Fredholm operators on graded Lie groups

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Motivation

- Observation: There are operators that are hypoelliptic but not elliptic, (sub-Laplacians, other Hörmander's sums of squares operators, ...),
- this lead to the study of operators on graded Lie groups in the 70s by Folland, Stein, Rothschild, ...
- development of pseudodifferential calculi for graded Lie groups (Christ–Geller–Głowacki–Polin, Fischer–Ruzhansky–Fermaninan-Kammerer, van Erp–Yuncken, ...)
- instead of ellipticity: P satisfies the Rockland condition $\Rightarrow$ P is hypoelliptic,
- however, P is not Fredholm. On $\mathbb{R}^n$: globally elliptic operators defined by Shubin and Helffer $\rightsquigarrow$ Fredholm operators, index formula, Weyl law, ...
- Analogue for graded Lie groups?

Graded Lie groups

Definition

A graded Lie group of step r is a simply connected Lie group G whose Lie algebra $\mathfrak{g}$ is graded, i.e.

$$\mathfrak{g} = \bigoplus_{j=1}^{r} \mathfrak{g}_j \quad \text{such that } [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

Here, we set $\mathfrak{g}_i = \{0\}$ for $i > r$.

- in particular, G is nilpotent, so that $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism,

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- in particular, G is nilpotent, so that $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism,
- $X \in \mathfrak{g}$ gives rise to a left-invariant differential operator

$$Xf(x) = \frac{d}{dt} f(x \cdot \exp(tX))\Big|_{t=0} \quad \text{for } f \in C^\infty(G).$$

New notion of order for left-invariant differential operators:

Declare order of $X \in \mathfrak{g}_i$ to be i.

Dilations

Dilations

Associated with the grading there is a dilation action of $\mathbb{R}_{>0}$ on $\mathfrak{g}$ and G:

$$\delta_\lambda(X) = \lambda^i X \quad \text{for } \lambda > 0 \text{ and } X \in \mathfrak{g}_i.$$

- a left-invariant differential operator P is m-homogeneous if $P(f \circ \delta_\lambda) = \lambda^m P(f) \circ \delta_\lambda$ for all $\lambda > 0$ and $f \in C^\infty(G)$,
- in particular, $X \in \mathfrak{g}_i$ defines an i-homogeneous operator.

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Examples of graded Lie groups

- $\mathbb{R}^n$ with arbitrary grading,
- Heisenberg group H with Lie algebra $\mathfrak{h}$ generated by X, Y, Z with $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$.
  Standard grading: $\mathfrak{h}_1 = \mathbb{R}X \oplus \mathbb{R}Y$ and $\mathfrak{h}_2 = \mathbb{R}Z$,
- upper triangular matrices.

Rockland operators

Let P be a homogeneous left-invariant differential operator on G.

- on $G = \mathbb{R}^n$: P is hypoelliptic $\Leftrightarrow p(\xi) = \widehat{P}(\xi) \neq 0$ for $\xi \neq 0$,

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- $\widehat{G}$: equivalence classes of irreducible representations $\pi\colon G \to \mathcal{U}(\mathcal{H}_\pi)$,
- π induces a representation $d\pi$ of $\mathfrak{g}$ as (possibly unbounded) operators on $\mathcal{H}_\pi$.

Definition

The operator P is a Rockland operator if for every $\pi \in \widehat{G} \setminus \{\pi_{\text{triv}}\}$ the operator $d\pi(P)$ is injective on $\mathcal{H}_\pi^\infty$.

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Theorem (Helffer-Nourrigat)

P is hypoelliptic $\Leftrightarrow$ P is a Rockland operator.

Example (Heisenberg group)

$P = X^2 + Y^2 + i\alpha Z$ is Rockland $\Leftrightarrow \alpha \notin 2\mathbb{Z} + 1$.

Hörmander calculus on graded Lie groups

Question

Is there an analogue of the Hörmander pseudodifferential calculus incorporating the new notion of order?

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Yes, different approaches:

- Christ–Geller–Głowacki–Polin: using distributional kernels having homogeneous expansions,
- Fischer–Ruzhansky and Fischer–Fermanian-Kammerer: using symbols $a(x, \pi)$ acting on $\mathcal{H}_\pi$ for $(x, \pi) \in G \times \widehat{G}$,
- van Erp–Yuncken: using a tangent groupoid and zoom action
  - building on work by Debord–Skandalis,
  - approach also works for filtered manifolds and even non-regular filtrations (Androulidakis–Mohsen–Yuncken).
- ...

