Improved bounds for planar sets avoiding the unit distance

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(joint work with T. Keleti, F. M. Oliveira Filho, I.Z. Ruzsa)

Sets avoiding the unit distance

Let $A \subset \mathbb{R}^n$ be measurable, such that $\|a - a'\| \neq 1$ for all $a, a' \in A$ (Euclidean norm). A is said to be "1-avoiding".

What is the maximal possible (upper) density of A?

Erdős conjectured

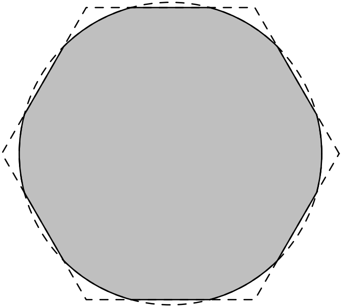
$m_1(\mathbb{R}^2) < 1/4$.

Upper density: $\overline{\delta}(A) = \limsup_{r\to\infty} \frac{\lambda(A\cap B(0,r))}{\lambda(B(0,r))}$ ($\lambda(\cdot)$ denotes Lebesgue measure)

$m_1(\mathbb{R}^n) = \sup\{\,\overline{\delta}(A) : A \subseteq \mathbb{R}^n \text{ is 1-avoiding and measurable}\}$.

Lower bounds by construction

- Hexagonal lattice arrangement of open disks of radius 1/2. $\overline{\delta}(A) = \pi/(8\sqrt{3}) = 0.2267\ldots$.
- Slight improvement by Croft (1967): shrink the lattice a bit, and replace disks by tortoises. $\overline{\delta}(A) = 0.22936\ldots$.



Upper bound for sets with block structure I.

Definition

$A \subset \mathbb{R}^n$ has *block structure* if $A = \bigcup_{i=0}^{\infty} A_i$, where $\|x - y\| < 1$ if x and y belong to the same block, and $\|x - y\| > 1$ if x and y belong to different blocks.

All known examples of "high" density in any dimension are sets with block structure (e.g. Croft's example).

Theorem (Keleti, M., Oliveira Filho, Ruzsa (2015))

If $A \subset \mathbb{R}^n$ has block structure then $\overline{\delta}(A) \leq \frac{1}{2^n} - \varepsilon_n$.

Remark: ε_n can be made effective (but very small even for $n = 2$).

Upper bound for sets with block structure II.

Theorem (Keleti, M., Oliveira Filho, Ruzsa (2015))

If $A \subset \mathbb{R}^n$ has block structure then $\overline{\delta}(A) \leq \frac{1}{2^n} - \varepsilon_n$.

Proof. Let $C_i = A_i + B_{1/2} = \{ a + b : a \in A_i,\ b \in B_{1/2} \}$.

Then $C_i \cap C_j = \emptyset$, for all $i \neq j$ (because A has block structure).

- Brunn-Minkowski: $\lambda(C_i)^{1/n} \geq \lambda(A_i)^{1/n} + \lambda(B_{1/2})^{1/n}$.
- Isodiametric inequality: $\lambda(A_i) \leq \lambda(B_{1/2})$.

Therefore, $\frac{\lambda(A_i)^{1/n}}{\lambda(C_i)^{1/n}} \leq \frac{\lambda(A_i)^{1/n}}{\lambda(A_i)^{1/n} + \lambda(B_{1/2})^{1/n}} \leq \frac{1}{2}$, and

$\overline{\delta}(A) \leq \frac{1}{2^n}$.

Upper bound for sets with block structure III.

Theorem (Keleti, M., Oliveira Filho, Ruzsa (2015))

If $A \subset \mathbb{R}^n$ has block structure then $\overline{\delta}(A) \leq \frac{1}{2^n} - \varepsilon_n$.

