Wick symbols of evolution operators

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Plan

Background

1. Problems in mathematical physics (arbitrary number of particles)
2. Pseudodifferential Weyl calculus in an infinite dimensional frame

Starting point :
a real, separable, infinite dimensional Hilbert space $(\mathfrak{h}, \cdot, | \,|)$ and its complexified space $\mathfrak{h}_{\mathbb{C}}$.
Two spaces built "on" $\mathfrak{h}$:

- Fock space
- Wiener space

The Fock space

$\mathcal{F}_s(\mathfrak{h})$ is the symmetric Fock space built on the complexified space of $\mathfrak{h}$:

$$\mathbb{C} \oplus \mathfrak{h}_{\mathbb{C}} \oplus \cdots \oplus \mathfrak{h}^{\otimes n,s} \oplus \ldots$$

Usual creation and annihilation operators :

$$f \in \mathfrak{h} \quad , \quad u = S_n(u_1 \otimes \cdots \otimes u_n)$$

$$\begin{aligned} a^*(f)u &= \sqrt{n+1} S_{n+1}(f \otimes u) \\ a(f)u &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} < f, u_j > S_{n-1}(u_1 \otimes \cdots \otimes \check{u}_j \otimes \cdots \otimes u_n), \\ \Phi(f) &= \frac{1}{\sqrt{2}}(a(f) + a^*(f)) \end{aligned}$$

The Wiener space

Construction (abridged)

$$\underbrace{B'}_{3} \subset \underbrace{\mathfrak{h} = (\mathfrak{h}, |\ |)}_{1} \subset \underbrace{B = (B, ||\ ||)}_{2} \longrightarrow \underbrace{(B, \mathcal{B}(B), \mu_{B,h/2})}_{4}$$

Bricks

Monomials

$a \in B'$ is

- a linear continuous function on B
- a random variable (denoted by ℓ_a) on $(B, \mu_{B,h/2})$, with the normal distribution

$$\ell_a \sim \mathcal{N}(0, \sigma^2 = \frac{h}{2}|a|^2).$$

The Wiener space : cylindrical functions

E finite dimensional subspace of B', orthonormal basis $(e_1, \dots e_d)$.
$\tilde{f} : B \to \mathbb{C}$ is cylindrical and based on E if it is written

$$\tilde{f} = f \circ \tilde{\pi}_E.$$

with the "projection" $\tilde{\pi}_E(u) = \sum_1^d \ell_{e_j}(u) e_j$ and f defined on E.

Integrals of cylindrical functions

$$\int_B \tilde{f}(u)\, \mu_{B,h/2}(u) = \int_E f(y) \underbrace{\frac{e^{-|y|^2/h}}{(\pi h)^{d/2}}\, dy}_{d\mu_{E,h/2}(y)} = \int_E f(y)\, d\mu_{E,h/2}(y).$$

Test functions

E finite dim. subspace of B'.

The space $\mathcal{D}$

$\mathcal{S}_E = \mathcal{S}_{E,h/2}$: space of functions $\varphi : E \to \mathbb{C}$ s.t. $x \mapsto \varphi(x)e^{-\frac{|x|^2}{2h}}$ is rapidly decreasing.
$\mathcal{D}_E$: space of functions $\tilde{f} : B \to \mathbb{C}$ s.t. $\tilde{f} = \varphi \circ \widetilde{\pi}_E$ with $\varphi \in \mathcal{S}_E$.
($\widetilde{\pi}_E$ is the "projection").
We set :

$$\mathcal{D} = \mathcal{D}_{B'} = \bigcup_{E \in \mathcal{F}(\mathcal{H})} \mathcal{D}_E.$$