Analogue of ellipticity on G

In these calculi, one has a notion of a principal symbol = family $(\sigma(P)_x)_{x\in G}$ of left-invariant homogeneous operators on G.

Theorem (Christ–Geller–Głowacki–Polin)

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- However $PQ - I, QP - I \in C^\infty(G \times G)$, not necessarily compact as operators on $L^2(G)$.
- Solution on $G = \mathbb{R}^n$: introduce a pseudodifferential calculus in which one has better control of the growth of the symbols as $|x| \to \infty$.

Operators on $\mathbb{R}^n$

Definition (Shubin, Helffer)

A function $p \in C^\infty(\mathbb{R}^{2n})$ belongs to the symbol class $\Gamma^m(\mathbb{R}^{2n})$ if for all $\alpha, \beta \in \mathbb{N}_0^n$ there is $C_{\alpha\beta} > 0$ such that for all $(x, \xi) \in \mathbb{R}^{2n}$:

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}(1 + |x| + |\xi|)^{m-|\alpha|-|\beta|}.$$

Obtain operators $\mathrm{Op}(p)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ using the standard quantization.

Example (Differential operators with polynomial coefficients)

$$p(x,\xi) = \sum_{|\alpha|+|\beta| \leq m} c_{\alpha\beta} x^\alpha (i\xi)^\beta \rightsquigarrow \mathrm{Op}(p) = \sum_{|\alpha|+|\beta| \leq m} c_{\alpha\beta} x^\alpha \partial_x^\beta$$

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- consider classical symbols: homogeneous expansion with respect to $\lambda \cdot (x,\xi) = (\lambda x, \lambda\xi)$ for $\lambda > 0$,
- principal symbol is a m-homogeneous function on $\mathbb{R}^{2n} \setminus \{0\}$,
- denote $\Psi_\Gamma^m = \{\mathrm{Op}(p)\colon p \in \Gamma^m(\mathbb{R}^{2n}) \text{ classical}\}$,
- the principal symbol map induces short exact sequences

$$0 \to \Psi_\Gamma^{m-1} \to \Psi_\Gamma^m \overset{\sigma_m}{\to} C^\infty(S^{2n-1}) \to 0.$$

Ellipticity

- Algebra structure: $\Psi_\Gamma^k \circ \Psi_\Gamma^l \subseteq \Psi_\Gamma^{k+l}$ and $\sigma_{k+l}(PQ) = \sigma_k(P)\sigma_l(Q)$.

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An operator $P \in \Psi^m_\Gamma$ is elliptic if its principal symbol $\sigma_m(P) \in C^\infty(S^{2n-1})$ is invertible.

Examples

- on $\mathbb{R}^n$: harmonic oscillator $-\Delta + |x|^2 \in \Psi^2_\Gamma$ (principal symbol $|\xi|^2 + |x|^2$),
- on $\mathbb{R}$: creation/annihilation operator $x \pm \partial_x \in \Psi^1_\Gamma$ (principal symbol $x \pm i\xi$).

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- denote by $\mathcal{K}^\infty$ integral operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$,
- $P \in \Psi^m_\Gamma$ elliptic $\Rightarrow$ there is a parametrix $Q \in \Psi^{-m}_\Gamma$ such that $PQ - 1, QP - 1 \in \mathcal{K}^\infty$.
- operators act on an adapted scale of Sobolev spaces,
- elliptic operators are Fredholm.

Some notation for graded Lie groups

- In the following: fix a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ such that
  - $X_1, \dots, X_{n_1}$ basis of $\mathfrak{g}_1$
  - $X_{n_1+1}, \dots X_{n_2}$ basis of $\mathfrak{g}_2$,
  - $\dots$
- identify $\mathbb{R}^n \to G$ via $(x_1, \dots, x_n) \mapsto \exp(x_1X_1 + \dots + x_nX_n)$,
- can talk of polynomials, Schwartz functions, etc. on G,

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- identify $\mathbb{R}^n \to G$ via $(x_1, \ldots, x_n) \mapsto \exp(x_1 X_1 + \ldots + x_n X_n)$,
- can talk of polynomials, Schwartz functions, etc. on G,
- the weights $q_1, \ldots, q_n \in \mathbb{N}$ of G are defined by $\delta_\lambda(X_i) = \lambda^{q_i} X_i$,
- homogeneous length of a multi-index $\alpha \in \mathbb{N}_0^n$: $[\alpha] = \alpha_1 q_1 + \ldots \alpha_n q_n$.

