Freiman's theorem in torsion-free groups

Mercede MAJ

UNIVERSITÀ DEGLI STUDI DI SALERNO

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- **Gregory Freiman**, Tel Aviv University
- **Marcel Herzog**, Tel Aviv University
- **Patrizia Longobardi**, Università di Salerno
- **Mercede Maj**, Università di Salerno
- **Yonutz Stanchescu**, Afeka Academic College Israel and The Open University of Israel
- **Alain Plagne**, Centre de mathématiques Laurent Schwartz, École polytechnique, France
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Basic definition

Definitions

If S is a subset of a group $G(+)$, then we denote

$$2S = S + S := \{x + y \mid x \in S, y \in S\}$$

$2S$ the double of S

If G is a multiplicative group, then we denote

$$S^2 = SS := \{xy \mid x \in S, y \in S\}$$

S^2 is also called the square of S

Background

Let S be a finite subset of a group G.

Remark

$$|S| \leq |S^2| \leq |S|^2.$$

The bounds are sharp.

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Background

Example

If S is a subgroup of G, then $S^2 = S$.

More generally, if $S = xH$, where H is a subgroup of G and $xH = Hx$, then $S^2 = xHxH = x^2H$ and $|S^2| = |S|$.

Background: some inverse results

Proposition

Let S be a non-empty finite subset of a group G.

$|S^2| = |S|$ if and only if $S = xH$, where $H \leq G$ and $xH = Hx$.

Background: some inverse results

Proposition

Let S be a non-empty finite subset of a group G.

If

$$|S^2| < \frac{3}{2}|S|,$$

then $H = SS^{-1}$ is a finite subgroup of G of order $|S^2|$, and $S \subset Hx = xH$ for some $x \in N_G(S)$.

Background: some inverse results

More generally

Proposition (Freiman-Hamidoune-Kneser)

Let S be a non-empty finite subset of a group G. Suppose that

$$|S^2| \leq (2-\epsilon)|S|, \ 0 \leq \epsilon < 1.$$

Then there exists a finite subgroup H of cardinality $\geq c(\epsilon)|S|$, such that S is covered by at most $C(\epsilon)$ right-cosets of H, where $c(\epsilon)$ and $C(\epsilon)$ depend only on ϵ .

Problem

What lower bound one can get on $|S^2|$ if we assume G torsion-free?

Background - direct results

Proposition

If S is a non-empty finite subset of the group of the integers, then we have

$$|2S| \geq 2|S| - 1.$$

Proof. We use the fact that the integers are totally ordered. Write $S = \{x_1, x_2, \cdots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$. Clearly

$$x_1 + x_1 < \ldots < x_1 + x_k < x_2 + x_k < \ldots < x_k + x_k$$

Hence $|2S| \geq 2k - 1$, as required. //

Background - direct results

More generally:

Theorem (J.H.B. Kemperman, Indag. Mat., 1956)

If S is a non-empty finite subset of a torsion-free group, then we have

$$|S^2| \geq 2|S| - 1.$$

Problem

Is this bound sharp?

An example

Definition

If $a, r \neq 1$ are elements of a multiplicative group G, a geometric left (rigth) progression with ratio r and length n is the subset of G

$$\{a, ar, ar^2, \cdots, ar^{n-1}\}(\{a, ra, r^2a, \cdots, r^{n-1}a\}).$$

If G is an additive group

$$\{a, a+r, a+2r, \cdots, a+(n-1)r\}$$

is called an arithmetic progression with difference r and length n.

An example

Example

If $S = \{a, ar, ar^2, \cdots, ar^{n-1}\}$ *is a geometric progression in a torsion-free group and* $ar = ra$, *then* $S^2 = \{a^2, a^2r, a^2r^2, \cdots, a^2r^{2n-2}\}$ *has order*

$$2|S| - 1.$$

Theorem (Freiman, Schein, Proc. Amer. Math. Soc. 1991)

If S is a finite subset of a torsion-free group, $|S| = k \geq 2$,

$$|S^2| = 2|S| - 1$$

if and only if

$$S = \{a, aq, \cdots, aq^{k-1}\}, \textit{and either } aq = qa \textit{ or } aqa^{-1} = q^{-1}.$$

In particular, if $|S^2| = 2|S| - 1$, then S is contained in a left coset of a cyclic subgroup of G.

