Quantum limits of some (perturbed) sub-Laplacians

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High frequency analysis: from operator theory to PDEs

Based on joint works with Gabriel Rivière and Chenmin Sun

Plan of the talk

1. Introduction. Sub-Laplacians

Let $d \in \mathbb{N}^*$ and let $\mathcal{M}$ be a smooth compact manifold of dimension d. We consider $m \geq 1$ smooth vector fields $X_1, \ldots, X_m$ on $\mathcal{M}$ (not necessarily independent) which satisfy the following:

Hörmander condition

The vector fields $X_1, \ldots, X_m$ and their iterated brackets

$$[X_i, X_j],\ [X_i, [X_j, X_k]], \ldots$$

span the tangent space $T_x\mathcal{M}$ at every point $x \in \mathcal{M}$.

Let μ be a **smooth volume** on $\mathcal{M}$, we define the **sub-Laplacian**:

$$\Delta_\mu := -\sum_{i=1}^{m} X_i^* X_i,$$

where the star denotes the transpose in $L^2(\mathcal{M}, \mu)$.

Theorem (Hörmander, 1967)

The operator Δ_μ is ***hypoelliptic****.*

- **Baouendi-Grushin operator** on $\mathcal{M} = \mathbb{T}^1_x \times \mathbb{T}^1_y$. Given $v \in \mathcal{C}^\infty(\mathbb{T}^1_x; \mathbb{R})$ s.t.

$$v(x) \sim x \text{ near } x = 0, \quad v(x) \neq 0 \text{ away from } x = 0,$$

we define $X_1 = \partial_x$, $X_2 = v(x)\partial_y$, and, for $\mu = dx \otimes dy$:

$$\Delta_G := -X_1^* X_1 - X_2^* X_2 = -\partial_x^2 - (v(x)\partial_y)^2.$$

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- **Sub-Riemannian contact Laplacians in 3D (particular case)**. Let (M, g) be a smooth, compact, oriented and boundaryless Riemannian surface, set

$$\mathcal{M} := SM = \{q = (m, \theta) \in TM \, : \, \|\theta\|_{g(m)} = 1\}.$$

On $\mathcal{M}$ we consider the vector fields X (geodesic) and V (rotation on the fiber) and set $X_\perp := [X, V]$. Let $\mu = \mu_L$ be the Liouville measure on $\mathcal{M}$, we define:

$$\Delta_{\mathrm{sR}} := -X_\perp^* X_\perp - V^* V.$$

1.1. Our models

- **Baouendi-Grushin operator** on $\mathcal{M} = \mathbb{T}^1_x \times \mathbb{T}^1_y$. Given $v \in \mathcal{C}^\infty(\mathbb{T}^1_x; \mathbb{R})$ s.t.

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$$\Delta_{\mathrm{sR}} := -X_\perp^* X_\perp - V^* V.$$

- **Perturbations of Δ_{sR}**. Let $Q, W \in \mathcal{C}^\infty(\mathcal{M}; \mathbb{R})$. We consider semiclassical perturbations of the form:

$$\widehat{P}_h := h^2 \Delta_{\mathrm{sR}} - ih^2 QX - \frac{ih^2 X(Q)}{2} + W, \quad h > 0.$$

Proposition

$\mathrm{Sp}_{L^2(\mathbb{T}^2)}(\Delta_G) = \{0 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots \to +\infty\}$.

Set $h_j := \lambda_j^{-1}$. *Let* $(\psi_h) \subset L^2(\mathcal{M})$ *normalized s.t.* $(h^2\Delta_G - 1)\psi_h = 0$. *Let* $\nabla_G := (\partial_x, v(x)\partial_y)$. *Then:*

$$\|h\nabla_G\psi_h\|_{L^2}^2 + \|h^2 D_y\psi_h\|_{L^2}^2 \lesssim 1.$$

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$$\|h\nabla_G\psi_h\|_{L^2}^2 + \|h^2 D_y\psi_h\|_{L^2}^2 \lesssim 1.$$

By Hörmander and Rothschild-Stein theorems, one also has:

Proposition

Assume that $\|Q\|_{\mathcal{C}^0} < 1$. *Then* $\exists h_0 > 0$ *s.t.* $\mathrm{Sp}_{L^2(\mathcal{M},\mu)}(\widehat{P}_h) = \{\lambda_h(j) \,:\, j \geq 0\}$ *for* $0 < h \leq h_0$, *with*

$$\min W + \mathcal{O}_Q(h) \leq \lambda_h(0) \leq \lambda_h(1) \leq \cdots \to +\infty.$$