Example

- $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is a $[\alpha]$-homogeneous differential operator,
- $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a $[\alpha]$-homogeneous function.

Shubin/Helffer type filtration

Definition

Filtration on the algebra of differential operators with polynomial coefficients on G: For $m \in \mathbb{N}_0$ set

$$\mathcal{A}_m = \left\{ \sum_{[\alpha]+[\beta]\leq m} c_{\alpha\beta} x^\alpha X^\beta \mid c_{\alpha\beta} \in \mathbb{C} \right\} \subseteq \mathcal{A}_{m+1} \subseteq \dots$$

- note $X^\beta x^\alpha - x^\alpha X^\beta \in \mathcal{A}_{[\alpha]+[\beta]-1}$ by homogeneity considerations,

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- note $X^\beta x^\alpha - x^\alpha X^\beta \in \mathcal{A}_{[\alpha]+[\beta]-1}$ by homogeneity considerations,
- principal symbol map $\sigma_m \colon \mathcal{A}_m \to \mathcal{A}_m/\mathcal{A}_{m-1}$.

Lemma

Let $\mathfrak{g}^*$ denote the commutative Lie algebra. Then there is an isomorphism

$$\bigoplus_{m=0}^{\infty} \mathcal{A}_m/\mathcal{A}_{m-1} \to \mathcal{U}(\mathfrak{g}^*) \otimes \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}^* \oplus \mathfrak{g})$$

induced by $\sum_{[\alpha]+[\beta]\leq m} c_{\alpha\beta} x^\alpha X^\beta \mapsto \sum_{[\alpha]+[\beta]=m} c_{\alpha\beta} (-i\partial_x)^\alpha X^\beta$.

$\rightsquigarrow$ Rockland condition on $\mathbb{R}^n \times G$

Examples of operators I

Fix a common multiple q of the weights $q_1, \dots, q_n \in \mathbb{N}$. Then

$$\|x\| = \left(\sum_{j=1}^{n} x_j^{\frac{2q}{q_j}} \right)^{\frac{1}{2q}}$$

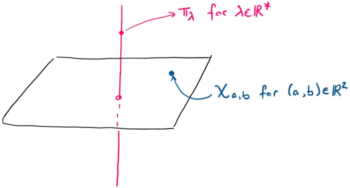
defines a homogeneous quasi-norm.

Example (Analogue of the harmonic oscillator)

The following operator satisfies the Rockland condition on $\mathbb{R}^n \times G$

$$P = \sum_{j=1}^{n} (-1)^{\frac{q}{q_j}} X_j^{\frac{2q}{q_j}} + \|x\|^{2q} \in \Psi_{\Gamma}^{2q}(G).$$

Examples of operators II



Representations of H

$\widehat{H}$ consists of

- characters $\chi_{a,b}$ on $\mathcal{H} = \mathbb{C}$ for $(a, b) \in \mathbb{R}^2$:
  $X \mapsto ia$, $Y \mapsto ib$, $Z \mapsto 0$,
- Schrödinger representations π_λ on $\mathcal{H} = L^2(\mathbb{R})$ for $\lambda \in \mathbb{R}^*$:
  $X \mapsto \sqrt{|\lambda|}\partial_u$, $Y \mapsto \pm i\sqrt{|\lambda|}u$, $Z \mapsto i\lambda 1$.

$\rightsquigarrow$ can check which operators of the form $-X^2 - Y^2 + \alpha Z + p(x, y, z)$ for $\alpha \in \mathbb{C}$ and a polynomial potential p satisfy the Rockland condition.

Groupoids to define a calculus

Lie Groupoid with arrow space $\mathcal{G}$, unit space $\mathcal{G}^{(0)}$

- range and source maps $r, s\colon \mathcal{G} \to \mathcal{G}^{(0)}$,
- multiplication $m\colon \{(\alpha, \beta)\colon s(\alpha) = r(\beta)\} \to \mathcal{G}$,
- convolution "$f * g(\gamma) = \int_{\alpha\beta=\gamma} f(\alpha)g(\beta)$" for $f, g \in C_c^\infty(\mathcal{G})$.