Background - inverse results

Theorem (Y.O. Hamidoune, A.S. Lladó, O. Serra, Combinatorica, 1998)

If S is a finite subset of a torsion-free group, $|S| = k \geq 4$, if

$$|S^2| \leq 2|S|,$$

then there exist $a, q \in G$ such that

$$S = \{a, aq, \cdots, aq^k\} \setminus \{c\}, \text{ with } c \in \{a, aq\}.$$

Doubling problems

Let G be a group and S a finite subset of G.
Let α, β real numbers

Problem

What is the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta?$$

Problems of this kind are called **inverse problems of doubling** type in additive number theory. The coefficient α, or more precisely the ratio $\frac{|S^2|}{|S|}$ is called the **doubling coefficient of S** .

Doubling problems

Inverse problems of doubling type have been first investigated by **G.A. Freiman**.

E. Breuillard, Y. O. Hamidoune, B. Green, M. Kneser, A.S. Lladó, A. Plagne, P.P. Palfy. Z. Ruzsa, O. Serra, Y.V. Stanchescu, T. Tao....

There are two main types of questions one may ask.

Doubling property

Problem

What is the general type of structure that S can have if

$$|S^2| \leq \alpha|S| + \beta?$$

How behaves this type of structure when α increases?

Studied recently by many authors

E.Breuillard, **B.Green**, **I.Ruzsa**, **T.Tao**...

Very powerful general results have been obtained (leading to a qualitatively complete structure theorem thanks to the concepts of nilprogressions and approximate groups).

Small doubling problems

But these results are not very precise quantitatively.

Problem

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets S which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ small.

Problems of this kind are called **inverse problems of small doubling** type.

Small doubling problems

Theorem (G. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|2S| \leq 3k - 4.$$

Then S is contained in an arithmetic progression of size $2k - 3$:

$$\{a, a + q, a + 2q, \cdots, a + (2k - 4)q\}.$$

Small doubling problems

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$.

Small doubling problems

Theorem (G. Freiman)

Let S be a finite set of integers with $k \geq 2$ elements and suppose that

$$|2S| \leq 3k - 3.$$

Then one of the following holds:
(i) S is contained in an arithmetic progression of size at most $2k-1$
(ii) S is a bi-arithmetic progression

$$S = \{a, a+q, a+2q, \cdots, a+(i-1)q\} \cup \{b, b+q, a+2q, \cdots, b+(j-1)q\}.$$

(iii) $k = 6$ and S has a determined structure.

Small doubling problems

Problem

Let G be any torsion-free group, S a finite subset of G, $|S| \geq 3$.

What is the structure of S if

$$|S^2| \leq 3|S| - 3?$$

Small doubling problems

Freiman studied also the case $|2S| = 3|S| - 2$, S a subset of the integers. He proved that, with the exception of some cases with $|S|$ small, then either S is contained in an arithmetic progression or it is the union of two arithmetic progressions with same difference.

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 11$, and

$$|S^2| \leq 3|S| - 2,$$

then S is contained in a geometric progression of length at most $2|S| + 1$ or it is the union of two geometric progressions with same ratio.

Small doubling problems

Small doubling problems have been studied in abelian groups by many authors.
Y. O. Hamidoune, B. Green, M. Kneser, A.S. Lladò, A. Plagne, P.P. Palfy. Z. Ruzsa, O. Serra, Y.V. Stanchescu...

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we study Freiman's Conjectures and more generally small doubling problems with doubling coefficient 3, in the class of orderable groups

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Orderable groups

Definition

Let G be a group and suppose that a total order relation $\leq$ is defined on the set G.
We say that $(G, \leq)$ is an *ordered group* if for all $a, b, x, y \in G$,

$$\textit{the inequality } a \leq b \textit{ implies that } xay \leq xby.$$

Definition

A group G is *orderable* if there exists a total order relation $\leq$ on the set G, such that $(G, \leq)$ is an ordered group.

Orderable groups

Remark

Any ordered group is torsion-free.

Theorem (F.W. Levi)

An **abelian group** *G is orderable if and only if it is torsion-free.*

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

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The class of orderable groups contains the class of **torsion-free nilpotent groups***.*

Definition

A group G is *nilpotent* if it has a *central* series, that is, there exists a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of normal subgroups of G such that G_{i+1}/G_i is contained in the center of G/G_i for all i.

