Combinatorial approaches of some ergodic and topological proofs

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Introduction

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$$D(D(A)) = A - A + A - A \supseteq B(S, \varepsilon).$$

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What about the structure of $D(A)$?

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and so $kB \subseteq A' - A'$ as we wanted.

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Question

Is there a set E *s.t. any coloring of* $\mathbb{N}$ *there exists a color class containing an INFINITE sub-pattern of* E *and* E^c *?*

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Theorem (Raimi)

There exists $E \subseteq \mathbb{N}$ *such that, whenever* $r \in \mathbb{N}$ *and* $\mathbb{N} = \bigcup_{i=1}^{r} D_i$ *there exist* $i \in \{1, 2, \ldots, r\}$ *and* $k \in \mathbb{N}$ *such that* $(D_i + k) \cap E$ *is infinite and* $(D_i + k) \setminus E$ *is infinite*

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One can perform it as

Theorem

There exists $E \subseteq \mathbb{N}$ such that, whenever $r-$coloring of integers, there exists a monochromatic subsets D_i and $k \in \mathbb{N}$ for which

$$k \in (E - D_i) \cap (E^c - D_i),$$

and the representation of k as a difference is infinite both in the two sets .(E^c is the complement of E with respect to $\mathbb{N}$.)

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Definition

Given a sequence $\{x_n\}_{n=1}^{\infty}$ *in* $\mathbb{N}$,

$$FS(x_0, \{x_n\}_{n=1}^{\infty}) = \left\{x_0 + \sum_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\right\}.$$

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Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ *be positive real numbers such that* $\sum_{i=1}^{r} \alpha_i = 1$.

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Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ *be positive real numbers such that* $\sum_{i=1}^{r} \alpha_i = 1$.
There exists a disjoint partition $\mathbb{N} = \bigcup_{i=1}^{r} E_i$

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i.e. $\exists F_m$ such the it contains an infinite copy of sub-pattern from each E_i.

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and hence $(x_0 + y) + F_m$ intersects all E_i in an infinite set.

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Thus we have that

A combinatorial proof II

Let c_{n+1} be any such c. Since $K \geq p_m$ and $0 \in FS_f(\{c_1, c_2, \ldots, c_n\})$ we have

$$c_{n+1} \cdot FP(\{c_1, c_2, \ldots, c_n\}) \subseteq \{0, c_{n+1}, 2c_{n+1}, \ldots, K \cdot c_{n+1}\} \subseteq B \setminus E.$$

Then by

$$FP(\{c_1, c_2, \ldots c_n\}) = FP(\{c_1, c_2, \ldots c_{n-1}\}) \cdot \{1, c_n\},$$

and by the inductive hypothesis $FP(\{c_1, c_2, \ldots, c_n, c_{n+1}\}) \subseteq B \setminus E$.
Moreover $K > f(n+1)$,

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$$\mathcal{F}_{n+1} \subseteq B \setminus E \subseteq A - A,$$

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as we wanted.

Merci pour l’attention

Merci pour l'attention (Thank you for your attention)