Weyl Calculus on graded groups

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PLAN OF THE TALK

- Quick overview of the Euclidean Weyl calculus, my motivations, and known results in the graded LG setting
- Preliminaries on Lie groups and the Kohn–Nirenberg quantization by F.–R. and F.–F.K.
- τ- quantizations and τ- calculus on graded Lie groups:

  Admissible quantizing functions & symmetry functions.

  Quantizations

  change of quantization

  Asymptotic formulas
- Candidate Weyl quantization and comparison with the Euclidean case
- The Weyl quantization in $\mathbb{H}_n$

Quantizations on $\mathbb{R}^n$

Given $\sigma \in \mathcal{S}'(T^*\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^{2n})$ one wants to find a formula, usually called **quantization**, to associate to σ an operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Examples of quantizations on $\mathbb{R}^n$

τ- quantizations

$$Op_\tau(\sigma)f(x) := \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\,\sigma\big((1-\tau)x+\tau y,\xi\big)\,f(y)\,dy\,d\xi \qquad \tau\in[0,1]$$

- $\tau = 0$ Kohn- Nirenberg quantization
- $\tau = \frac{1}{2}$ Weyl quantization

Feynman quantization on $\mathbb{R}^n$

$$Op^F(\sigma) f(x) = \int_{\mathbb{R}^{2n}} e^{i(x-y)\xi} \frac{\sigma(x,\xi)+\sigma(y,\xi)}{2} f(y)\, dy\, d\xi$$

Why is the Weyl quantization so important?

σ = Hamiltonian

1. Weyl quantizations of real Hamiltonians are selfadjoint operators

$$Op^W(\sigma)^* = Op^W(\sigma^*)$$

2. The Weyl quantization is symplectic invariant, that is

$$\forall S \in Sp(2n, \mathbb{R}^n) \quad \exists!\, U_S \in \mathcal{U}(L^2(\mathbb{R}^n)) \quad \text{s.t.}$$

$$Op^w(\sigma \circ S) = U_S^{-1}\, Op^w(\sigma)\, U_S$$

Remark 1 2. Is the most important property of the Weyl quantization. Indeed, property 1. is enjoyed by the Feynman quantization too.

Remark 2 1. & 2. on $\mathbb{R}^n$ are satisfied only by the Weyl quantization.

Definition. Global Hörmander symbol classes on $\mathbb{R}^n$

Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$. The Hörmander symbol class of symbols of order m and type (ρ, δ), $S^m_{\rho,\delta}(\mathbb{R}^n)$, is the set of all $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ s.t. $\forall\, \alpha, \beta \in \mathbb{N}_0^n$ $\exists\, C_{\alpha,\beta}$ s.t.

$$|\partial_\xi^\alpha D_x^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m - \rho[\alpha] + \delta[\beta]} \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

$S^m_{\rho,\delta}(\mathbb{R}^n)$ is a Fréchet space and $C_{\alpha,\beta}$ depend only on the seminorms of σ.

Weyl quantization of $\sigma \in S^m_{\rho,\delta}(\mathbb{R}^n)$

$$Op^W(\sigma)f(x) := Op^{\frac{1}{2}}(\sigma)f(x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi$$

enjoys properties 1. and 2.

The Weyl calculus on $\mathbb{R}^n$. Asymptotic formulas

The symbol of the adjoint

Let $\sigma \in S^m_{\rho,\delta}(\mathbb{R}^n)$ and let $\sigma^{(*)}$ be the symbol of $Op^W(\sigma)^*$. Then,

$$\sigma^{(*)}(x,\xi) = \sigma^*(x,\xi).$$

The symbol of the composition

Let $\sigma_1 \in S^{m_1}_{\rho,\delta}(\mathbb{R}^n)$ and $\sigma_2 \in S^{m_2}_{\rho,\delta}(\mathbb{R}^n)$. Let also $\sigma \in S^{m_1+m_2}_{\rho,\delta}(\mathbb{R}^n)$ be the symbol of $Op^W(\sigma) = Op^W(\sigma_1) \circ Op^W(\sigma_2)$. Then, $\forall M \in \mathbb{N}_0$

$$\sigma \sim \sum_{j=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}^{2n} \cup \{0\} \\ |\alpha| = j}} \left(D_x^{\alpha_1} \partial_\xi^{\alpha_2} \sigma_1(x,\xi)\right)\left(D_x^{\alpha_2} \partial_\xi^{\alpha_1} \sigma_2(x,\xi)\right) \frac{(-1)^{|\alpha_2|}}{2^{|\alpha|}}, \text{ i.e., } \sigma - \sum_{\substack{\alpha \in \mathbb{N}_0^{2n} \\ |\alpha| < M}} \left(D^{\alpha_1} \partial_\xi^{\alpha_2} \sigma_1\right)\left(D^{\alpha_2} \partial_\xi^{\alpha_1} \sigma_2\right) \in S^{m_1+m_2-M}_{\rho,\delta}(\mathbb{R}^n)$$

Motivations

Poisson bracket in $\mathbb{R}^n$ $\{\sigma_1, \sigma_2\}(x,\xi) := \sum_{j=1}^{n} \left[(D_j \sigma_1)(\partial_{\xi_j} \sigma_2) - (\partial_{\xi_j} \sigma_1)(D_j \sigma_2) \right](x,\xi) = H_{\sigma_1} \sigma_2 (x,\xi)$

Hence, for $M=2$,

$$\sigma_1 \circ_w \sigma_2 = \sigma = \sigma_1(x,\xi)\sigma_2(x,\xi) + \frac{1}{2}\{\sigma_1, \sigma_2\}(x,\xi) + r_{m_1+m_2-2}(x,\xi)$$

Feature 1 = easy to use.