Let $(\psi_h^j) \subset L^2(\mathcal{M}, \mu)$ *normalized s.t.* $(\widehat{P}_h - \lambda_h(j))\psi_h^j = 0$. *Then:*

$$\|hX_\perp\psi_h^j\|_{L^2}^2 + \|hV\psi_h^j\|_{L^2}^2 + \|h^2 X\psi_h^j\|_{L^2}^2 \leq C_{Q,W}(1 + |\lambda_h(j)|)^2.$$

- **Baouendi-Grushin type operators**. Let $\Delta_\gamma := -\partial_x^2 + |x|^{2\gamma}\partial_y^2$ on $\mathcal{M} = (-1,1)_x \times \mathbb{T}_y$. Observability and controllability of the Schrödinger equation Letrouit, Sun (2020) ($\gamma \geq 1$) and Burq, Sun (2019) ($\gamma = 1$).
- **Sub-Riemannian contact sub-Laplacians (general case)**. Let $\mathcal{N}$ be a smooth compact 3-dimensional manifold so that there exists

$$T\mathcal{N} \supset \mathcal{D} = \mathrm{Span}(X_2, X_3) = \ker \alpha, \quad \alpha \wedge d\alpha \neq 0,$$

  The study of quantum limits for Δ_{sR} was undertaken by Colin-de-Verdière, Hillairet, Trélat (2015 ...).
- **Heisenberg sub-Laplacians**. The spectral asymptotics and quantum evolution of sub-Laplacians in groups of Heisenberg type have been studied by Bahouri, Fermanian-Kammerer, Gallagher (2012), Letrouit (2020), Fermanian-Kammerer, Fischer (2021), Fermanian-Kammerer, Letrouit (2021) ...
- **Magnetic Laplacians**. The fine structure of eigenvalues and eigenfunctions of semiclassical magnetic operators has been widely studied. Some recents works include those of Helffer, Hérau, Raymond, Vu Ngoc, Morin, Krejcirik, Abou Alfa ...

Definition

A probability measure $\nu \in \mathcal{P}(\mathbb{T}^2)$ is a **quantum limit** of Δ_G if there exists a normalized sequence $(\psi_h) \subset L^2(\mathbb{T}^2)$ satisfying $(h^2\Delta_G - 1)\psi_h = 0$ such that, for every $a \in \mathcal{C}(\mathbb{T}^2)$,

$$\lim_{h\to 0^+} \int_{\mathbb{T}^2} a\,|\psi_h|^2 dxdy = \int_{\mathbb{T}^2} a\, d\nu.$$

On the other hand, there exists a subsequence and a positive Radon measure $w \in \mathcal{M}_+(T^*\mathcal{M})$ (**semiclassical measure**) such that, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\lim_{h\to 0^+} \left\langle \operatorname{Op}_h(a)\psi_h, \psi_h \right\rangle_{L^2(\mathbb{T}^2)} = \int_{T^*\mathbb{T}^2} a\, dw.$$

Moreover, let $H_G(x, y, \xi, \eta) := \xi^2 + v(x)^2\eta^2$ be the **principal symbol** of Δ_G defined on $(x, y, \xi, \eta) \in T^*\mathbb{T}^2$. Then:

- $\operatorname{supp} w \subset H_G^{-1}(1)$,
- $\{H_G, w\} = 0$.

Definition

A probability measure $\nu \in \mathcal{P}(\mathcal{M})$ is a **quantum limit** of $\widehat{P}_h$ if there exists a normalized sequence $(\psi_h) \subset L^2(\mathcal{M}, \mu)$ satisfying

$$(\widehat{P}_h - \lambda_h)\psi_h = 0, \quad \lambda_h \to \lambda_0 \geq \min W,$$

such that, for every $a \in \mathcal{C}(\mathcal{M})$,

$$\lim_{h \to 0^+} \int_{\mathcal{M}} a \, |\psi_h|^2 d\mu = \int_{\mathcal{M}} a \, d\nu.$$

There exists a subsequence and a positive Radon measure $w \in \mathcal{M}_+(T^*\mathcal{M})$ (**semiclassical measure**) such that, for every $a \in \mathcal{C}_c^\infty(T^*\mathcal{M})$,

$$\lim_{h \to 0^+} \left\langle \operatorname{Op}_h(a)\psi_h, \psi_h \right\rangle_{L^2(\mathcal{M},\mu)} = \int_{T^*\mathcal{M}} a \, dw.$$

Moreover:

- $\operatorname{supp} w \subset \mathcal{E}^{-1}(\lambda_0) := \{(q,p) \in T^*\mathcal{M} \, : \, \sigma(X_\perp)^2 + \sigma(V)^2 + W(q) = \lambda_0\}$,
- $\{\sigma(X_\perp)^2 + \sigma(V)^2 + W, w\} = 0$.