An example of symbol classes

Defined by a quadratic form

A : linear, selfadjoint, nonnegative, trace class application on $\mathfrak{h}^2$. For all $U = (u, v) \in \mathfrak{h}^2$, set

$$Q_A(u, v) = \langle A(u, v), (u, v) \rangle,$$

where $\langle \, , \, \rangle$ denotes the scalar product in $\mathfrak{h}^2$.
$S(Q_A) = S(Q_A, \mathfrak{h}^2)$ is defined as the class of all functions $F \in C^\infty(\mathfrak{h}^2)$ s.t. there exists $C(F) > 0$ satisfying :

$$\forall (x, \xi) \in \mathfrak{h}^2, \; |F(x, \xi)| \leq C(F),$$

$$\forall m \in \mathbb{N}^*, \forall (x, \xi) \in \mathfrak{h}^2, \; \forall (U_1, \dots, U_m) \in (\mathfrak{h}^2)^m,$$
$$|(d^m F)(x, \xi)(U_1, ..., U_m)| \leq C(F) \prod_{j=1}^{m} Q_A(U_j)^{\frac{1}{2}}.$$

From Fock to Wiener

The Segal isomorphism

$$\begin{array}{ccc} \mathcal{F}_s(\mathfrak{h}) & \longrightarrow & L^2(B, \mu_{B,h/2}) \\ S_n(a_1 \otimes \cdots \otimes a_n) & \mapsto & \frac{1}{\sqrt{n!}} \left(\frac{2}{h}\right)^{n/2} : \ell_{a_1} \ldots \ell_{a_n} : \end{array}$$

This isomorphism is an isometry of the Fock space on $L^2_{\mathbb{R}}(B, \mathcal{B}(B), \mu_{B,h/2})$.
$: \ell_{a_1} \ldots \ell_{a_n} :$ is a “Wick product”.

Coherent states

For $X = (x, \xi) \in \mathfrak{h}^2$, corresponding to $z = x + i\xi \in \mathfrak{h}_{\mathbb{C}}$,

On the Wiener side

$$\Psi_{X,h}(u) = \Psi_{x+i\xi,h}(u) = e^{\frac{1}{h}(\ell_x + i\ell_\xi)(u) - \frac{1}{2h}|x|^2 - \frac{i}{2h}x\cdot\xi},$$

On the Fock side

$$\Psi_{X,h} = \Psi_{x+i\xi,h} = e^{-\frac{|X|^2}{4h}} \sum_{n\geq 0} \frac{(x+i\xi)\otimes\cdots\otimes(x+i\xi)}{(2h)^{n/2}\sqrt{n!}}, \qquad \Psi_{0,h} = \Omega,$$

They satisfy :

$$< \Psi_{X,h}, \Psi_{Y,h} >= e^{-\frac{1}{4h}|X-Y|^2 + \frac{i}{2h}(\xi\cdot y - x\cdot\eta)}.$$

Coherent states, Wick symbol

On a finite dim. space $E \subset \mathfrak{h}$,

Decomposition of the identity

$\forall f, g \in L^2(E, \mu_{E,h/2}), \langle f, g\rangle_{L^2(E,\mu_{E,h/2})} =$

$$= \int_{E^2} \langle f, \Psi_{(x,\xi),h}\rangle_{L^2(E,\mu_{E,h/2})} \langle \Psi_{(x,\xi),h}, g\rangle_{L^2(E,\mu_{E,h/2})} \frac{1}{(2\pi h)^d} d\lambda(x,\xi),$$

with the Lebesgue measure.

Wick symbol of an operator : definition

A : bounded operator on $L^2(B, \mu_{B,h/2})$ or on $\mathcal{F}_s(\mathfrak{h})$.

$$\sigma_h^{Wick}(A)(X) = \langle A\Psi_{X,h}, \Psi_{X,h}\rangle.$$

Plan

The test functions : first condition

If a function $\tilde{f}$ is based on $E \subset B'$, then it is in $\mathcal{D}_E$ if and only if

Decay

$$\begin{array}{rcl} E^2 & \to & \mathbb{C} \\ X & \mapsto & \langle \tilde{f}, \Psi_{X,h} \rangle \end{array}$$

belongs to the Schwartz space $\mathcal{S}(E^2)$.

Measurability criterion

Proposition

$\tilde{f} \in L^2(B, \mu_{B,h/2})$. E is a finite dim. subspace of B'
Then $\tilde{f}$ is cylindrical on E if and only if

$$||\tilde{f}||^2_{L^2(B,\mu_{B,h/2})} = \int_{E^2} \left|\langle \tilde{f}, \Psi_{X,h}\rangle\right|^2 \frac{1}{(2\pi h)^d}\, d\lambda(X).$$

($d\lambda(X)$ is the Lebesgue measure on E^2).