Gaining the ε_n is more technical, but the idea is clear:

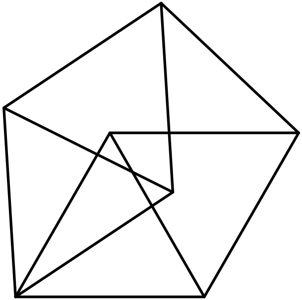
- if the isodiametric inequality is sharp then A_i must be close to being balls of radius 1/2 (stability lemma!)
- then all C_i are close to being unit balls
- but unit balls cannot pack the space very densely

Stability lemma (Maggi, Ponsiglione, Pratelli, 2014)

$E \subset \mathbb{R}^n$, $\lambda(E) > 0$, $\mathrm{diam} E = 2$. Then there exist $x, y \in \mathbb{R}^n$ such that $E \subset B(x, 1+r)$ and $B(y, 1) \subset E + B_r$, where $r = K_n \left(\frac{\lambda(B_1)}{\lambda(E)} - 1 \right)^{1/n}$ for some constant K_n that depends only on n.

General upper bounds in the plane

Moser-spindle (1961): $m_1(\mathbb{R}^2) \leq 2/7 = 0.285\ldots$.



Székely (1984): $m_1(\mathbb{R}^2) \leq 12/43 = 0.279\ldots$.
Vallentin, Oliveira Filho (2010): $m_1(\mathbb{R}^2) \leq 0.268\ldots$.

Theorem (Keleti, M., Oliveira Filho, Ruzsa (2015))

$m_1(\mathbb{R}^2) \leq 0.258\ldots$

For $\mathbb{R}^n$: Bachoc, Passuello, Thiery (2015): $m_1(\mathbb{R}^n) \leq (1 + o(1))1.268^{-n}$

Ingredients of the proof I.

Delsarte's method (Fourier formulation)

$\mathcal{G}$ finite Abelian group, $0 \in S = -S \subset \mathcal{G}$ symmetric set.

$\Delta(S) = \max\{|A| : (A - A) \cap S = \{0\}\} = ?$

(Independence number of the Cayley graph corresponding to $S \subset \mathcal{G}$.)

Intuition for 1-avoiding sets: $\mathcal{G} = \mathbb{R}^2$, $S =$ unit circle $\cup\{0\}$

Observation: $f(x) = |A \cap (A - x)| =$(number of solutions to $x = a - a'$) is a positive definite function. $\hat{f}(\mathbf{1}) = \sum f(x) = |A|^2, f(0) = |A|$.

Delsarte LP-bound

$\Delta(S) \leq$

$\sup\{\frac{\hat{f}(\mathbf{1})}{f(0)} : \; f(x) \geq 0 \; \forall x \in \mathcal{G}, f(x) = 0 \; \forall x \in S \setminus \{0\}, \hat{f}(\gamma) \geq 0 \; \forall \gamma \in \hat{\mathcal{G}}\} =$

$\inf\{\frac{h(0)}{\hat{h}(\mathbf{1})} : \; h(x) \leq 0 \; \forall x \in S^c, \hat{h}(\gamma) \geq 0 \; \forall \gamma \in \hat{\mathcal{G}}\}$

Ingredients of the proof II.

Delsarte LP-bound:

$\Delta(S) = \max\{|A| : (A - A) \cap S = \{0\}\} \leq$

$\sup\{\frac{\hat{f}(\mathbf{1})}{f(0)} : \ f(x) \geq 0 \ \forall x \in \mathcal{G}, f(x) = 0 \ \forall x \in S \setminus \{0\}, \hat{f}(\gamma) \geq 0 \ \forall \gamma \in \hat{\mathcal{G}}\}$

Improvement by Oliveira Filho, Vallentin: extra linear conditions on f.

Lemma (Oliveira Filho, Vallentin, 2010)

Let $A \subset \mathcal{G}$ be S-avoiding and let $V \subset \mathcal{G}$. For $f(x) = |A \cap (A - x)|$ we have $\sum_{y \in V} f(y) \leq \alpha(V)|A|$, where $\alpha(V)$ is the independence number of the subgraph on V.