Let $\pi : T^*\mathcal{M} \to \mathcal{M}$ be the canonical projection, the measure

$$\nu_\infty := \nu - \pi_* w$$

does not vanish in general due to escape of mass at infinity. Notice that $H_G^{-1}(1)$ and $\mathcal{E}^{-1}(\lambda_0)$ are not compact.

Goal: To describe the measure ν_∞.

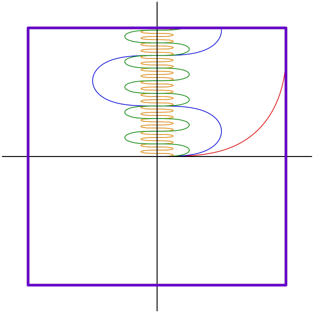
Theorem (Colin de Verdière, Hillairet, Trélat, 2015)

Let $\widehat{P}_h = h^2\Delta_{\mathrm{sR}}$ $(Q, W = 0)$, then the measure ν_∞ satisfies $X(\nu_\infty) = 0$. In the more general 3D contact case, X is replaced by the Reeb vector field on $T\mathcal{M}$.

Question: How stable is this property under hypoelliptic perturbations of Δ_{sR}?

Classical dynamics on $H_G^{-1}(1)$

Let $H_G(x, y, \xi, \eta) = \xi^2 + v(x)^2\eta^2$. The level-set $H_G^{-1}(1)$ is not compact in the regime $|\eta x| \sim 1$ as $x \to 0$. Classical dynamics in this regime is characteristic of the motion of charged particles subjected to a magnetic field (fast rotation coupled to a drift).



$$\dot{x}(t) = 2\xi,$$

$$\dot{y}(t) = 2\eta v(x)^2 \sim 2\eta x^2,$$

$$\dot{\xi}(t) = -2v'(x)v(x)\eta^2 \sim -2x\eta^2,$$

$$\dot{\eta}(t) = 0.$$

Proposition

$\operatorname{supp} \nu_\infty \subset \{0\}_x \times \mathbb{T}_y.$

Main result (Δ_G)

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function near zero, define for $\lambda > 0$, $\chi_\lambda := \chi(\cdot/\lambda)$.

Theorem (A., Sun, 2022; A., Rivière, 2023)

There exists a non-negative Radon measure $\mu_\infty \in \mathcal{M}_+(\mathbb{T}_y \times \mathbb{R}_\eta)$ such that, for every $a \in \mathcal{C}_c^\infty(\mathbb{T}_y \times \mathbb{R}_\eta)$:

$$\lim_{R\to\infty} \lim_{h\to 0^+} \left\langle \operatorname{Op}_h \left((1 - \chi_R(\eta) a(y, h\eta) \right) \psi_h, \psi_h \right\rangle_{L^2(\mathbb{T}^2)} = \int_{\mathbb{T}_y \times \mathbb{R}_\eta} a \, d\mu_\infty .$$

Moreover,

$$\mu_\infty = \overline{\mu}_\infty + \sum_{k=0}^{\infty} \mu_{k,\infty}^{+} + \mu_{k,\infty}^{-},$$

where:

- $\operatorname{supp} \overline{\mu}_\infty \subset \mathbb{T}_y \times \{0\}, \quad \operatorname{supp} \mu_{k,\infty}^{\pm} \subset \left\{ (y,\eta) \in \mathbb{T}_y \times \mathbb{R}_\eta \ : \ \eta = \pm \dfrac{1}{2k+1} \right\}.$
- $\partial_y \overline{\mu}_\infty = 0, \quad \partial_y \mu_{k,\infty}^{\pm} = 0.$
- $\nu_\infty(x,y) = \delta_0(x) \otimes \int_{\mathbb{R}} \mu_\infty(y, d\eta).$

Remarks

- The measure $\overline{\mu}_\infty$ captures the **sub-critical** sub-elliptic regime:

$$h^{-1} \ll |D_y| \ll h^{-2}.$$

- The measures $\mu_{k,\infty}^{\pm}$ capture the **critical** sub-elliptic regime:

$$|D_y| \sim h^{-2}.$$

Notice, in particular, that by projection the measure ν_∞ inherit the invariance property

$$\partial_y \nu_\infty = 0,$$

which is analogous to the invariance $X(\nu_\infty) = 0$ in the 3D contact case.