On $\mathbb{R}^n$ the Weyl calculus is extremely powerful to attack problems in PDEs,
given $\sigma \in S^m_{\rho,\delta}(\mathbb{R}^n)$ polyhomogeneous, i.e. $\sigma(x,\xi) \sim \sigma_m(x,\xi) + \sigma_{m-1}(x,\xi) + \dots$
where σ_j is homogeneous of degree j, then $\exists!$ $\sigma^{Weyl} \in S^m_{\rho,\delta}(\mathbb{R}^n)$ $Op^{KN}(\sigma) = Op^W(\sigma^{Weyl})$

$$\sigma^{Weyl} \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} D_x^\alpha \partial_\xi^\alpha \sigma(x,\xi) = \sigma_m(x,\xi) + \underbrace{\sigma^s_{m-1}(x,\xi)}_{\text{subprincipal symbol}} + \dots$$

σ^s_{m-1} is invariant on the double characteristic set of Σ, that is $\{(x,\xi) \in \mathbb{R}^{2n} \mid \partial_\xi \sigma_m(x,\xi) = 0,\ \sigma_m(x,\xi) = 0\}$.

Feature 2 = gives info on the invariants of the operators.

Feature 3: Local solvability of PDEs $\longleftrightarrow$ Validity of a priori estimates.

$$\|P^* u\|_{H^s} \gtrsim \|u\|_{H^{s'}} \quad \forall u \in C_0^\infty(V)$$

$\updownarrow$

Gårding, sharp-Gårding, Melin, Hörmander, Rotschild-Stein, Fefferman-Phong inequalities satisfied by P^*

$\underbrace{\qquad\qquad}$ Gårding-type inequalities

Def $P \in \Psi^m(\Omega)$, $\Omega \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$. P is l.s. at x_0 if $\exists\, V_{x_0}$ s.t.

$\forall f \in C^\infty(\Omega)$, $\exists\, u \in \mathcal{D}'(\Omega)$ with $Pu \overset{\mathcal{D}'}{=} f$ in V

On $\mathbb{R}^n$ Gårding-type inequalities are (usually) proved via the Weyl calculus.

Remark Local solvability problems on 2-step groups for double-characteristics operators were considered by Müller-Ricci '96, Ann. of Math.

What about Weyl quantizations on graded groups? And general τ-quantizations?

On H_n

Dynin '76: for $\sigma \in \mathcal{S}'(\mathcal{O}_\pi)$

$$Op^W(\sigma)f(x) = c \iint_{H_n/Z(H_n)} \hat{\sigma}(p,q)\left(\pi^{pr}(p,q)f\right)(x)\, dq\, dp \qquad \pi \in \widehat{H_n}$$

corresponding to the Euclidean version: $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$

$$Op^W(\sigma)f(x) = c \iint_{\mathbb{R}^n \times \mathbb{R}^n} \hat{\sigma}(p,q)\, e^{iq(x+\frac{p}{2})} f(x+p)\, dq\, dp.$$

Good news: Dynin's quantization satisfies

1. $Op^W(\sigma)^* = Op^W(\sigma^*)$
2. $\forall S \in Aut_0(H_n)\ \exists\ U_S \in \mathcal{U}(L^2(H_n))$ s.t.

$$Op^W(\sigma \circ S) = U_S^{-1}\, Op^W(\sigma)\, U_S$$

Bad news:

The symbolic calculus has limitations: $\sigma_1 \circ \sigma_2$ has problems in the asymptotic exp.

On 2-step Lie groups

Dynin's result were generalized by Folland for general 2-step Lie groups.

Mantoiu – Ruzhansky 2017:

τ-quantizations for measurable $\tau : G \to G$
G = type I unimodular locally compact, including symmetric quantizations.

Limitations:
- few examples of symmetry function
- defined only for HS operators
- the calculus was not developed

Our contribution:

- We developed a τ-calculus, for a class of functions τ, on any graded group G.
- We focused on symmetric τ-quantizations on any graded G to identify the possible Weyl quantization
- We considered the case of H_n and established that our candidate is the Weyl quantization in this setting.

Preliminaries

Fourier transform

Let $\widehat{G}$ = dual of G = set of equivalent classes of strongly continuous irreducible unitary representations of G

$$\hat{f}(\pi) \equiv \mathcal{F}_G f(\pi) := \int_G f(x)\pi(x)^* \, dx \qquad \pi(x)^* = \pi(x^{-1})$$

operator-valued

$$\hat{f}(\pi) : H_\pi \longrightarrow H_\pi \qquad H_\pi = \text{Representation space (Hilbert)}$$

Plancherel

$$f(x) = \int_{\widehat{G}} \mathrm{Tr}(\pi(x)\hat{f}(\pi))\, d\mu(\pi) = \iint_{\widehat{G}\times G} \mathrm{Tr}(\pi(y^{-1}x) f(y))\, dy\, d\mu(\pi)$$

$d\mu(\pi)$ Plancherel measure.

Exponential coordinates

$G = \exp_G(\mathfrak{g})$ $\quad \exp_G$: exponential map $\quad \exp_G : \mathfrak{g} \simeq T_e \mathbb{R}^n \longrightarrow G$

$x, y \in G \cong (\mathbb{R}^n, \circ_G)$

$$x \cdot y := x \circ_G y = \exp_G(x_1 X_1 + \dots + x_n X_n) \exp(y_1 X_1 + \dots + y_n X_n)$$

$$= \dots + \quad + \left(x_j + y_j + P_j(x_1, y_1, \dots, x_{j-1}, y_{j-1}) \right) X_j, \dots \Big) + \dots$$

$$x \cdot y = (x_1 + y_1, \quad , x_j + y_j + P_j(x, y), \dots)$$

(BCH)

We will work in exponential coordinates

Invariant derivatives

G homog. LG $\quad \mathfrak{g} = \mathrm{Span}\{X_1, \dots, X_n\}$ $\quad \{X_j\}_{j=1,\dots,n}$ eigenv. of A

X_j left-invariant v.f $\quad X_x f(\ell_y(x)) = (Xf)(\ell_y(x))$ $\quad \ell_y(x) = y \cdot x$

One can also define the right-invariant v.f $\tilde{X}_j$ similarly.