Elements of the proof

$(\ell_{e_1},\dots,\ell_{e_d})$: orthonormal basis of E (scalar product of $\mathfrak{h}$)
The conditional expectation of $\tilde{f}$ with respect to $\sigma(\ell_{e_1},\dots,\ell_{e_d})$ is given by

$$E(\tilde{f}|\sigma(\ell_{e_1},\dots,\ell_{e_d}))(u)=\int_{E^\perp}\tilde{f}(\tilde{\pi}_E(u)+u_{E^\perp})\;d\mu_{E^\perp,h/2}(u_{E^\perp}).$$

It can be rewritten as

$$E(\tilde{f}|\sigma(\ell_{e_1},\dots,\ell_{e_d}))=\int_{E^2}\langle\tilde{f},\Psi_{(x,\xi),h}\rangle_{L^2(B,\mu_{B,h/2})}\Psi_{(x,\xi),h}\,\frac{1}{(2\pi h)^d}d\lambda(x,\xi)\;.$$

The $L^2(B,\mu_{B,h/2})$ norms satisfy

$$\begin{aligned}||E(\tilde{f}|\sigma(\ell_{e_1},\dots,\ell_{e_d}))|| &= \int_{E^2}\left|\langle\tilde{f},\Psi_{(x,\xi),h}\rangle_{L^2(B,\mu_{B,h/2})}\right|^2\frac{1}{(2\pi h)^d}d\lambda(x,\xi)\\ &\leq ||\tilde{f}||^2.\end{aligned}$$

Plan

Operators and spaces

Spaces

Basic space : $\mathfrak{h} = L^2(\mathbb{R}^3)$, the Fock space : $\mathcal{F}_s(\mathfrak{h})$.

- $\omega \in L^2_{loc}(\mathbb{R}^3)$, nonnegative.
- $f \in \mathfrak{h} = L^2(\mathbb{R}^3)$ s.t. $\omega^{-1/2} f \in \mathfrak{h}$.

Operators and domains

- On $\mathfrak{h}$: M_ω, multiplication by ω,
- On $\mathcal{F}_s(\mathfrak{h})$: $d\Gamma(M_\omega)$,
- On $\mathcal{F}_s(\mathfrak{h})$: $\Phi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f))$.

One checks that

$$D(d\Gamma(M_\omega)) \subset D(a^\sharp(f)) \subset D(\overline{\Phi(f)}) \subset \mathcal{F}_s(\mathfrak{h}).$$

Self adjunction

Then :

- $d\Gamma(M_\omega)$ is self-adjoint on $D(d\Gamma(M_\omega))$;
- $\Phi(f)$ is self-adjoint on $D(\overline{\Phi(f)})$;
- for $g \in \mathbb{R}$, $d\Gamma(M_\omega) + g\Phi(f)$ is self-adjoint on

$$D(d\Gamma(M_\omega)) \cap D(\overline{\Phi(f)}) = D(d\Gamma(M_\omega))$$

The unitary operators are well defined :

$$e^{itd\Gamma(M_\omega)}, \; e^{it\Phi(f))}, \; e^{it(d\Gamma(M_\omega)+g\Phi(f))}.$$

Reduced operator

$$e^{-itd\Gamma(M_\omega)} e^{it(d\Gamma(M_\omega)+g\Phi(f))}$$

Action on the coherent states

ω, f as before.
$g, t, h \in \mathbb{R}$ and $h > 0$.
$z = x + i\xi \in \mathfrak{h}_{\mathbb{C}}$. Then

$$e^{it(d\Gamma(M_\omega)+g\Phi(f))}\Psi_{z,h} =$$

$$e^{i\frac{gt}{2\sqrt{h}}\mathrm{Re}\left(\bar{f}\cdot ze^{it\frac{\omega}{2}}\,\mathrm{sin}_c(t\frac{\omega}{2})\right)}\,e^{i\frac{g^2t^2}{2}f\cdot\bar{f}\left(\frac{\mathrm{sin}_c(t\omega)-1}{t\omega}\right)}\Psi_{e^{it\omega}z+igt\sqrt{h}\,fe^{it\frac{\omega}{2}}\,\mathrm{sin}_c(t\frac{\omega}{2}),h},$$

where $\mathrm{sin}_c(u) = \frac{\sin(u)}{u}$.