The "quantized" support property of the measures $\mu_{k,\infty}^{\pm}$ reflects a **two-microlocal phenomenon**.

Main result $(\widehat{P}_h)$

We associate to each smooth function $f \in \mathcal{C}^\infty(\mathcal{M})$ the vector field in the contact plane $\mathcal{D} = \mathrm{span}(X_\perp, V)$:

$$\Omega_f := V(f)X_\perp - X_\perp(f)V.$$

Theorem (A., Rivière, 2023)

Let $Q, W \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ such that $\|Q\|_{\mathcal{C}^0} < 1$, let $\lambda_0 > \max_{q \in \mathcal{M}} W(q)$. Then the measure ν_∞ decomposes as

$$\nu_\infty = \overline{\nu}_\infty + \sum_{k=0}^{\infty} (\nu_{k,\infty}^+ + \nu_{k,\infty}^-),$$

where $\overline{\nu}_\infty, \nu_{k,\infty}^\pm$ are non-negative Radon measures on $\mathcal{M}$ verifying:

$$Y_W(\overline{\nu}_\infty) = 0, \quad Y_{W,Q,k}^\pm(\nu_{k,\infty}^\pm) = 0,$$

with

- $Y_W = X + \Omega_{\log(\lambda_0 - W)}$,
- $Y_{W,Q,k}^\pm = \big(\pm(2k+1) + Q\big) Y_W - \Omega_Q$.

Remarks

Let $H_1 := \sigma(X)$.

- The measure $\overline{\nu}_\infty$ captures the **sub-critical** sub-elliptic regime:

$$h^{-1} \ll |H_1| \ll h^{-2}.$$

- The measures $\nu^{\pm}_{k,\infty}$ capture the **critical** sub-elliptic regime:

$$|H_1| \sim h^{-2}.$$

The proof relies in the study of a suitable lift μ_∞ of ν_∞ to the phase space via introducing a new variable $E \in \mathbb{R}$ which parameterizes the escape of mass along the degenerated direction X as $h \to 0^+$, so that:

$$\nu_\infty(q) = \int_{\mathbb{R}} \mu_\infty(q, dE).$$

Sketch of proof (Δ_G)

Let us consider test functions $a \in \mathcal{C}_c^\infty(\mathbb{R}_\rho \times \mathbb{T}_y \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function near zero, define for $\lambda > 0$, $\chi_\lambda := \chi(\cdot/\lambda)$. Set, for $R > 0$ sufficiently large and $\epsilon > 0$:

$$a^1_{R,\epsilon,h}(x,y,\xi,\eta) := (1 - \chi_R(\eta))\chi_\epsilon(h\eta)a(x|\eta|, y, \xi, h\eta),$$

$$a^2_{R,\epsilon,h}(x,y,\xi,\eta) := (1 - \chi_R(\eta))(1 - \chi_\epsilon(h\eta))\chi_\epsilon(h\eta)a(x|\eta|, y, \xi, h\eta)$$

Let (ψ_h) satisfy $(h^2\Delta_G - 1)\psi_h = 0$, define:

$$I^j_{R,\epsilon,h}(a) := \left\langle \mathrm{Op}_h(a^j_{R,\epsilon,h})\psi_h, \psi_h \right\rangle_{L^2(\mathbb{T}^2)}, \quad j = 1, 2.$$

Proposition (Existence of two-microlocal semiclassical measures)

There exist $\overline{\mu}_\infty \in \mathcal{M}_+(\mathbb{R}_\rho \times \mathbb{T}_y \times \mathbb{R}_\xi)$, $M_\infty \in \mathcal{M}_+(\mathbb{T}_y \times \mathbb{R}^*_\eta; \mathcal{L}^1(L^2(\mathbb{R}_x)))$ *such that:*