$\nu_1, \dots, \nu_n$ = eigenvalues of A = weights

$\alpha \in \mathbb{N}^n \cup \{0\}$ $\quad X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ $\quad [\alpha] := \sum_{j=1}^{n} \alpha_j \nu_j$ homogeneous lenght of α

X^α is of homogeneous degree $[\alpha]$, i.e. $X_x^\alpha (f \circ D_r)(x) = r^{[\alpha]} (X^\alpha f)(D_r(x))$

In general a linear operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is of hom. degree ν iff

$$T(f \circ D_r) = r^\nu (Tf) \circ D_r$$

X^α are the left-invariant derivatives, $\tilde{X}^\alpha$ the right-invariant ones.

Remark $X^\alpha X^\beta$ do not commute, however $X^\alpha X^\beta = \sum_{[\gamma]=[\alpha]+[\beta]} X^\gamma$

Homogeneous polynomials

$f: G \longrightarrow \mathbb{C}$ is a hom. polynomial of degree ν if $f \circ \exp$ is a polynomial on $\mathbb{R}^n \cong \mathfrak{g}$ and

$$f \circ D_r = r^\nu f \qquad \forall\, r>0.$$

Special basis of P = space of h. polynomials $\forall\, \alpha \in \mathbb{N}^n \cup \{0\}$ $\exists\, q_\alpha$ hom. polynomial of degree $[\alpha]$ s.t. $\forall\, \beta \in \mathbb{N}_0^n$

$$X^\beta q_\alpha(0) = \delta_{\alpha\beta}(0) = \begin{cases} 1 & \beta=\alpha \\ 0 & \text{otherwise} \end{cases}$$

Difference operators = Derivations wrt representations

$$\Delta_\pi^\alpha \widehat{f}(\pi) = \widehat{\tilde{q}_\alpha f}(\pi)$$

q_α element of the special basis.

$$\tilde{q}_\alpha(x) = q_\alpha(x^{-1})$$

Taylor expansion Let $|\cdot| : G \longrightarrow \mathbb{R}$ be a quasi-norm on G.

The (left) Taylor polynomial of order $M \in \mathbb{N}$ of $f : G \longrightarrow \mathbb{C}$ at $x \in G$ is

$$P^f_{x,M}(y) = \sum_{[\alpha] \leq M} q_\alpha(y)(X^\alpha f)(x)$$

$$R^f_{x,M}(y) = f(xy) - P^f_{x,M}(y)$$

Fisher–Rozhansky / Fisher–Fermanian-Kammerer global Hörmander symbol classes

A symbol σ is a family of operators

$$\sigma = \{ \sigma(x,\pi) \mid (x,\pi) \in G \times \widehat{G} \}$$

with $\sigma(x,\pi) : \mathcal{H}_\pi^\infty \longrightarrow \mathcal{H}_\pi \quad \forall x \in G$ and a.e. $\pi \in \widehat{G}$

$S^m_{\rho,\delta}(G)$ – classes $\quad \sigma \in S^m_{\rho,\delta}(G)$ if $\forall \alpha, \beta \in \mathbb{N}^n \cup \{0\}$

$$\sup_{(x,\pi)\in G\times\widehat{G}} \left\| \Delta_\pi^\alpha X_x^\beta \sigma(x,\pi) \cdot \pi(I+R)^{-\frac{(m-\rho[\alpha]+\delta[\beta])}{\nu}} \right\|_{\mathcal{L}(\mathcal{H}_\pi)} < \infty$$

Kohn–Nirenberg quantization on G

$$\mathrm{Op}^{KN}(\sigma) f(x) = \int_{G\times\widehat{G}} \mathrm{Tr}\left(\pi(y^{-1}x)\, \sigma(x,\pi) f(y)\right) dy\, d\mu(\pi)$$

τ - quantization on Graded groups.

Mantoiu - Ruzhansky on unimodular type I locally compact groups

$$Op^{\tau}(\sigma)f(x) = \int_{G\times\widehat{G}} \mathrm{Tr}\left(\pi(y^{-1}x)\,\sigma(x\tau(y^{-1}x)^{-1},\pi)\,f(y)\right)dy\,d\mu(\pi)$$

where $\tau: G \longrightarrow G$ is a measurable function, the quantizing function.

$\sigma = \{\sigma(x,\pi) \mid (x,\pi)\in G\times\widehat{G}\}$ with $\sigma(x,\pi): H_\pi^{-\infty} \longrightarrow H_\pi^{\infty}$ $\forall x\in G$ and a.e. $\pi\in\widehat{G}$.

However $Op^{\tau}(\sigma)\in HS(L^2(G))$ (Hilbert-Schmidt op. on L^2), and no calculus was developed.

Note that

- $G=\mathbb{R}^n$ KN ($\tau(x)=x$) and Weyl ($\tau(x)=\frac{x}{2}$)
- G unim. type I with $\tau(x)=e_G$ K-N quantization by F-R. / F-FK

Def $\tau: G\longrightarrow G$ is a symmetry function iff $\tau(x)=\tau(x^{-1})x$.

If τ is a symmetry function $Op^{\tau}(\sigma^*) = Op^{\tau}(\sigma)^*$ Mantoiu-Ruzhansky in an abstract setting

Our choice of τ- functions on graded groups.

