Combinatorial properties of Nil–Bohr sets of integers

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- Arise naturally in higher order Fourier analysis ($\to$ Bohr sets)

SG_d^* sets

- A purely combinatorial construction
- Refinement of the notion of IP* sets

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Theorem ("$\Longrightarrow$", K.)

Any Nil–Bohr$_0$ of step d is $\mathrm{SG}_{d'}^*$, where $d' = \binom{d+2}{2}$.

Problem: Let $A \subset \{1, 2, \ldots, N\}$, $|A| = \delta N$ ($\delta = \text{const.}$, $N \to \infty$). Study k-term arithmetic progressions in A. In particular: show that some exist. [We will pretend that: $[N] = \mathbb{Z}/N\mathbb{Z}$, N prime. We use $e(t) = e^{2\pi i t}$.]

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Structure: If A fails to be uniform then ...

$\exists$ large Fourier coefficient: $\hat{f}_A(a) = \mathbb{E}_{x \in [N]} f_A(x) e(\frac{ax}{N}) = \Omega(1)$.	$\exists$ $(k-2)$-step *nilsequence* ψ, (bounded complexity), $\lvert \psi \rvert \leq 1$, which correlates with f_A: $\mathbb{E}_{x \in [N]} f_A(x) \psi(x) = \Omega(1)$.

Definition (nilsequences)

Let G be a (d-step) nilpotent Lie group, and $\Gamma < G$ a *cocompact* discrete subgroup.

1. The space $X = G/\Gamma$ is a *nilmanifold*.
2. For $g \in G$, the map $T_g \colon X \to X$, $x \mapsto gx$ is a *nilrotation*.
3. If $F \colon X \to \mathbb{R}$ is a (smooth) function, $x_0 \in X$, then $\psi(n) = F(g^n x_0)$ is a (d-step) *nilsequence*.

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A reassuring example: Take $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$. Then $G/\Gamma = \mathbb{T}$, the unit circle, equipped with rotations $x \mapsto x + \theta$. The additive characters $n \mapsto e(n\theta)$ are 1-step nilsequences.

Nilmanifolds, nilsequences, and Nil–Bohr sets

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Definition (Nil–Bohr sets)

Let ψ be a (d-step) nilsequence, and $V \subset \mathbb{R}$ an open set.
A set $A = \{n : \psi(n) \in V\}$ is called a (d-step) Nil–Bohr set (if $\neq \emptyset$).
If $\psi(0) \in V$ (i.e. $0 \in A$), then A is called a Nil–Bohr$_0$ set.

Slogan: A set is either uniform or resembles a Nil–Bohr set.

Examples

1. **Linear phases**: Let $\theta \in \mathbb{R}$.
   - $\psi(n) = e(n\theta)$ is 1-step nilsequence.
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2. **Polynomial phases**: Let $p \in \mathbb{R}[x]$ with $\deg(p) = d$, $p(0) = 0$.
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3. **Generalised polynomial phases**:
   - Generalised polynomials = polynomials + floor function.
     Example: $g(n) = \sqrt{3}n^2 \cdot \left\lfloor \sqrt{2}n \lfloor en \rfloor \right\rfloor \cdot \left\lfloor \sqrt{5}n^3 \right\rfloor + \pi n^2$.
   - $\psi(n) = e(g(n))$ is (morally) a nilsequence.
   - $A = \{n \in \mathbb{N} : g(n) \in (-\frac{1}{10}, \frac{1}{10}) \bmod 1\}$ is a Nil–Bohr set (with any luck).

Warning: We're skipping technicalities here.

Finite sums. For $\vec{n} = (n_i)_{i=1}^{\infty}$, $n_i \in \mathbb{N}$, define:

$$\mathrm{FS}(\vec{n}) = \Big\{ \sum_{i \in \alpha} n_i \ : \ \alpha \subset \mathbb{N},\ \text{finite},\ \alpha \neq \emptyset \Big\}.$$

Convenient to write: $\mathcal{F} := \{\alpha \subset \mathbb{N},\ \text{finite},\ \neq \emptyset\}$ and $n_\alpha := \sum_{i\in\alpha} n_i$, so that $\mathrm{FS}(\vec{n}) = \{n_\alpha \ : \ \alpha \in \mathcal{F}\}$.

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- A set $A \subset \mathbb{N}$ is IP is there is $\vec{n}$ with $A \supset \mathrm{FS}(\vec{n})$.
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Theorem (Hindman)

- *If A is an* IP *set, $A = A_1 \cup A_2 \cup \cdots \cup A_r$ then $\exists j :\ A_j$ is* IP.
- *If $B_1, B_2 \ldots, B_r$ are* IP* *sets then $B = B_1 \cap B_2 \cap \cdots \cap B_r$ is* IP*.

