exists s, C_s > 0, \qquad \forall \varepsilon > 0 \qquad \varepsilon^s \|\mathscr{L}^{\frac{s}{2}} \psi_0^\varepsilon\|_{L^2(G)} + \varepsilon^{-s} \|\mathscr{L}^{-\frac{s}{2}} \psi_0^\varepsilon\|_{L^2(G)} \leq C_s.$$

Let $(\psi^\varepsilon)_{\varepsilon>0}$ be the associated solutions to the Schrödinger equation

$$i\varepsilon^\tau \partial_t \psi^\varepsilon = -\frac{\varepsilon^2}{2} \mathscr{L} \psi^\varepsilon, \quad (\psi^\varepsilon)_{|t=0} = \psi_0^\varepsilon \qquad \text{(with parameter } \tau > 0\text{)}.$$

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A parenthesis: compare with the Euclidean case, i.e. $\mathscr{L} = \Delta$ on $\mathbb{R}^d$:

Any weak limits of $|\psi^\varepsilon(x,t)|^2 dx\, dt$ is of the form $d\varrho_t(x) \otimes dt$, with

- if $\tau \in (0,1)$, then $\varrho_t = \varrho_0$,
- if $\tau = 1$, then $\varrho_t(x) = \int_{\mathbb{R}^d} \mu_0(x - t\xi, d\xi)$,
- if $\tau > 1$, then $\varrho_t = 0$.

Result for densities

$G = \exp \mathfrak{g}$ is step-2: $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$.

Any weak limits of $|\psi^\varepsilon(x,t)|^2 dx\, dt$ is of the form $d\varrho_t(x) \otimes dt$, with $\varrho_t = \varrho_t^{\mathfrak{v}^*} + \varrho_t^{\mathfrak{z}^*}$ satisfying

- If $\tau \in (0,1)$, $\varrho_t^{\mathfrak{v}^*} = \varrho_0^{\mathfrak{v}^*}$ and $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$.
- If $\tau = 1$, then $\varrho_t^{\mathfrak{v}^*}(x) = \int_{\mathfrak{v}^*} \varsigma_0 \left(x \mathrm{Exp}(t\omega \cdot V), d\omega \right)$ and $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$.
- If $\tau \in (1,2)$, then $\varrho_t^{\mathfrak{v}^*} = 0$ and $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$.
- If $\tau = 2$, then $\varrho_t^{\mathfrak{v}^*} = 0$ and $\varrho_t^{\mathfrak{z}^*} = \sum_{n=0}^{\infty} \int_{\mathfrak{z}^* \setminus \{0\}} \gamma_{n,t}(x, d\lambda)$ with

$$\partial_t \gamma_{n,t} - (n + \frac{d}{2}) \mathcal{Z}^{(\lambda)} \gamma_{n,t} = 0,$$

  where $\mathcal{Z}^{(\lambda)}$ livf $\leftrightarrow |\lambda|^{-1} \lambda \in \mathfrak{z}^*$.
- If $\tau > 2$, then $\varrho_t = 0$.

This is a consequence of a more precise result: Euclidean case

For $\mathscr{L} = \Delta$ on $\mathbb{R}^d$:

Any weak limits of $|\psi^\varepsilon(x,t)|^2 dx\,dt$ is of the form $d\varrho_t(x) \otimes dt$, with

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- if $\tau = 1$, then $\varrho_t(x) = \int_{\mathbb{R}^d} \mu_0(x - t\xi, d\xi)$,
- if $\tau > 1$, then $\varrho_t = 0$.

is a consequence of

On $\mathbb{R}^d$, any semi-classical measure of $(\psi^\varepsilon)_{\varepsilon>0}$ is of the form $d\mu_t(x,\xi) \otimes dt$ s.t.

- if $\tau \in (0,1)$, then $\partial_t \mu_t = 0$,
- if $\tau = 1$, then $\partial_t \mu_t(x,\xi) = \xi \cdot \nabla_x \mu_t(x,\xi)$
- if $\tau > 1$, then $\mu_t = 0$.

This is a consequence of the more precise result:

On G, any semi-classical measures of $(\psi^\varepsilon)_{\varepsilon>0}$ is an operator valued measure $\Gamma_t d\gamma_t \otimes dt$ satisfying:

(i) $\Gamma_t(x,\lambda) = \sum_{n\in\mathbb{N}} \Gamma_{n,t}(x,\lambda)$ with $\Gamma_{n,t}(x,\lambda) := \Pi_n^{(\lambda)} \Gamma_t(x,\lambda) \Pi_n^{(\lambda)}$, where the $\Pi_n^{(\lambda)}$'s are the eigenprojectors for $\widehat{\mathscr{L}}$.

(ii) Above $\lambda \neq 0$,

1. if $\tau \in (0,2)$, then $\partial_t \left(\Gamma_{n,t}(x,\lambda) d\gamma_t(x,\lambda)\right) = 0$,
2. if $\tau = 2$, then $\left(\partial_t - \frac{2n+d}{2|\lambda|} \mathcal{Z}^{(\lambda)}\right) \left(\Gamma_{n,t}(x,\lambda) d\gamma_t(x,\lambda)\right) = 0$,
3. if $\tau > 2$, then $\Gamma_{n,t} d\gamma_t = 0$.

(iii) Above $\lambda = 0$, $d\varsigma_t(x,\omega) = \Gamma_t(x,(0,\omega)) d\gamma_t(x,(0,\omega)) \mathbf{1}_{\lambda=0}$ satisfies:

1. if $\tau \in (0,1)$, then ς_t is constant in t,
2. if $\tau = 1$, then $\varsigma_t(x,\omega) = \varsigma_0 \left(\mathrm{Exp}(t \sum_{j=1}^d \omega_j V_j) x, \omega\right)$,
3. if $\tau > 1$, ς_t is supported on $G \times \{\omega = 0\}$.

Invariance of semi-classical measures on two-step nilmanifolds M

Fermanian+F.+Flynn '22

We consider a sequence of eigenfunctions on $M = \Gamma \backslash G$:

$$\mathscr{L}\psi_j = \lambda_j \psi_j, \quad j = 0, 1, 2, \ldots, \qquad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \|\psi_j\|_{L^2(M)} = 1,$$

and describe some properties of the weak limits ϱ of $|\psi_j(x)|^2 dx$.

Slightly more precisely, $G = \exp \mathfrak{g}$ step two, so $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$.
Then $\varrho = \varrho^{\mathfrak{v}^*} + \varrho^{\mathfrak{z}^*}$ with invariance properties for $\varrho^{\mathfrak{v}^*}$ and $\varrho^{\mathfrak{z}^*}$ depending on the group structure.

Acknowledgments and Future Directions

Quantum limits for subelliptic operators

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Postdocs: Dr Steven Flynn, Dr Søren Mikkelsen.

Future Directions

- More sR geometry
  (eg symbolic pseudodifferential calculus on sR manifolds with applications)
- Much more geometry (with tensor calculi adapted to sR)
- More operator algebra to understand quantum limits as states
  (eg classification of C^*-algebras generated by pseudodifferential calculi on $\mathbb{R}^n$ and on G)

Thank you for your attention.