Admissible τ functions : (HP) condition

$\tau : G \longrightarrow G$ satisfies (HP) if in exponential coordinates

$\tau(x) = \left(c_1^\tau(x), c_2^\tau(x), \ldots, c_n^\tau(x) \right)$, where each $c_j^\tau(x)$ is

either $c_j^\tau(x) = 0 \ \forall x \in G$

or

$c_j^\tau(x) = c_j^\tau(x_1, \ldots, x_j) = c_j^\tau x_j + d_j^\tau(x_1, \ldots, x_{j-1})$, $c_j^\tau \neq 0$ and d_j^τ a HP of deg ν_j

These τ- functions allowed us to develop a τ-calculus for the global $S^m_{\rho,\delta}(G)$ Hörmander classes, $\forall m \in \mathbb{R}$, $\forall \rho, \delta \in \mathbb{R}$ s.t

$0 \leq \rho < \frac{1}{\nu_n} \leq 1$ if $\tau \neq e_G$ and $0 \leq \delta < \rho \leq 1$ if $\tau = e_G$

Remark 1

Mantoiu-Ruzhansky provide an example of symmetry functions on any exponential G, i.e.

$$\tau_{MR}(x) = \int_0^1 \exp(s \log(x))\, ds$$

which, on H_n, gives

$$\overset{\circ}{\tau}_{MR}(x) = \left(\frac{x_1}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j=1}^{n} \frac{x_j x_{j+n}}{24}\right).$$

On H_n we find a whole family of symmetry functions, i.e

$$\tau(x) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2} + \sum_{j,k=1}^{2n} c_{j,k}\, x_j x_k\right)$$

for any choice of $c_{j,k} \in \mathbb{R}$, $j,k = 1, \dots, n$. Which one is the Weyl-quantizing function?

Note that $\tau(xy) \neq \tau(x)\tau(y)$ in general, this gives a lot of technical problems.

Remark 2 $\boxed{0 \leq \delta < \dfrac{\rho}{\nu_n} \leq 1}$ Needed for the calculus, not for the quantization

- $G = \mathbb{R}^n \to \nu_n = 1$ and we recover the Euclidean τ-quantizations.
- When $\tau(x) = e_G$ we do not have the restriction on (ρ, δ) and we get back the K-N quantization by F-R and F-FK

Remark 3 If $G = H_n$, $0 \leq \delta < \rho \leq 1$. What about general G? Open problem.

Remark 4 The (HP) condition is somehow natural and reflects the homogeneous structure of G. Indeed,

$\tau(x)_j =$ HP of degree ν_j, that is

$$X_k \tau(x)_j = \begin{cases} \text{HP of degree } \nu_j - \nu_k & \text{if } \nu_k \leq \nu_j \\ 0 & \text{if } \nu_k > \nu_j \end{cases}$$

τ-quantization: first properties and the (ρ,δ)-condition

Proposition (Continuity property)

$0 \le \delta < \rho \le 1$, $0 < \rho \le 1$, $\delta < \frac{1}{\nu_n}$. Let $\sigma \in S^m_{\rho,\delta}(G)$, then $\forall f \in \mathcal{S}(G)$ $\mathrm{Op}^\tau(\sigma) \in \mathcal{S}(G)$ and $\mathrm{Op}^\tau(G) : \mathcal{S} \longrightarrow \mathcal{S}$ is continuous, i.e.

$$\|\mathrm{Op}^\tau(\sigma) f\|_{\mathcal{S},N_1} \le C \|\sigma\|_{S^m_{\rho,\delta,a,b}} \|f\|_{\mathcal{S},N_2} \quad \text{for some } N_1, N_2, a, b \in \mathbb{N}.$$

Remark: In principle one could use $0 \le \delta < \min\{\rho, \frac{1}{\nu_n}\}$, in practice $\delta < \frac{\rho}{\nu_n}$ is needed to prove the results for general graded groups.

Theorem (F. – Rottensteiner – Ruzhansky) (Change of quantization)

$0 \le \delta < \min\{\rho, \frac{1}{\nu_n}\}$, τ quantizing function satisfying (HP) s.t. $\tau \neq e_G$,

$T : \mathcal{S}(G) \longrightarrow \mathcal{S}(G)$ continuous linear operator.

If σ, σ_τ are two symbols s.t.

$$T = Op^\tau(\sigma_\tau) = Op^{KN}(\sigma)$$

Then $\sigma \in S^m_{\rho,\delta}(G) \Longleftrightarrow \sigma_\tau \in S^m_{\rho,\delta}(G)$, $\sigma \longmapsto \sigma_\tau$ is a Frechet sp. isomorphism

and

$$\sigma_\tau \sim \sum_{j=0}^{\infty} \Big(\sum_{[\alpha]=j} c^{\tau, KN}_{\alpha,\alpha'} \Delta^{\alpha'} X^{\alpha} \sigma \Big), \quad \sigma \sim \sum_{j=0}^{\infty} \Big(\sum_{[\beta]=j} c_{\beta',\beta} \Delta^{\beta} \sigma_\tau \Big)$$

i.e. $\forall M, N \in \mathbb{N}_0$

$$R_M^{\tau, KN} := \sigma - \sum_{j=0}^{M} \Big(\sum_{[\alpha]=j} c^{\tau, KN}_{\alpha,\alpha'} \Delta^{\alpha'} X^{\alpha} \sigma \Big) \in S^{m-(\rho-\delta)(M+1)}_{\rho,\delta}(G)$$

$$R_N^{KN,\tau} = \sigma_\tau - \sum_{j=0}^{N} \Big(\sum_{[\beta]=j} c_{\beta',\beta} \Delta^{\beta} \sigma_\tau \Big) \in S^{m-(\rho-\delta)(N+1)}_{\rho,\delta}(G)$$