The product $\cdot$ is in the sense of $\mathfrak{h} = L^2(\mathbb{R}^3)$.

Ingredients of the proof

$$e^{itd\Gamma(M_\omega)}\Psi_{z,h} = \Psi_{e^{it\omega}z,h}$$

$$e^{is\Phi(f)}\Psi_{z,h} = e^{\frac{is}{2\sqrt{h}}\mathrm{Re}(\bar{f}\cdot z)}\Psi_{z+is\sqrt{h}f,h}$$

Trotter's Formula

Trotter's Formula

$\forall \Psi \in \mathcal{F}_s(\mathfrak{h}), \forall t \in \mathbb{R},$

$$\lim_{n \to 0} \left(e^{i \frac{t}{n} d\Gamma(M_\omega)} e^{i \frac{t}{n} g\Phi(f)} \right)^n \Psi = e^{it(d\Gamma(M_\omega) + g\Phi(f))} \Psi.$$

Wick symbols

The classical operators

$$\sigma_h^{Wick}(e^{itd\Gamma(M_\omega)})(x,\xi) = e^{-\frac{1}{2h}(|x|^2+|\xi|^2)+\frac{1}{2h}\int_{\mathbb{R}^3}(x^2+\xi^2)(k)e^{it\omega(k)}dk}$$

$$\sigma_h^{Wick}(e^{is\Phi(f)})(x,\xi) = e^{-\frac{s^2}{4}(|\alpha|^2+|\beta|^2)+i\frac{s}{\sqrt{h}}(\alpha\cdot x+\beta\cdot\xi)}$$

As before, $|x|^2 = |x|^2_{\mathfrak{h}} = \int_{\mathbb{R}^3}\dots$.

Reduced operator

$$\sigma_h^{Wick}(e^{-itd\Gamma(M_\omega)}e^{it[d\Gamma(M_\omega)+g\Phi(f)]})(x,\xi) =$$

$$e^{i\frac{gt}{2\sqrt{h}}\left(\bar{f}ze^{it\frac{\omega}{2}}+f\bar{z}e^{-it\frac{\omega}{2}}\right)\mathrm{sin}_c(\frac{t\omega}{2})}\; e^{i\frac{g^2t^2}{2}f.\bar{f}\frac{\mathrm{sinc}(t\omega)-1}{t\omega}}\; e^{-\frac{g^2t^2}{4}|f\,\mathrm{sin}_c(\frac{t\omega}{2})|^2_{\mathfrak{h}}}$$

Relationship with the symbol classes

Corollary

The Wick symbol of the operator

$$e^{-itd\Gamma(M_\omega)} e^{it[d\Gamma(M_\omega)+g\Phi(f)]}$$

belongs to the classes $S(Q_A)$ for a convenient A.

Proof

This symbol is of the form

$$F : (x, \xi) \mapsto Ce^{i(x\cdot y+\xi\cdot\eta)},$$

$$d^m(F)(x,\xi)\cdot(U_1,\dots U_m) = i^m Ce^{i(x\cdot y+\xi\cdot\eta)}\prod_{j=1}^{m}(u_j\cdot y + v_j\cdot\eta).$$

$(U_j = (u_j, v_j))$
The “convenient “ A is the rank one (and hence trace-class) operator defined on $\mathfrak{h}^2$ by

$$A((u,v)) = 2((u\cdot y)y, (v\cdot\eta)\eta).$$

Plan

What remains to do ?

- Are the test functions in $\mathcal{D}$ preserved by the reduced operator (measurability) ?
- Wick symbol of a real evolution operator

$$e^{-it\mathbb{H}} A e^{it\mathbb{H}}$$

- Treat Pauli Fierz type operators on $\mathbb{C}^2 \otimes \mathcal{F}_s(\mathfrak{h})$ like

$$\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_{int} = \underbrace{I \otimes d\Gamma(\omega) + \sigma_3 \otimes I}_{\mathbb{H}_0} + \underbrace{g\sigma_1 \otimes \Phi(f)}_{\mathbb{H}_{int}} .$$

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Thank you for your attention.