Finite sums and bounded gaps. For $\vec{n} = (n_i)_{i=1}^{\infty}$, $n_i \in \mathbb{N}$, define:

$$\mathrm{SG}_d(\vec{n}) = \Big\{ \sum_{i \in \alpha} n_i \ : \ \alpha \subset \mathbb{N}, \text{ finite}, \ \alpha \neq \emptyset, \text{ gaps} \leq d \Big\} = \Big\{ n_\alpha \ : \ \alpha \in \mathcal{S}_d \Big\},$$

where gaps of $\alpha = \{a_1 < a_2 < \cdots < a_r\}$ are $a_{i+1} - a_i$, $i = 1, \ldots, r-1$, and $\mathcal{S}_d = \{\alpha \in \mathcal{F} \ : \ \text{gaps} \leq d\}$.

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- A set $A \subset \mathbb{N}$ is SG_d is there is $\vec{n}$ with $A \supset \mathrm{SG}_d(\vec{n})$.
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Fact

We have the chain of implications:

- $\mathrm{SG}_1 \Longleftarrow \mathrm{SG}_2 \Longleftarrow \mathrm{SG}_3 \ldots \Longleftarrow \mathrm{IP}$;
- $\mathrm{SG}_1^* \Longrightarrow \mathrm{SG}_2^* \Longrightarrow \mathrm{SG}_3^* \ldots \Longrightarrow \mathrm{IP}^*$.

Theorem (Host-Kra)

Suppose that $A \subset \mathbb{N}$ is SG_d^*. Then A contains a strongly piecewise Nil–Bohr$_0$ set of step d.
In particular, there are Nil–Bohr$_0$ set B of step d and a thick set $T = \bigcup_{i=1}^{\infty}[n_i, m_i]$, $m_i - n_i \to \infty$ such that $A \supset B \cap T$.

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Conjecture: If A is a Nil–Bohr$_0$ set of step d, then A is SG_d^*.

Basic facts:

- Any Nil–Bohr$_0$ set is IP^*. (Fact about distal dynamical systems.)
- Any Bohr$_0$ set is SG_1^*, i.e. intersects $S - S$, for $S \subset \mathbb{N}$, infinite.

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Theorem (K.)

- *Any d-step Nil–Bohr$_0$ set A is $\mathrm{SG}_{d'}^*$, where $d' = \binom{d+2}{2}$.*
- *For $A = \{n : p(n) \in (-\varepsilon, \varepsilon) \bmod 1\}$, $p(x) \in \mathbb{R}[x]$, this holds $d' = d$.*

Proof: Basics

- **Setup:** Let G/Γ be a d-step nilmanifold, $a \in G$; $\vec{n} = (n_i)_{i=1}^{\infty}$, $n_i \in \mathbb{N}$; and $k \geq \binom{d+2}{2}$. Need to show that $e\Gamma \in \mathrm{cl}\{a^{n_\alpha}\Gamma : \alpha \in \mathcal{S}_k\}$, where $n_\alpha = \sum_{i\in\alpha} n_i$.

  Hence, we study functions of the form

$$f\colon \mathcal{F}_\emptyset \to G/\Gamma, \quad f(\alpha) = a^{n_\alpha}\Gamma. \tag{$*$}$$

- **Some useful operations:**
  - *Subsequences*: For $(\beta_i)_{i=1}^{\infty}$, $\beta_i \in \mathcal{F}$, disjoint, consider

$$\tilde{f}(\alpha) := f(\beta_\alpha), \quad \beta_\alpha = \bigcup_{i\in\alpha} \beta_i.$$

    [Will insist that $\alpha \mapsto \beta_\alpha$ maps $\mathcal{S}_l$ to $\mathcal{S}_k$ for some $l \leq k$.]

  - *Pointwise limits:* Given $f_m\colon \mathcal{F}_\emptyset \to G/\Gamma$, consider

$$\tilde{f}(\alpha) = \lim_{m\to\infty} f_m(\alpha).$$

Proof: Polynomials

- **Problem:** The class of functions given by

$$f\colon \mathcal{F}_\emptyset \to G/\Gamma, \quad f(\alpha) = a^{n_\alpha}\Gamma \tag{$*$}$$

  is closed under subsequences, but not under pointwise limits.
- **Solution:** Introduce the class of *polynomial maps* from $\mathcal{F}$ to G/Γ with respect to pre-filtration $G_\bullet = G_0 \supseteq G_1 \supseteq G_2 \dots$ (i.e. $G_0 = G$, $[G_i, G_j] \subset G_{i+j}$, $G_{d+1} = \{e\}$).
  A function $f\colon \mathcal{F} \to G$ is polynomial w.r.t. $G_\bullet$ if either $f = e$ and $G_1 = \{e\}$, or for any $\beta \in \mathcal{F}$, the discrete derivative

$$\Delta_\beta f(\alpha) := f(\beta)^{-1} f(\alpha \cup \beta) f(\alpha)^{-1}, \qquad (\alpha \cap \beta = \emptyset)$$