Kernels and change of coordinates

$$Op^{kN}(\sigma) f(x) = \int \mathrm{Tr}\left(\pi(y^{-1}x)\sigma(x,\pi) f(y) dy\right) d\mu(\pi) = \int k_\sigma(x, y^{-1}x) f(y)\, dy = \int ker_\sigma^{kN}(x,y) f(y)\, dy$$

$$Op^{\tau}(\sigma) f(x) = \int_{G\times\widehat{G}} \mathrm{Tr}\left(\pi(y^{-1}x)\,\sigma(x\tau(y^{-1}x)^{-1},\pi) f(y)\right) dy\, d\mu(\pi)$$

$$= \int k_\sigma(x\tau(y^{-1}x)^{-1}, y^{-1}x) f(y)\, dy = \int_G \underbrace{ker_\sigma^{\tau}(x,y)}_{\text{distributional kernel}} f(y)\, dy$$

$k_\sigma(x,y) = \mathcal{F}^{-1}_{\pi\to y}(\sigma(x,\pi))$ associated kernel

change of coordinates

$cv^\tau : G\times G \longrightarrow G\times G$ $\quad cv^\tau(x,y) = (x\tau(y^{-1}x)^{-1}, y^{-1}x)$, $\quad (cv^\tau)^{-1}(x,y) = (x\tau(y), x\tau(y)y^{-1})$

cv^τ $(cv^\tau)^{-1}$ associated pullbacks $\quad (cv^\tau \circ k_\sigma)(x,y) = k_\sigma(cv^\tau(x,y)) = k_\sigma(x\tau(y^{-1}x)^{-1}, y^{-1}x)$

$((cv^\tau)^{-1}\circ k_\sigma)(x,y) = k_\sigma((cv^\tau)^{-1}(x,y)) = \dots$

$$\Rightarrow \boxed{ker_\sigma^\tau(x,y) = k_\sigma(cv^\tau(x,y))}$$

Relation between symbols and the distributional kernel in the τ-quantization

$$\sigma(x,\pi) = ((\mathrm{Id}\otimes\mathcal{F}_G)\circ k_\sigma)(x,\pi) = \mathcal{F}_{y\to\pi}\left(((cv^\tau)^{-1}\circ ker_\sigma^\tau)(x,y)\right)$$

τ - Calculus. Asymptotic formula for the adjoint.

Theorem (F. - Rottensteiner - Ruzhansky)

$m \in \mathbb{R}$, $0 \leq \delta < \min(\rho, \frac{1}{\nu_n})$, $\tau : G \to G$ quantizing function satisfying (HP)

Let $T : \mathcal{S}(G) \to \mathcal{S}(G)'$ linear cont op. s.t. $T = Op^{\tau}(\sigma_\tau)$ for some $\sigma \in S^m_{\rho,\delta}(G)$.

Let T^* be the formal adjoint given by $T^* = Op(\sigma^{(*)})$ for a uniquely determined $\sigma_\tau^{(*)} \in S^m_{\rho,\delta}(G)$. Then $\sigma_\tau \mapsto \sigma_\tau^{(*)}$ is a Frechet space isomorphism from $S^m_{\rho,\delta}(G)$ onto itself and

$$\sigma^{(*)} \sim \sum_{j=0}^{\infty} \Big(\sum_{[\alpha]=j} \sum_{[\alpha']=[\alpha]} C^{(*),\tau}_{\alpha',\alpha} \Delta_\pi^{\alpha'} X_\tau^{\alpha} \sigma^* \Big)$$

with $C^{\tau}_{\alpha',\alpha} \in \mathbb{R}$ uniquely determined by the equations

$$q_\alpha(\tau(y)y^{-1}\tau(y^{-1})^{-1}) = \sum_{[\alpha']=[\alpha]} C^{(*),\tau}_{\alpha',\alpha} \tilde{q}_{\alpha'}(y).$$

When $\tau = e_G$: $\sigma^{(*)} \sim \sum_{j=0}^{\infty} \Big(\sum_{[\alpha]=j} \Delta_\pi^{\alpha} X_\tau^{\alpha} \sigma^* \Big)$ in the KN - calculus by Fisher-Ruzhansky

If moreover τ is a symmetry function then $Op^{\tau}(\sigma)^* = Op^{\tau}(\sigma^*)$

Asymptotic formula for the composite symbol

Theorem (F. – Rottensteiner – Ruzhansky)

$m_1, m_2 \in \mathbb{R}$, $0 \le \delta < \frac{\rho}{\nu_n} \le 1$, τ satisfying (H), $\sigma_1 \in S^{m_1}_{\rho,\delta}(G)$, $\sigma_2 \in S^{m_2}_{\rho,\delta}(G)$.

Then there exists a uniquely determined $\sigma \in S^{m_1+m_2}_{\rho,\delta}(G)$ s.t

$$Op^{\tau}(\sigma) = Op^{\tau}(\sigma_1) \cdot Op^{\tau}(\sigma_2).$$

The τ-composition of symbols

$$\circ_\tau \quad S^{m_1}_{\rho,\delta}(G) \times S^{m_2}_{\rho,\delta}(G) \longrightarrow S^{m_1+m_2}_{\rho,\delta}(G)$$

$$(\sigma_1, \sigma_2) \longmapsto \sigma_1 \circ_\tau \sigma_2 := \sigma$$

is a bilinear, continuous map

and

$$\sigma \sim \sum_{i,j=}^{\infty} \left(\sum_{\substack{[\alpha]=i \\ [\beta]=j}} \; \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha] \\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2} c_{\beta_1,\beta_2} \left(\Delta^{\alpha_2} \Delta^{\beta_1} X^{\alpha} \sigma_1 \right) \left(\Delta^{\beta_2} \Delta^{\alpha_1} X^{\beta} \sigma_2 \right) \right),$$

that is, $\forall\, M, N \in \mathbb{N}_0$

$$\sigma - \sum_{\substack{[\alpha]\le M \\ [\beta]\le N}} \; \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha] \\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \left(\tilde{\Delta}^{\alpha_2} \Delta^{\beta_1} X^{\alpha} \sigma_1\right)\left(\tilde{\Delta}^{\beta_2} \Delta^{\alpha_1} X^{\beta} \sigma_2\right) \in S^{m_1+m_2-(\rho-\delta)(\min\{M,N\}+1)}_{\rho,\delta} \qquad (G)$$

and were c_{α_1,α_2} c_{β_1,β_2} are uniquely determined by the equations

$$q_\alpha(p_1(y,z)) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\, \tilde{q}_{\alpha_1}(y)\, \tilde{q}_{\alpha_2}(z^{-1}y)$$

$$q_\beta(p_2(y,z)) = \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2}\, \tilde{q}_{\beta_1}(z^{-1}y)\, \tilde{q}_{\beta_2}(z) \,.$$