- $\displaystyle \lim_{R\to+\infty} \lim_{\epsilon\to 0} \lim_{h\to 0^+} I^1_{R,\epsilon,h}(a) = \int_{\mathbb{R}_\rho \times \mathbb{T}_y \times \mathbb{R}_\xi} a(\rho, y, \xi, 0)\, d\overline{\mu}_\infty,$
- $\displaystyle \lim_{R\to+\infty} \lim_{\epsilon\to 0} \lim_{h\to 0^+} I^2_{R,\epsilon,h}(a) = \mathrm{Tr} \int_{\mathbb{T}_y \times \mathbb{R}^*_\eta} \mathrm{Op}_1^{\mathbb{R}_x}\left(a\left(x|\eta|, y, \xi, \eta\right)\right) dM_\infty.$

Properties of the two-microlocal semiclassical measures adapted to Δ_G

The measure $\overline{\mu}_\infty$ satisfies:

- $\operatorname{supp} \overline{\mu}_\infty \in \{(\rho, y, \xi) \,:\, \rho^2 + \xi^2 = 1\}$,
- $\{\rho^2 + \xi^2, \overline{\mu}_\infty\} = 0$,
- $\partial_y \overline{\mu}_\infty = 0$.

The measure M_∞ satisfies:

- $\operatorname{supp} M_\infty \subset \bigcup_{k=0}^{\infty} \left\{ (y, \eta) \,:\, |\eta| = \frac{1}{2k+1} \right\}$,
- $[D_x^2 + \eta^2 x^2, M_\infty] = 0$,
- $\partial_y M_\infty = 0$.

We start from the equation

$$\left\langle \chi(h^2\Delta_G - 1)(1 - \chi_R(hD_y)(1 - \chi_\epsilon(h^2 D_y)\psi_h, \psi_h \right\rangle_{L^2(\mathbb{T}^2)} = 0.$$

Taking limits $h \to 0^+$, $\epsilon \to 0$, and $R \to +\infty$, we arrive to:

$$\mathrm{Tr} \int_{\mathbb{T}_y \times \mathbb{R}^*_\eta} \chi(h^2 D_x^2 + x^2\eta^2 - 1) dM_\infty = 0.$$

Considering an orthonormal basis $\{\varphi_k(\eta)\}_{k=0}^\infty$ in $L^2(\mathbb{R}_x)$ given by eigenfunctions of $D_x^2 + \eta^2 x^2$, and defining

$$\mu_{k,\infty}(y,\eta) := \langle M_\infty(y,\eta)\varphi_k(\eta), \varphi_k(\eta)\rangle_{L^2(\mathbb{R}_x)},$$

we get:

$$\sum_{k=0}^\infty \int_{\mathbb{T}_y \times \mathbb{R}^*_\eta} \chi(|\eta|(2k+1) - 1) d\mu_{k,\infty}(y,\eta) = 0.$$

The main difficulty regarding the 3D contact ($\mathcal{M} = SM$) case is the following:

- The operator Δ_{sR} is *not in normal form*, due to the commutation relations:

$$[X, X_\perp] = -KV, \quad [X, V] = X_\perp, \quad [X_\perp, V] = -X,$$

where K is the sectional curvature of M (seen on $\mathcal{M} = SM$ via pullback).

Notice that in the case of the Baouendi-Grushin operator,

$$[\partial_x, v(x)\partial_y]\big|_{x=0} = \partial_y, \quad [\partial_y, \partial_x] = 0, \quad [\partial_y, v(x)\partial_y] = 0.$$

Similarly, in the case of the 3-dimensional Heisenberg group,

$$Z_H = -[X_H, Y_H] = \partial_z, \quad [Z_H, X_H] = 0, \quad [Z_H, Y_H] = 0,$$

where $X_H = \partial_x$ *and* $Y_H = \partial_y - x\partial_z$.