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Two groupoids with arrow space $G \times G$ and unit space $\mathcal{G}^{(0)} = G$:

Example (Pair groupoid of G)

- $r(x, y) = x$, $s(x, y) = y$ and $(x, y)(y, z) = (x, z)$,
- $f * g\,(x, z) = \int_G f(x, y)g(y, z)\,dy \rightsquigarrow$ composition of kernels.

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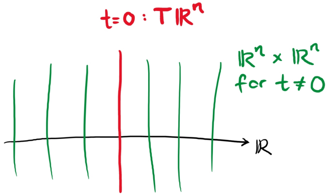
Example (Noncommutative tangent space $T_H G$)

- $r(x,v) = s(x,v) = x$ and $(x,v)(x,w) = (x, v\cdot w)$,
- $(f * g)(x,v) = \int_G f(x,w)g(x,w^{-1}v)\,dw$

  $\rightsquigarrow$ $G = \mathbb{R}^n$: under Fourier transform product of principal symbols

Tangent groupoid for Hörmander classes on $\mathbb{R}^n$



$$\mathbb{TR}^n = T\mathbb{R}^n \times 0 \cup (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^*$$

with smooth structure

$$\Phi\colon \mathbb{TR}^n \xrightarrow{\sim} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R},$$

$$(x, y, t) \mapsto (x, \tfrac{y-x}{t}, t) \qquad t \neq 0,$$

$$(x, v, 0) \mapsto (x, v, 0)$$

Homogeneity and zoom action

- want homogeneity of the symbols wrt. $\lambda \cdot (x, \xi) = (x, \lambda\xi)$ for $\lambda > 0$,
- zoom action of $\mathbb{R}_{>0}$

$$\alpha_\lambda(x, y, t) = (x, y, \tfrac{t}{\lambda}) \qquad t \neq 0,$$

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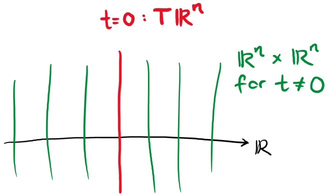
Tangent groupoid for Γ-classes on $\mathbb{R}^n$



$$\mathbb{T}_\Gamma \mathbb{R}^n = T\mathbb{R}^n \times 0 \cup (\mathbb{R}^n \times \mathbb{R}^n) \times \mathbb{R}^*$$

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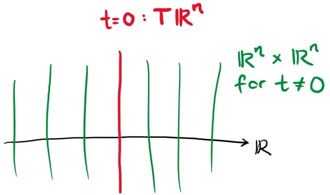

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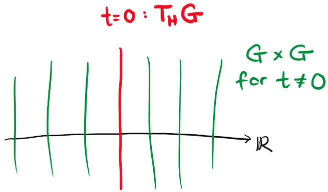
Γ-tangent groupoid for G

Similarly, we define for a graded Lie group G using the dilations $(\delta_\lambda)_{\lambda\in\mathbb{R}}$:



$$\mathbb{T}_\Gamma G = T_H G \times 0 \cup (G \times G) \times \mathbb{R}^*$$

with smooth structure

$$\Phi\colon \mathbb{T}_\Gamma G \xrightarrow{\sim} G \times G \times \mathbb{R},$$

$$(x, y, t) \mapsto (x, \delta_{t^{-2}}(x^{-1}y), t) \quad t \neq 0,$$

$$(x, v, 0) \mapsto (x, v, 0)$$

Γ-zoom action

For $\lambda > 0$ set

$$\alpha_\lambda(x, y, t) = (\delta_{\lambda^{-1}}(x), \delta_{\lambda^{-1}}(y), \tfrac{t}{\lambda}) \qquad t \neq 0,$$

$$\alpha_\lambda(x, v, 0) = (\delta_{\lambda^{-1}}(x), \delta_\lambda(v), 0).$$

Some possible modifications

Remark

More generally, we can consider two commuting dilations, one to define the order of left-invariant differential operators, the other for the order of polynomials.