Remark The proof is based on the use of kernels

Kernel of the composition Let $Op(\sigma_1) Op(\sigma_2) = Op(\sigma)$, then

$$k_\sigma(x,y) = \int_G k_{\sigma_1}\big(x \underbrace{\tau(y)\tau(z^{-1}y)^{-1}}_{p_1(y,z)}, z^{-1}y\big)\, k_{\sigma_2}\big(z \underbrace{\tau(y^{-1}z)^{-1}}_{p_2(y,z)}, y^{-1}z\big)\, dz .$$

Remark: For $\tau = e_G$ we find the composite kernel of the K-N quantization

Sketch of the proof of the asymptotic formula for the composite symbol

Assume $\sigma_1, \sigma_2 \in S^{-\infty}_{\rho,\delta} = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}$.

Step 1. We expand k_{σ_1} and k_{σ_1} using Taylor expansion

$$k_{\sigma_1}(x p_1(x,y), z^{-1}y) = \sum_{[\alpha] \le M} q_\alpha(p_1(y,z))\, X^\alpha_{x_1 = x} k_{\sigma_1}(x_1, z^{-1}y) + R^{k_{\sigma_1}(\cdot, z^{-1}y)}_{x,M}(p_1(y,z))$$

$$k_{\sigma_2}(x p(x,y), z) = \sum_{[\beta] \le N} q_\beta(p_2(y,z))\, X^\beta_{x_2 = x} k_{\sigma_2}(x_1, z) + R^{k_\sigma(\cdot, z)}_{x,M}(p_2(y,z))$$

We use a crucial property of the q_j's, i.e.

$$q_\alpha(p_1(y,z)) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\, \tilde{q}_{\alpha_1}(z)\, \tilde{q}_{\alpha_2}(z^{-1}y), \quad q_\beta(p_2(y,z)) = \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2}\, \tilde{q}_{\alpha_1}(z^{-1}y)\, \tilde{q}_{\alpha_2}(z)$$

and get

$$k_\sigma(x,y) = k_{T_0}(x,y) + k_{T_1}(x,y) + k_{T_2}(x,y) + k_{T_3}(x,y)$$

who is $\sigma(x,\pi)$ then?

$$\sigma(x,\pi) = T_0(x,\pi) + T_1(x,\pi) + T_2(x,\pi) + T_3(x,\pi)$$

where

$$T_0(x,\pi) = \sum_{\substack{[\alpha]\le M \\ [\beta]\le N}} \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha] \\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2}\, c_{\beta_1,\beta_2} \left(\Delta^{\alpha_2}\Delta^{\beta_1} X^{\alpha}\sigma_1\right)\left(\Delta^{\beta_2}\Delta^{\alpha_1} X^{\beta}\sigma_2\right), \text{ and}$$

$$\sigma(x,\pi) - T_0(x,\pi) = T_1(x,\pi) + T_2(x,\pi) + T_3(x,\pi) = \text{Reminder}$$

Hence we need

$T_j(x,\pi)$, $\forall\, j=1,2,3$ to belong to $S^{m_1+m_2-(\rho-\delta)(\min\{M,N\}+1)}_{\rho,\delta}(G)$.

where

$$T_1(x,\pi) := \sum_{\substack{[\beta]\leq N \\ [\beta_1]+[\beta_2]=[\beta]}} \int_{G\times G} \tilde{q}_{\beta_1}(z^{-1}y)\,\tilde{q}_{\beta_2}(z)\, X^{\beta}_{x_2=x} k_{\sigma_2}(x_2,z)\, R^{k_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}(p_1(y,z))\, dz\, \pi^*(y)\, dy$$

$T_2(x,\pi)$ similar to T_1 (reversing the roles of σ_1 and σ_2)

$$T_3(x,\pi) := \int_{G\times G} R^{k_{\sigma_1}(\cdot,z^{-1}y)}_{x,M}(p_1(y,z))\, R^{k_{\sigma_2}(\cdot,z)}_{x,N}(p_2(y,z))\, dz\, \pi^*(y)\, dy.$$

We need that $\forall \alpha_0, \beta_0$, $\forall$ $0 \leq \delta < \frac{1}{\nu_n}$, $\exists$ $C > 0$ and $a_1, a_2, b_1, b_2 \in \mathbb{N}$ s.t.

$$\sup_{(x,\pi) \in G \times \widehat{G}} \| \Delta^{\alpha_0} X^{\beta_0} T_j(x,\pi) \, \pi(I+R)^{-\frac{m_1+m_2-(\rho-\delta)L - \rho[\alpha_0]+\delta[\beta_0]}{\nu}} \|_{\mathcal{L}(H_\pi)} \leq C \|\sigma_1\|_{S^{m_1}_{\rho,\delta,a_1,b_1}} \|\sigma_2\|_{S^{m_2}_{\rho,\delta,a_2,b_2}} \quad (T_j)$$

If this is true we conclude the proof for $\sigma_1, \sigma_2 \in S^{-\infty}_{\rho,\delta}$.