The center $Z(G)$ of a group G is the set of elements that commute with every element of G:

$$Z(G) = \{x \in G | gx = xg, for\ all\ g \in G\}.$$

The length of a shorter central series of G is the *nilpotent class* of G.
Abelian groups are nilpotent of class 1.

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Orderable groups

Free groups are orderable. Pure braid groups are orderable.

The group

$$\langle x, c \mid x^{-1}cx = c^{-1} \rangle$$

is not an orderable group.

More information concerning orderable groups may be found, for example, in

R. Botto Mura and A. Rhemtulla, *Orderable groups*,
Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1977.

A.M.W. Glass, *Partially ordered groups*,
World Scientific Publishing Co., Series in Algebra, v. 7, 1999.

Small doubling in orderable groups

Theorem (Freiman, Herzog, Longobardi, - , J. Austral. Math. Soc., 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 3$.

Assume that

$$t = |S^2| \leq 3|S| - 4.$$

Then $\langle S \rangle$ is abelian. Moreover, there exists $a, q \in G$, such that $qa = aq$ and S is a subset of

$$\{a, aq, aq^2, \cdots, aq^{t-k}\}.$$

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Then $< S >$ is abelian.

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Theorem (Freiman, Herzog, Longobardi, - , Plagne, Stanchescu, 2015)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 3$. If

$$|S^2| \leq 3|S| - 3,$$

then $\langle S \rangle$ is abelian, at most 3-generated and one of the following holds:

- (1) $|S| = 6$;
- (2) *S is a subset of a geometric progression of length at most $2|S| - 1$*;
- (3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$

Small doubling in orderable groups

What about $\langle S \rangle$ if S is a subset of an orderable group and

$$|S^2| \leq 3|S| - 2?$$

Is it abelian? Is it abelian if $|S|$ is big enough?

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Small doubling in orderable groups

Remark

There exists an ordered group G with a subset S of order k (for any k) such that $\langle S \rangle$ is not abelian and $|S^2| = 3k - 2$.

Example

Let

$$G = \langle a, b \mid b^{-1}ab = a^2 \rangle,$$

the Baumslag-Solitar group $B(1,2)$ and

$$S = \{b, ba, ba^2, \cdots, ba^{k-1}\}.$$

Then

$$S^2 = \{b^2, b^2a, b^2a^2, b^2a^3, \cdots, b^2a^{3k-3}\}.$$

Thus $\langle S \rangle$ is non-abelian and $|S^2| = 3k - 2$.

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Small doubling problems

Problem

Let G be an order able group, S a finite subset of G, $|S| \geq 3$.

What is the structure of S if

$$|S^2| \leq 3|S| - 2?$$

Problem

Let G be an orderable group, S a finite subset of G, $|S| \geq 3$.

What is the structure of $\langle S \rangle$ if

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The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Stanchescu, 2015)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

- (1) $\langle S \rangle$ *is an abelian group, at most 4-generated;*
- (2) $\langle S \rangle = \langle a, b \mid ba = cab,\ ac = ca,\ cb = bc \rangle$. *In particular $\langle S \rangle$ is a nilpotent group of class 2;*
- (3) $\langle S \rangle = \langle a, b \mid a^b = a^2 \rangle$. *Therefore $\langle S \rangle$ is the Baumslag-Solitar group $B(1,2)$;*
- (4) $\langle S \rangle = \langle a \rangle \times \langle b, c \mid c^b = c^2 \rangle$;
- (5) $\langle S \rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a \rangle$.

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The structure of S if $|S^2| = 3|S| - 2$

If $\langle S \rangle$ is abelian, then the structure of $|S^2| = 3|S| - 2$ can be obtained using some previous results by Freiman and Stanchescu.