Example

$\mathbb{R}^n$ with different weights $\rightsquigarrow$ anisotropic calculus (Boggiatto–Nicola):

- weights $(q_1, \ldots, q_n)$, $(w_1, \ldots, w_n)$ and corresponding homogeneous quasi-norms $\|\cdot\|_q$ and $\|\cdot\|_q$
- order m: symbol estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}(1 + \|x\|_q + \|\xi\|_w)^{m - \langle q,\alpha\rangle - \langle w,\beta\rangle}.$$

Pseudo-differential calculus

Follow the approach of van Erp–Yuncken to define a corresponding pseudodifferential calculus:

Definition

An operator $P\colon \mathcal{S}(G) \to \mathcal{S}(G)$ belongs to $\Psi^m_\Gamma(G)$ if there is an essentially m-homogeneous extension $\mathbb{P} \in \mathcal{K}(\mathbb{T}_\Gamma G)$, i.e. $\mathbb{P}_1 = k_P$ and

$$\alpha_{\lambda *}(\mathbb{P}) - \lambda^m \mathbb{P} \in \mathcal{S}(\mathbb{T}_\Gamma G) \quad \text{for all } \lambda > 0.$$

Here, $\mathcal{K}(\mathbb{T}_\Gamma G)$ denotes a certain space of fibred distributions.

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Example (Differential operator with polynomial coefficients)

$P = \sum_{[\alpha]+[\beta]\leq m} c_{\alpha\beta} x^\alpha X^\beta$ can be extended to

$$\mathbb{P}_t = \begin{cases} t^m \sum_{[\alpha]+[\beta]\leq m} t^{-[\alpha]+[\beta]} c_{\alpha\beta} x^\alpha X^\beta & \text{for } t \neq 0, \\ \sum_{[\alpha]+[\beta]=m} c_{\alpha\beta} x^\alpha X^\beta_v & \text{for } t = 0. \end{cases}$$

Properties of the calculus

Analogously to the results of van Erp–Yuncken for filtered manifolds one can show

- there is a well-defined principal cosymbol:

$$\sigma_m(P) = [\mathbb{P}_0] \in \mathcal{K}(T_H G)/\mathcal{S}(T_H G)$$

  for any essentially m-homogeneous extension $\mathbb{P}$,
- the principal symbol map induces short exact sequences,

$$0 \to \Psi_\Gamma^{m-1} \to \Psi_\Gamma^m \xrightarrow{\sigma_m} \Sigma_\Gamma^m \to 0.$$

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- the principal symbol map induces short exact sequences,
$$0 \to \Psi_\Gamma^{m-1} \to \Psi_\Gamma^m \overset{\sigma_m}{\to} \Sigma_\Gamma^m \to 0.$$
- $P \circ Q \in \Psi_\Gamma^{k+l}(G)$ for $P \in \Psi_\Gamma^k(G)$ and $Q \in \Psi_\Gamma^l(G)$ and $\sigma_{k+l}(PQ) = \sigma_k(P) * \sigma_l(Q)$
- $\bigcap\limits_{m\in\mathbb{Z}} \Psi_\Gamma^m(G) = \mathcal{K}^\infty$ (all operators with kernel in $\mathcal{S}(G \times G)$),
- recovers for $G = \mathbb{R}^n$ the calculus of Shubin/ Helffer.

Remark

The C^*-completion of the order zero extension can also be obtained using generalized fixed point algebras. In particular, $\Psi_\Gamma^0 \subseteq \mathbb{B}(L^2G)$ and $\Psi_\Gamma^{-1} \subseteq \mathbb{K}(L^2G)$.

Rockland condition and Fredholm properties

One can define a corresponding Sobolev scale $H^s(G)$ on G.

Proposition

Let $P \in \Psi^m_\Gamma(G)$. Then $P\colon H^s(G) \to H^{s-m}(G)$ is bounded for all $s \in \mathbb{R}$.

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Definition

An operator $P \in \Psi^m_\Gamma(G)$ satisfies the two-sided Rockland condition if $(\mathrm{ev}_x, \pi)(\sigma_m(P))$ and $(\mathrm{ev}_x, \pi)(\sigma_m(P^t))$ are injective on $\mathcal{H}^\infty_\pi$ for all $(x, \pi) \in G \times \widehat{G} \setminus \{(0, \pi_{\mathrm{triv}})\}$.

Using the result of Christ-Geller-Głowacki-Polin:

Theorem

Let $P \in \Psi^m_\Gamma(G)$ satisfy the two-sided Rockland condition, then

(1) *there is a parametrix $Q \in \Psi^{-m}_\Gamma(G)$ such that $PQ - 1, QP - 1 \in \mathcal{K}^\infty$,*

(2) *$P\colon H^s(G) \to H^{s-m}(G)$ is Fredholm for all $s \in \mathbb{R}$.*

How to compute the index?