Proof. $|A| \geq |\cup_{y \in V} (A \cap (A - y))| \geq \frac{1}{\alpha(V)} \sum_{y \in V} |A \cap (A - y)|$ because each $a \in A$ can be covered at most $\alpha(V)$ times.

Consequence: improved bound on $\Delta(S)$.

Ingredients of the proof III.

Delsarte LP-bound:

$\Delta(S) = \max\{|A| : (A-A)\cap S = \{0\}\} \le$

$\sup\{\frac{\hat{f}(\mathbf{1})}{f(0)} : \ f(x)\ge 0\ \forall x\in\mathcal{G}, f(x)=0\ \forall x\in S\setminus\{0\}, \hat{f}(\gamma)\ge 0\ \forall\gamma\in\hat{\mathcal{G}}\}$

Improvement by Székely: extra linear conditions on f.

Lemma (Székely, 1984)

Let $A\subset\mathcal{G}$ be S-avoiding, and let $C\subset\mathcal{G}$. For $f(x)=|A\cap(A-x)|$ we have $\sum_{x\ne y, x,y\in C} f(x-y)\ge |C||A|-|\mathcal{G}|$.

Proof. Inclusion-exclusion principle: $|G|\ge|\cup_{x\in C}(A-x)|\ge \sum_{x\in C}|A-x| - \sum_{x\ne y}|(A-x)\cap(A-y)| = |C||A| - \sum_{x\ne y,x,y\in C} f(x-y)$.

Consequence: improved bound on $\Delta(S)$.

Application to $\mathbb{R}^2$

Let $A \subset \mathbb{R}^2$ be measurable, periodic 1-avoiding. Autocorrelation function: $f(x) = \delta(A \cap (A - x))$ (density).

- Linear conditions on f: Delsarte, Oliveira Filho, Vallentin, Székely.
- Radialize f by averaging over rotations. $\tilde{f}(x) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} f(\xi\|x\|) d\omega(\xi)$, where ω is the surface measure of the unit sphere. The linear conditions remain true for $\tilde{f}$.
- Write $f(x) = \sum_{u \in 2\pi L^*} |\widehat{\mathbf{1}}_A(u)|^2 e^{iu\cdot x}$, and
- $\tilde{f}(x) = \sum_{u \in 2\pi L^*} |\widehat{\mathbf{1}}_A(u)|^2 \Omega_n(\|u\|\|x\|) = \sum_{t \geq 0} \kappa(t)\Omega_n(t\|x\|)$

where $\Omega_n(\|x\|) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} e^{ix\cdot\xi}\, d\omega(\xi)$, and $\kappa(t)$ is the sum of $|\widehat{\mathbf{1}}_A(u)|^2$ over all u such that $\|u\| = t$.

Linear duality

So, $\tilde{f}(x) = \sum_{t\geq 0} \kappa(t)\Omega_n(t\|x\|)$.

Let $\delta := \delta(A)$, and $\tilde{\kappa}(t) = \kappa(t)/\delta$ (normalization).

Then $\tilde{\kappa}(0) = \delta$, and we get an LP problem for $\tilde{\kappa}(t)$:

- max $\tilde{\kappa}(0)$ subject to
- $\sum_{t\geq 0} \tilde{\kappa}(t) = 1$
- $\sum_{t\geq 0} \tilde{\kappa}(t)\Omega_2(t) = 0$
- $\sum_{t\geq 0} \tilde{\kappa}(t) \sum_{x\in V} \Omega_2(t\|x\|) \leq \alpha(V)$ for V
- $\sum_{t\geq 0} \tilde{\kappa}(t) \sum_{\{x,y\}\in C} \Omega_2(t\|x-y\|) \geq |C| - \delta^{-1}$ for C.
- $\tilde{\kappa}(t) \geq 0$ for all $t \geq 0$.

Choose your sets V and C cleverly, apply weak duality, and known estimates for $\Omega_2(t)$ to produce a witness function testifying the upper bound $m_1(\mathbb{R}^2) \leq 0.258\ldots$.