Theorem

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is abelian, then one of the following possibilities occurs:

- (1) $|S| \leq 11$;
- (2) *S is a subset of a geometric progression of length at most $2|S| + 1$;*
- (3) *S is contained in the union of two geometric progressions with the same ratio.*

The structure of S if $|S^2| = 3|S| - 2$

If $\langle S \rangle$ is nilpotent of class 2, we have the following

Theorem (Freiman, Herzog, Longobardi, - , 2016)

Let G be a torsion-free nilpotent group of class 2 and let S be a subset of G of finite size $k \geq 4$. Then

$$|S^2| = 3k - 2,$$

if and only if

$$S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\},$$

with $1 + i + 1 + j = k$ and $ab = bac$ or $ba = abc$, $c > 1$.

Proof

Proof. Write $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, and suppose for example $ab = bac$.

G is nilpotent of class 2, thus $G/Z(G)$ is abelian. Then $abZ(G) = baZ(G)$, thus $c \in Z(G)$.

Then we have

$$S^2 = \{a, ac, \cdots, ac^i\}^2 \cup \{b, bc, \cdots, bc^j\}^2 \cup \{ba, bac, \cdots, bac^{i+j}, abc^{i+j}\} =$$

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Thus $|S^2| = 2(i+1) - 1 + 2(j+1) - 1 + i + j + 2 = 3i + 3j + 4 = 3(i + j + 2) - 2 = 3|S| - 2$, as required. //

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G is nilpotent of class 2, thus $G/Z(G)$ is abelian. Then $abZ(G) = baZ(G)$, thus $c \in Z(G)$.
Then we have

$$S^2 = \{a, ac, \cdots, ac^i\}^2 \cup \{b, bc, \cdots, bc^j\}^2 \cup \{ba, bac, \cdots, bac^{i+j}, abc^{i+j}\} =$$

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Thus $|S^2| = 2(i+1) - 1 + 2(j+1) - 1 + i + j + 2 = 3i + 3j + 4 = 3(i + j + 2) - 2 = 3|S| - 2$, as required. //

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The structure of S if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Robinson, Stanchescu, J. Algebra, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is non-abelian, then one of the following statements holds:

- (1) $|S| \leq 4$;
- (2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, *where* $c^x = c^2$ *or* $(c^2)^x = c$;
- (3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, *with* $1 + i + 1 + j = k$ *and* $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$.

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

Some methods

Remark

Any orderable group is an R-group.

A group G is an **R-group** if, with $a, b \in G$,

$$a^n = b^n, \; n \neq 0, \; \text{implies } a = b.$$

Remark

If G is an orderable group, $a, b \in G$ and if $a^n b = ba^n$ for some positive integer n, then $ab = ba$.

Let $(G, \leq)$ be an ordered group, $S = \{x_1, x_2, \cdots, x_{k-1}, x_k\}$ a subset of G, $|S| = k$, $|S^2| \leq 3k - v, v \in \{1, 2, 3, 4\}$.
Suppose $x_1 < x_2 < \cdots < x_{k-1} < x_k$.
Write

$$T = \{x_1, \cdots, x_{k-1}\}.$$

We show that either

$$|T^2| \leq 3(k-1) - v, \text{ or } \langle T \rangle \text{ is abelian.}$$

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Proofs of Theorems concerning the structure of $\langle S\rangle$

Let $(G,\leq)$ be an ordered group, $S=\{x_1,x_2,\cdots,x_{k-1},x_k\}$ a subset of G, $|S|=k$, $|S^2|\leq 3k-v, v\in\{1,2,3,4\}$.
Suppose $x_1<x_2<\cdots<x_{k-1}<x_k$.
Write

$$T=\{x_1,\cdots,x_{k-1}\}.$$

We show that either

$$|T^2|\leq 3(k-1)-v, \textit{ or } \langle T\rangle \textit{ is abelian.}$$

Proof

Theorem

Let $(G, \leq)$ be an ordered group and let $S = \{x_1, x_2, \cdots, x_k\}$ be a finite subset of G of size $k \geq 2$, with $x_1 < x_2 \cdots < x_k$.
Assume that

$$|S^2| \leq 3k - 3.$$

Then $< S >$ is abelian.

Proof

Suppose that $S = \{x_1, x_2, \cdots, x_k\}$ is a subset of an ordered group, $x_1 < x_2 < \cdots < x_k$.

Assume $|S^2| \leq 3|S| - 3$.

We want to show that $\langle S \rangle$ is abelian.

If $k = 2$ or $k = 3$, we prove directly the result.