Phase-space distribution in the sub-elliptic regime

Let $H_1 = \sigma(X)$, $H_2 = \sigma(X_\perp)$, and $H_3 = \sigma(V)$. We consider the cut-offs:

$$\chi_R^B := \chi\left(\frac{H_1 + H_2 + H_3}{R}\right), \qquad \tilde{\chi}_R^B := 1 - \chi_R^B, \quad R > 0,$$

$$\chi_\epsilon^C := \chi\left(\frac{\epsilon H_1}{\sqrt{H_2^2 + H_3^2 + 1}}\right), \quad \tilde{\chi}_\epsilon^C := 1 - \chi_\epsilon^C, \quad \epsilon > 0,$$

and, let $(x, y, z) \in \mathcal{U}_0 \subset \mathcal{M}$ be a local chart, define the distributions:

$$\mu_h^{R,\epsilon} : a \in \mathcal{C}_c^\infty(\mathcal{U}_0 \times \mathbb{R}) \mapsto \left\langle \operatorname{Op}_h(a(x, y, z, hH_1)\tilde{\chi}_R^B \tilde{\chi}_\epsilon^C)\psi_h, \psi_h \right\rangle_{L^2(\mathcal{U}_0,\mu)}.$$

By estimates of pseudo-differential calculus, we can take limits

$$\lim_{\epsilon \to 0} \lim_{R \to +\infty} \lim_{h \to 0^+} \langle \mu_h^{R,\epsilon}, a \rangle = \int_{\mathcal{U}_0 \times \mathbb{R}} a(x, y, z, E) d\mu_\infty(x, y, z, E),$$

where the measure μ_∞ satisfies

$$\nu_\infty = \int_{\mathbb{R}} \mu_\infty(\cdot, dE).$$

Properties of μ_∞

Let us define:

$$\mathcal{H}(q, E) := \frac{\lambda_0 - W(q)}{E} - Q(q), \quad (q, E) \in \mathcal{M} \times \mathbb{R}^*,$$

$$\mathcal{X}_{W,Q} := \frac{\lambda_0 - W}{E} X + \Omega_{\mathcal{H}} + EX(\mathcal{H})\partial_E,$$

and notice that $\mathcal{X}_{W,Q}(\mathcal{H}) = 0$.

Theorem (A., Rivière, 2023)

The measure μ_∞ decomposes as

$$\mu_\infty = \overline{\mu}_\infty + \sum_{k=0}^{\infty} \left(\mu^+_{k,\infty} + \mu^-_{k,\infty} \right),$$

where

- $\operatorname{supp} \overline{\mu}_\infty \subset \mathcal{M} \times \{0\}$, $\operatorname{supp} \mu^\pm_{k,\infty} \subset (\pm\mathcal{H})^{-1}(2k+1) \cap \mathbb{R}^*_\pm$,
- $Y_W(\overline{\mu}_\infty) = 0$, $\mathcal{X}_{W,Q}(\mu^\pm_{k,\infty}) = 0$.

We start from the algrebaic relations:

$$\{H_1, H_2\} = -KH_3, \quad \{H_1, H_3\} = H_2, \quad \{H_2, H_3\} = -H_1.$$

The pair (H_2, H_3) behaves in some sense like a system of coordinates in $T^*\mathbb{R}$ for the classical harmonic oscillator (notice that $\sigma(h^2\Delta_{\mathrm{sR}}) = H_2^2 + H_3^2$). Defining it in terms of complex coordinates:

$$Z = H_2 + iH_3, \quad \overline{Z} = H_2 - iH_3,$$

we can rewrite write $H_2^2 + H_3^2 = |Z|^2$ and, using that $\{Z, \overline{Z}\} = 2iH_1$, we obtain the algebraic identity

(A) $$\left\{ |Z|^2, \frac{Z^k \overline{Z}^l}{2i(l-k)} \right\} = H_1 Z^k \overline{Z}^l, \quad k \neq l,$$

which will allow us to cancel the bad terms in the normal form procedure.

Wigner equation

The idea consists in considering the *Wigner equation*

$$\left\langle [\widehat{P}_h, \mathrm{Op}_h(a\tilde{\chi}_R^B\tilde{\chi}_\epsilon^C)]\psi_h, \psi_h \right\rangle_{L^2(\mathcal{U}_0,\mu)} = 0,$$

and trying to obtain an ODE for μ_∞ in the limit $\epsilon \to 0$, $R \to +\infty$, $h \to 0^+$.