Lemma 1 If r satisfies (HP) then

$$|p_1(y,z)| \lesssim |z| + \sum_{j=1}^{n} \sum_{\substack{[\alpha]+[\beta]=\nu_j \geq 2 \\ \alpha,\beta \in \mathbb{N}^n}} |z|^{\frac{[\alpha]}{\nu_j}} |z^{-1}y|^{\frac{[\beta]}{\nu_j}}$$

$$|p_2(y,z)| \lesssim |z^{-1}y| + \sum_{j=1}^{n} \sum_{\substack{[\alpha]+[\beta]=\nu_j \geq 2 \\ \alpha,\beta \in \mathbb{N}^n}} |z|^{\frac{[\alpha]}{\nu_j}} |z^{-1}y|^{\frac{[\beta]}{\nu_j}}$$

Combining Lemma 1, Taylor's remainders properties and estimates, generalization of kernel estimates we proved (T_j) $\forall j = 1, \dots, 3$ and thus the result.

$\square$

Example 1: G graded and $\tau = e_G$ = KN quantization

$$q_\alpha(p_1(y,z)) = \sum_{[\alpha_1]+[\alpha_2]=[\alpha]} c_{\alpha_1,\alpha_2}\, \tilde{q}_{\alpha_1}(y)\, \tilde{q}_{\alpha_2}(z^{-1}y) \qquad p_1(y,z) = e_G \implies c_{\alpha_1,\alpha_2} = \delta_{\alpha_1,0}\,\delta_{\alpha_2,0} \quad \forall \alpha \in \mathbb{N}_0$$

$$q_\beta(p_2(y,z)) = \sum_{[\beta_1]+[\beta_2]=[\beta]} c_{\beta_1,\beta_2}\, \tilde{q}_{\beta_1}(z^{-1}y)\, \tilde{q}_{\beta_2}(z) \qquad p_2(y,z) = y^{-1}z \implies c_{\beta_1,\beta_2} = \delta_{\beta_2,0} \quad \forall \beta \in \mathbb{N}_0$$

$$\implies \sigma_1 \circ_{\tau=e_G} \sigma_2 = \sigma_1 \circ_{KN} \sigma_2 \sim \sum_{j=0}^{\infty} \sum_{[\beta]=j} (\Delta^\beta \sigma_1)(X^\beta \sigma_2)$$

Example 2 $G = \mathbb{R}^n$ $\tau(x) = \frac{x}{2} = \left(\frac{x_1}{2}, \dots, \frac{x_n}{2}\right)$ = Weyl quantization

The homogeneous Poisson bracket on graded Lie groups

Def. G-homogeneous order

G graded, σ smooth symbol. We say that σ is G-homogeneous of degree $k \in \mathbb{N}_0$ and write $\mathrm{Ord}_G(\sigma) = k$ if

$$X_x^\alpha \, \sigma(D_r(x), \pi) = r^{k+[\alpha]} X_{y=D_r(x)}^\alpha \sigma(y,\pi) \quad \forall r > 0 \text{ and a.e } (x,\pi) \in G\times\widehat{G}.$$

Note that, $\forall \sigma_1, \sigma_2$

$$\mathrm{Ord}_G\left(\left(X^\alpha \sigma_1(x,\pi)\right)\left(X^\beta \sigma_2(x,\pi)\right)\right) = \left(k_1+[\alpha]\right)+\left(k_2+[\beta]\right)$$

$$\Longrightarrow \quad \underset{\sigma_1 \circ_G \sigma_2}{\overset{\parallel}{\sigma}} \sim \sum_{i,j=}^{\infty} \left(\sum_{\substack{[\alpha]=i \\ [\beta]=j}} \; \sum_{\substack{[\alpha_1]+[\alpha_2]=[\alpha] \\ [\beta_1]+[\beta_2]=[\beta]}} c_{\alpha_1,\alpha_2} c_{\beta_1,\beta_2} \underbrace{\left(\Delta^{\alpha_2}\Delta^{\beta_1}X^\alpha\sigma_1\right)\left(\Delta^{\beta_2}\Delta^{\alpha_1}X^\beta\sigma_2\right)}_{:= I} \right),$$

$$\mathrm{Ord}_G(I) = k_1+[\alpha]+k_2+[\beta]$$

Def Homogeneous Poisson bracket

G graded $\quad \sigma_j \in S^{m_j}_{\rho,\delta}(G) \quad j=1,2 \quad 0 \le \delta \le \rho \le 1$

We define the homogeneous Poisson bracket as the symbol

$$\{\sigma_1,\sigma_2\}_{hom} := (-i) \sum_{[\alpha]=1} \left((X^\alpha \sigma_1)(\Delta^\alpha \sigma_2) - (\Delta^\alpha \sigma_1)(X^\alpha \sigma_2) \right) \in S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}(G)$$

Remark 1 $\quad \mathrm{Ord}_G \{\sigma_1,\sigma_2\}_{hom} = k_1+k_2+1 \quad \forall$ graded $G \quad \mathrm{Or}_G(\sigma_j) = k_j$

Remark 2 $\quad G=\mathbb{R}^n \quad \{\sigma_1,\sigma_2\}_{hom} = \{\sigma_1,\sigma_2\}$

Proposition (F - Rottensteiner - Ruzhansky)

$\sigma_j \in S^{m_j}_{\rho,\delta}(G)$, $j=1,2$, & satisfies (HP) and is a symmetry function, $0 \le \delta < \frac{\rho}{\sqrt{n}} \le 1$.

then the uniquely determined summand in the asymptotic expansion of $\sigma_1 \circ_G \sigma_2$ of order $m_1+m_2-(\rho-\delta)$ and G-homogeneous order k_1+k_2+1 is $\frac{1}{2}\{\sigma_1,\sigma_2\}_{hom}$.