  is polynomial w.r.t. shifted pre-filtration $G_{\bullet+1} = G_1 \supseteq G_2 \supseteq \dots$.
  Likewise, $\bar{f}\colon \mathcal{F} \to G/\Gamma$ is polynomial w.r.t. $G_\bullet$ if $\bar{f}(\alpha) = f(\alpha)\Gamma$, $f\colon \mathcal{F} \to G$ polynomial.
  - *Generalization:* Functions in $(*)$ are polynomials w.r.t. the lower central series $G_0 = G_1 = G$, $G_{i+1} = [G_i, G]$.
  - *Closure properties*: Polynomials w.r.t. a given filtration are closed under both subsequences and pointwise limits.
  - *Abelian case:* For $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$, $G_0 = G_1 = \dots = G_d = \mathbb{R}$, $G_{d+1} = \{0\}$, these are the maps

$$\alpha \mapsto \sum_{\gamma \subset \alpha, |\gamma| \le d} a_\gamma, \quad a_\gamma \in \mathbb{R}.$$

Lemma

Let $g\colon \mathcal{F}_\emptyset \to G/\Gamma$ be a polynomial with respect to filtration $G_\bullet$ of length $\leq d$, with $g(\emptyset) = e\Gamma$. Let r be the least index s.t. $G_r \neq G$, $k \geq r$. Then, there exist a polynomial sequence $\tilde{g}\colon \mathcal{F}_\emptyset \to G/\Gamma$ (limit of subsequences of g) such that

- $\{\tilde{g}(\alpha) : \alpha \in \mathcal{S}_{k-r}\} \subseteq \mathrm{cl}\{g(\alpha) : \alpha \in \mathcal{S}_k\}$,
- $\tilde{g}(\alpha) \in \pi(G_r)$ *for any* $\alpha \in \mathcal{F}$, *where* $\pi\colon G \to G/\Gamma$ *is the quotient.*

Proof of Main theorem, assuming the Lemma.

- **Claim:** With notation above, $e\Gamma \in \mathrm{cl}\{g(\alpha) : \alpha \in \mathcal{S}_k\}$, provided that $k \geq r + (r+1) + \cdots + (d+1)$.
- Apply Lemma to produce $\tilde{g}$; suffice to show $e\Gamma \in \mathrm{cl}\{\tilde{g}(\alpha) : \alpha \in \mathcal{S}_{k-r}\}$.
- Can construe $\tilde{g}$ as polynomial on the simpler sub-nilmanifold $\tilde{G}/\tilde{\Gamma} = G_r/G_r \cap \Gamma$ w.r.t. pre-filtration $\tilde{G}_j = G_j \cap G_r$.
- Apply the inductive claim to $\tilde{g}$, where $\tilde{k} = k - r$, $\tilde{r} \geq r + 1$ (except if $r = d + 1$ — then we are done). □

Lemma

If $g\colon \mathcal{F}_\emptyset \to G/\Gamma$ *is a polynomial w.r.t.* $G_\bullet$ *of length* $\le d$, $g(\emptyset) = e\Gamma$, $G_r \neq G$, *then there exist a polynomial sequence* $\tilde{g}\colon \mathcal{F}_\emptyset \to G/\Gamma$ *such that*

- $\{\tilde{g}(\alpha) : \alpha \in \mathcal{S}_{k-r}\} \subseteq \mathrm{cl}\{g(\alpha) : \alpha \in \mathcal{S}_k\}$,
- $\tilde{g}(\alpha) \in \pi(G_r)$ *for any* $\alpha \in \mathcal{F}$ *(*$\pi\colon G \to G/\Gamma$ *is the quotient map).*

Proof of the Lemma.

- Quotient out G_r: can assume that $G_r = \{e\}$. W.l.o.g. $G/\Gamma = \mathbb{R}^m/\mathbb{Z}^m = \mathbb{T}^m$, and $g(\alpha) = \sum_{\gamma\subset\alpha,\ |\gamma|\le d} a_\gamma$, $a_\gamma \in \mathbb{R}^m$.
- Repeatedly pass to limits of subsequences of $g(\alpha)$ to obtain "simplest possible sequence". May assume that:
  - a_γ are k-periodic: $a_{\gamma+k} = a_\gamma$,
  - $a_\gamma = 0$ whenever γ has diameter $> k$.
- Let Σ be the closure of the set of subsequences $h(\alpha) = g(\beta_\alpha)$ where $\alpha \mapsto \beta_\alpha$ maps $\mathcal{S}_{k-r}$ to $\mathcal{S}_k$, and β_i's are somewhat "generic".
- Let Δ be the set of maps δ such that $h + \delta \in \Sigma$ whenever $h \in \Sigma$. Find elements of Δ by modifying a few β_i's. Conclude that $\Sigma \subset \Delta$. □

The End

Thank You for your attention!