Let us first consider the term $[h^2\Delta_{\mathrm{sR}}, \mathrm{Op}_h(a\tilde{\chi}_R^B\tilde{\chi}_\epsilon^C)]$. If we forget "naifly" the cut-offs and use formally the symbolic calculus

$$[\mathrm{Op}_h(\cdot), \mathrm{Op}_h(\cdot)] = \frac{h}{i}\mathrm{Op}_h(\{\cdot,\cdot\}) + O(h^3)$$

we find the first term

$$\begin{aligned}\{|Z|^2, a(\cdot, hH_1)\} &= \{|Z|^2, a(\cdot, E)\}|_{E=hH_1} + \{|Z|^2, hH_1\}\partial_E a \\ &= Z\{\overline{Z}, a(\cdot, E)\}|_{E=hH_1} + \overline{Z}\{Z, a(\cdot, E)\}|_{E=hH_1} \\ &\quad + \frac{ih(1-K)}{2}(Z^2 - \overline{Z}^2)\partial_E a\end{aligned}$$

A small deformation of a

To improve the commutation relation $\{|Z|^2, a(\cdot, E)\}|_{E=hH_1}$ in the sub-elliptic regime $|H_2|, |H_3| \ll |H_1|$, we set:

$$\tilde{a} := a + a_1 + a_2,$$

with

$$a_1 := \frac{Z}{2iH_1}\{\overline{Z}, a\} - \frac{\overline{Z}}{2iH_1}\{Z, a\}, \quad a_2 := -\frac{(Z^2 X_{\overline{Z}}^2 + \overline{Z}^2 X_Z^2)(a)}{8H_1^2}.$$

This gives:

$$\{|Z|^2, \tilde{a}\} = \frac{|Z|^2}{H_1} X(a) + \frac{1}{H_1} R_a,$$

with

$$R_a = \sum_{|\alpha| \geq 2} R_{a,\alpha}(x, y, z, E) \left(\frac{H_2}{H_1}\right)^{\alpha_2} \left(\frac{H_3}{H_1}\right)^{\alpha_3}, \quad \alpha = (\alpha_2, \alpha_3).$$

A small deformation of H_1

To improve the commutation relation $\{|Z|^2, H_1\}$ also in the sub-elliptic regime $|H_2|, |H_3| \ll |H_1|$, we set:

$$\tilde{H}_1 := H_1(1 + P_2 + P_3)$$

with

$$P_j = \sum_{|\alpha|=j} P_{j,\alpha}(x,y,z) \left(\frac{H_2}{H_1}\right)^{\alpha_2} \left(\frac{H_3}{H_1}\right)^{\alpha_3}, \quad j = 2, 3,$$

we get:

$$\{|Z|^2, \tilde{H}_1\} = H_2^2 R_1 + H_3^2 R_2 + H_2 H_3 R_3$$

with

$$R_j = \sum_{|\alpha|\geq 2} R_{j,\alpha}(x,y,z) \left(\frac{H_2}{H_1}\right)^{\alpha_2} \left(\frac{H_3}{H_1}\right)^{\alpha_3}, \quad j = 1, 2, 3.$$

Conclusion

Defining the distribution

$$\tilde{\mu}_h^{R,\epsilon} : a \in \mathcal{C}_c^\infty(\mathcal{U}_0 \times \mathbb{R}) \mapsto \left\langle \mathrm{Op}_h(\tilde{a}(x,y,z,h\tilde{H}_1)\tilde{\chi}_R^B \tilde{\chi}_\epsilon^C)\psi_h, \psi_h \right\rangle_{L^2(\mathcal{U}_0,\mu)},$$

(further technical work is required regarding the cut-offs) we prove that the weak limit of $\tilde{\mu}_h^{R,\epsilon}$ is still μ_∞.

From the modified Wigner equation

$$\left\langle [\widehat{P}_h, \mathrm{Op}_h(\tilde{H}_1 \tilde{a}(x,y,z,h\tilde{H}_1)\tilde{\chi}_R^B \tilde{\chi}_\epsilon^C)]\psi_h, \psi_h \right\rangle_{L^2(\mathcal{U}_0,\mu)} = 0,$$

we arrive to the ODE:

$$\int_{\mathcal{U}_0 \times \mathbb{R}} \mathcal{X}_{W,Q}(Ea) d\mu_\infty(x,y,z,E) = 0, \quad \forall a \in \mathcal{C}_c^\infty(\mathcal{U}_0 \times \mathbb{R}).$$

Restricting this equation to the different components of the support of μ_∞, we get:

$$Y_W(\overline{\mu}_\infty) = 0, \quad \mathcal{X}_{W,Q}(\mu_{k,\infty}^{\pm}) = 0.$$

Finally, by projection on $\mathcal{M}$ of these identities, we obtain that

$$Y_W(\overline{\nu}_\infty) = 0, \quad Y_{W,Q,k}^{\pm}(\nu_{k,\infty}^{\pm}) = 0.$$