Suppose $k > 3$ and argue by induction on k. Write $T = \{x_1, \cdots, x_{k-1}\}$. Then either $\langle T \rangle$ is abelian or $|T^2| \leq 3|T| - 3$, by the previous remarks. By induction we can assume that $\langle T \rangle$ is abelian.

If $x_i x_k \in T^2$, for some $i < k$, then $x_k \in \langle T \rangle$ and $\langle S \rangle \subseteq \langle T \rangle$ is abelian , as required. Hence we can assume that $x_1 x_k, \cdots, x_{k-1} x_k, x_k^2 \notin T^2$, then $|T^2| \leq |S^2| - k = 3k - 3 - k = 2(k-1) - 1$. Then

$$T = \{a, ac, \cdots, ac^{k-2}\}$$

is a geometric progression with $ac = ca$.

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Suppose that $S = \{x_1, x_2, \cdots, x_k\}$ is a subset of an ordered group, $x_1 < x_2 < \cdots < x_k$.
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$$T = \{a, ac, \cdots, ac^{k-2}\}$$

is a geometric progression with $ac = ca$.

Proof

Write

$$V = \{x_2, \cdots, x_k\}.$$

Considering the order opposite to $<$ and arguing on V as we did on T we get that V is abelian.

Moreover $|V| \leq 3$, since $k > 3$. Then there exist $i \neq j$ such that $x_k(ac^i) = (ac^i)x_k$ and $x_k(ac^j) = (ac^j)x_k$. Then $x_k(c^{i-j}) = (c^{i-j})x_k$ and $x_k c = cx_k$, since we are in an ordered group.

From $x_k(ac^j) = (ac^j)x_k$, we get that also $x_k a = ax_k$.

Thus $x_k \in C_G(T)$ and $\langle S \rangle$ is abelian, as required. //

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Stanchescu, 2015)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

- (1) $\langle S\rangle$ *is an abelian group, at most 4-generated;*
- (2) $\langle S\rangle = \langle a, b \mid ba = abc,\ ac = ca,\ cb = bc\rangle$. *In particular* $\langle S\rangle$ *is a nilpotent group of class 2;*
- (3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. *Therefore* $\langle S\rangle$ *is the Baumslag-Solitar group* $B(1,2)$;
- (4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;
- (5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$.

Definition

A group G is *soluble* if it has an *abelian* series, that is, there exists a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of subgroups of G such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, for all i.

The length of a shorter abelian series of G is the *solubility length* of G.
Abelian groups are soluble of length 1. Soluble group of length at most 2 are called *metabelian*.

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The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2,$$

then $\langle S\rangle$ is metabelian.

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

and $\langle S\rangle$ is nilpotent, then it is nilpotent of class at most 2.

Problem

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that

$$|S^2| = 3|S| - 1$$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

In fact we have:

Theorem (Freiman, Herzog, Longobardi,- , Plagne, Stanchescu, 2015)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

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An example

Example

For any $k \geq 3$, there exists an ordered group, with a subset S of finite size k, such that $\langle S \rangle$ is not soluble and

$$|S^2| = 4k - 5.$$

Let

$$G = \langle a \rangle \times \langle b, c \rangle,$$

where $\langle a \rangle$ is infinite cyclic and $\langle b, c \rangle$ is free of rank 2.
For any $k \geq 3$, define

$$S = \{a, ac, \cdots, ac^{k-2}, b\}.$$

Then

$$|S^2| = 4k - 5.$$

Problems

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$

Theorem (K.J. Böröczky, P.P. Palfy, O. Serra, Bull. London Math. Soc., 2012)

The conjecture of Freiman holds if

$$|S^2| \leq 2|S| + \frac{1}{2}|S|^{\frac{1}{6}} - 3$$

Conjecture

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian and, if it is nilpotent, it is nilpotent of class 2

It follows from results of **E. Breuillard, B. Green and T. Tao** that

Remark

If $|S^2| \leq 3|S| - 4$, then there exists a nilpotent subgroup H of nilpotency class $c(3)$ and generated by at most $d(3)$ elements such that $S \subseteq ZH$, for some subset Z of the group such that $|Z| \leq s(3)$.

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Thank you for the attention !

M. Maj
Dipartimento di Matematica
Università di Salerno
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
E-mail address : mmaj@unisa.it

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