The Weyl quantization. A candidate for general graded Lie groups.

- On any G graded $\tau(x) = \left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_n}{2}\right) = \exp\left(\frac{1}{2}\log(x)\right)$ is a symmetry quantizing function

canonical symmetry function

- On $\mathbb{R}^n$ and $\mathbb{H}^n$ τ^W has special properties

On $\mathbb{R}^n$

$\forall\ S \in Sp(2n, \mathbb{R})$ $\quad Op^W(\sigma \circ S) = U_S^{-1}\, Op^W(\sigma)\, U_S$ $\quad \forall\ U_S = Mp(2n, \mathbb{R}) \subseteq \mathcal{U}(L^2(\mathbb{R}^n))$

On general G graded

Theorem (F.– Rottensteiner – Ruzhansky)

If $\tau(x) = \exp\left(\frac{1}{2}\log(x)\right)$, then $\forall\ S \in Aut(G)$ $\exists$ a unitary $U_S \in \mathcal{U}(L^2(G))$ s.t.

$$Op^\tau(\sigma \circ S) f = U_S^{-1}\, Op^\tau(\sigma)\, U_S f \qquad \forall f \in L^2(G)$$

$\forall \sigma \in S^0_{\rho,\delta}(G)$ $\quad 0 \le \delta < \frac{1}{\nu_n}$.

Remark

The proof works for any admissible τ that commutes with $S \cdot \in \mathrm{Aut}(G)$, thus also $\tau = e_G$.

On $\mathbb{R}^4$ it works for any $\tau(x) = \tau x$.

To find the function τ giving the Weyl quantization we must ask for both invariance property and the involution property.

$\tau = \exp\left(\frac{1}{2}\log(x)\right)$ satisfies both on any graded G. Good candidate!

The Weyl quantization on the Heisenberg group.

On $\mathbb{H}_n$ we know exactly $\mathrm{Aut}(\mathbb{H}_n)$. So we could check that the only symmetry quantizing function commuting with $\mathrm{Aut}(\mathbb{H}_n)$ is

$$\tau(x) = \exp\left(\frac{1}{2}\log(x)\right)$$

Theorem (F.- Rottensteiner- Ruzhansky)

$G = H_n$, $0 \leq \delta < \frac{\rho}{\sqrt{n}} \leq 1$, $\tau : H_n \to H_n$. Among all the symmetry quantizing functions $\tau(x) = \exp(\frac{1}{2}\log(x))$ is the only one satisfying both the following conditions

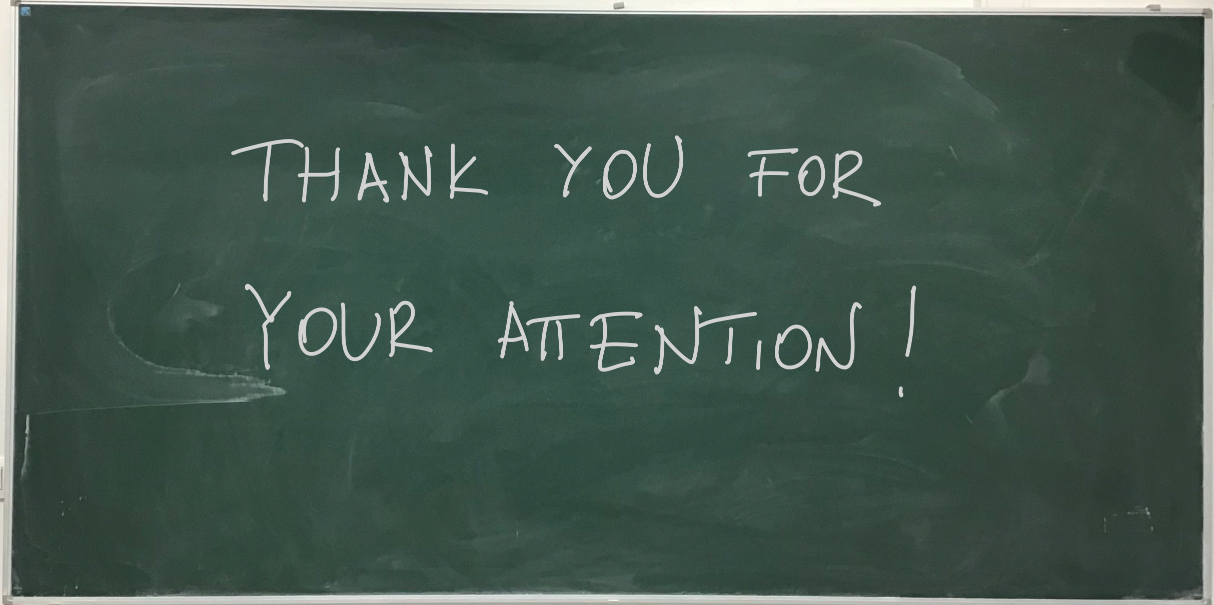
$\forall \sigma \in S^{0}_{\rho,\delta}(G)$, $0 \leq \delta < \frac{\rho}{\sqrt{n}} \leq 1$, $\forall f \in L^2(G)$

- $Op^{\tau}(\sigma)^{*} = Op^{\tau}(\sigma^{*})$ — Preservation of involution

- $\forall A \in Aut(G)\ \exists U_S \in L^2(G)$ s.t. — Automorphic invariance

$$Op^{\tau}(\sigma \circ U_S) f = U_S^{-1}\, Op^{\tau}(\sigma)\, U_S f$$

Conclusion $\tau(x) = \left(\frac{x_1}{2}, \dots, \frac{x_{2n}}{2}, \frac{x_{2n+1}}{2}\right)$ gives the Weyl quantization on H_n and is the candidate symmetry function giving the Weyl quantization on any graded Lie group.


THANK YOU FOR
YOUR ATTENTION!