On $\mathbb{R}^n$:

Theorem (Elliott–Natsume–Nest)

Let $\mathrm{Op}(a) \in \Psi_\Gamma^m(\mathbb{R}^n)$ *be elliptic of positive order. Then*

$$\mathrm{ind}(\mathrm{Op}(a)) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} \mathrm{tr}(p_a (dp_a)^{2n}), \tag{1}$$

where

$$p_a = \begin{pmatrix} (1+a^*a)^{-1} & (1+a^*a)^{-1}a^* \\ a(1+a^*a)^{-1} & a(1+a^*a)^{-1}a^* \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Idea of the proof:

- use the tangent groupoid,
- Fredholm index as a pairing of a cyclic cocycle ω_1 and a K-theory class associated with $\mathrm{Op}(a)$,
- extend cocycle to $(\omega_t)_{t\in\mathbb{R}}$ on $\mathbb{T}_\Gamma\mathbb{R}^n$,
- pairing at $t = 0$ gives right hand side of (1),
- result of the pairing depends continuously on t.

Index formula on G

Using that $G \cong \mathbb{R} \rtimes \mathbb{R} \ldots \rtimes \mathbb{R}$, we construct a cocyle $(\omega_t)_{t\in\mathbb{R}}$ on $\mathbb{T}_\Gamma G$ s.t.:

Theorem (E–Nest–Schmitt)

Let $P \in \Psi^m_\Gamma(G)$ be of positive order and satisfy the two-sided Rockland condition. Then

$$\mathrm{ind}(P) = (\omega_0 \# \mathrm{tr})(p_{\mathbb{P}_0}, \ldots, p_{\mathbb{P}_0}).$$

Example (Heisenberg group)

One computes for $f_i \in \mathcal{S}(H \times H)$

$$\omega_0(f_0, \ldots, f_6) = \sum_{\sigma \in S_6} \mathrm{sgn}(\sigma) \int_H f_0 * D_{\sigma(1)} f_1 * \ldots * D_{\sigma(6)} f_6(x, 0)\, \mathrm{d}x$$
$$+ \text{ extra terms (explicitly computable)}$$

where $D_1 = \partial_{x_1}, D_2 = v_1, D_3 = \partial_{x_2}, D_4 = v_2, D_5 = \partial_{x_3}, D_6 = v_3$.

Rewriting the cocycle using Fourier transform

Recall on $\mathbb{R}^n$:

$$\operatorname{ind}(\operatorname{Op}(a)) = \frac{1}{(2\pi i)^n n!} \int_{T^*\mathbb{R}^n} \operatorname{tr}(p_a (dp_a)^{2n})$$

.

Plancherel Theorem

For $f \in \mathcal{S}(G)$ and the *Plancherel measure* μ on $\widehat{G}$

$$f(0) = \int_{\widehat{G}} \operatorname{Tr}\left(\widehat{f}(\pi)\right) \mathrm{d}\mu(\pi).$$

- denote by $\Delta_{v_i} \widehat{f}(\pi) = \widehat{v_i \cdot f}(\pi)$ (difference operators),

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- denote by $\Delta_{v_i}\widehat{f}(\pi) = \widehat{v_i \cdot f}(\pi)$ (difference operators),
- the Plancherel measure on $\widehat{H}$ is supported within the Schrödinger representations π_λ for $\lambda \in \mathbb{R}\backslash\{0\}$,
- using this, the cocycle can be rewritten, for example,

$$\int_H f_0 * \partial_{x_1} f_1 * \ldots * v_3 f_6(x,0)\,\mathrm{d}x$$

$$= (2\pi)^{-4}\int_{H\times\mathbb{R}\backslash\{0\}} \mathrm{sgn}(\lambda)\,\mathrm{Tr}(\widehat{f_0(x)}(\pi_\lambda)\partial_{x_1}\widehat{f_1(x)}(\pi_\lambda)\ldots\Delta_{v_3}\widehat{f_6(x)}(\pi_\lambda))\,\mathrm{d}x\,\mathrm{d}\lambda.$$

Open questions/ future directions

- We can show for several differential operators on the Heisenberg group that their index is zero. Is there a differential operator with non-zero Fredholm index?
- What about higher step groups?
- Is there a Weyl law for operators on G?

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Merci!

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