Dilates and Baumslag-Solitar groups

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Papers

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Direct and inverse problems

in additive number theory and in non – abelian group theory

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A small doubling structure theorem in a Baumslag – Solitar group

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Basic definition

Definition

If X, Y are subsets of a (semi)group G, then we denote

$$XY = \{xy \mid x \in X, y \in Y\} \quad \text{and} \quad X^2 = \{x_1x_2 \mid x_1, x_2 \in X\} \ .$$

If $X = \{x\}$, then we denote XY by xY and if $Y = \{y\}$, then we write Xy instead of $X\{y\}$.

If G is an additive group, then we denote

$$X + Y = \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad 2X = \{x_1 + x_2 \mid x_1, x_2 \in X\} \ .$$

$X + Y$ is also called the (Minkowski) sumset of X and Y.

Direct and Inverse problems

Gregory A. Freiman, Structure theory of set addition, *Astérisque*, **258** (1999), 1-33

"Thus a **direct problem** *in additive number theory is a problem which, given summands and some conditions, we discover something about the set of sums. An* **inverse problem** *in additive number theory is a problem in which, using some knowledge of the set of sums, we learn something about the set of summands."*

Background

Remark

Let X and Y be finite on-empty sets of integers.

Obviously

$$|X+Y| \geq |X|+|Y|-1$$

and

$$|X+Y| = |X|+|Y|-1$$

if and only if

X and Y are arithmetic progressions with the same difference, unless one of them is a singeton.

Small doubling property

Let G be a group and S a finite subset of G.
Let $S^2 = \{s_1 s_2 \mid s_1, s_2 \in S\}$.

Problem

What if the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta,$$

for some small $\alpha \geq 1$ and small $|\beta|$?

Definition

The subset S of G is said to satisfy the *small doubling property* if

$$|S^2| \leq \alpha|S| + \beta,$$

where α and β denote real numbers, $\alpha \geq 1$.

A remark

Let X and Y be finite sets of integers with k and h elements, respectively. Assume that

X and Y are arithmetic progressions with the same difference:

$X = \{a, a+d, a+2d, ..., a+(k-1)d\}$ and

$Y = \{b, b+d, b+2d, ..., b+(h-1)d\}$ for some $k, h > 1$ and $d \neq 0$.

If, for instance, $h \leq k$, then

$$Y \subseteq \{(b-a)+x \mid x \in X\} = (b-a)+X.$$

Dilates

Subsets of $\mathbb{Z}$ of the form

$$r * A := \{rx \mid x \in A\},$$

where r is a **positive** integer and A is a **finite** subset of $\mathbb{Z}$, are called *r-dilates*.

Minkowski sums of dilates are defined as follows:

$$r_1 * A + ... + r_s * A = \{r_1x_1 + ... + r_sx_s \mid x_i \in A,\ 1 \leq i \leq s\}.$$

Dilates

These sums have been recently studied in different situations by *Bukh, Cilleruelo, Hamidoune, Plagne, Rué, Silva, Vinuesa* and others.

In particular, they examined sums of two dilates of the form

$$A + r * A = \{a + rb \mid \; a, b \in A\}$$

and solved various *direct* and *inverse* problems concerning their sizes.

Dilates

For example, it was shown by **J. Cilleruelo, M. Silva, C. Vinuesa** (A sumset problem, *J. Comb. Number Theory* **2** (2010), no. 1, 79–89) that

$$|A + 2 * A| \geq 3|A| - 2$$

and that

Theorem

$$|A + 2 * A| = 3|A| - 2$$

if and only if

A is an arithmetic progression.

Dilates

$$|A + 2 * A| \geq 3|A| - 2$$

J. Cilleruelo, M. Silva, C. Vinuesa,
A sumset problem,
J. Comb. Number Theory **2** (2010), no. 1, 79–89

announced in

M. B. Nathanson,
Inverse problems for linear forms over finite sets of integers,
J. Ramanujan Math. Soc. **23** (2008), no. 2, 151–165.

Dilates

Theorem (Y.O. Hamidoune, A. Plagne, 2002)

Let A be a finite set of integers. Then

$$|A + r * A| \geq 3|A| - 2$$

for any integer $r \geq 2$.

Theorem (M.B. Nathanson, 2008)

Let A be a finite set of integers. Then

$$|A + r * A| \geq \frac{7}{2}|A| - 2$$

for any integer $r \geq 3$.

Dilates

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa, 2010)

For any finite set A of integers we have

$$|A + 3 * A| \geq 4|A| - 4.$$

Furthermore if $|A + 3 * A| = 4|A| - 4$, *then* $A = 3 * \{0, ..., n\} \cup (3 * \{0, ..., n\} + 1)$ *or* $A = \{0, 1, 3\}$ *or* $A = \{0, 1, 4\}$ *or* A *is an affine transform of one of these sets.*

Dilates

Let A be **finite** subset of $\mathbb{Z}$.

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

$$|A + 3 * A| \geq 4|A| - 4.$$

Question

What about $|A + r * A|$, *where* $r \geq 4$?

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If $r \geq 3$, *then* $|A + r * A| \geq 4|A| - 4$.

Dilates: $|A + 2 * A| \geq 3|A| - 2$

J. Cilleruelo, M. Silva, C. Vinuesa

Let A be a finite set of integers and let $r > 1$. Divide A into residue classes modulo r, and define $\hat{A}$ to be the projection of A into $\mathbb{Z}/r\mathbb{Z}$.

Lemma

For arbitrary finite non empty sets B and $A = \bigcup_{i \in \hat{A}} (r * A_i + i)$ we have

(i) $|A + r * B| = \sum_{i \in \hat{A}} |A_i + B|$

(ii) $|A + r * B| \geq |A| + |\hat{A}|(|B| - 1)$

(iii) *Furthermore, if equality holds in (ii), then either $|B| = 1$ or $|A_i| = 1$ for all $i \in \hat{A}_i$ or B and all the sets A_i with more than one element are arithmetic progressions with the same difference.*

Dilates: $|A + r * A| \geq 3|A| - 2$, for any $r \geq 2$

Proof. Let $A = \{x_1, x_2, \cdots, x_k\}$ and assume $x_1 < x_2 < \cdots < x_k$. Clearly

$$x_1 + rx_1 < x_2 + rx_1 < x_1 + rx_2 < x_2 + rx_2 < x_3 + rx_2 < x_2 + rx_3 < \dots$$

$$\dots < x_{k-1} + rx_{k-1} < x_k + rx_{k-1} < x_{k-1} + rx_k < x_k + rx_k.$$

Then, for each i such that $1 \leq i \leq k - 1$ we have the three elements $x_i + rx_i < x_{i+1} + rx_i < x_i + rx_{i+1}$ and one more element $x_k + rx_k$. Therefore $|A + r * A| \geq 3(k - 1) + 1 = 3|A| - 2$, as required. //

Dilates

A similar argument shows that:

Theorem

If A is a finite set of integers and r, s are positive integers, $r \neq s$, then

$$|s * A + r * A| \geq 3|A| - 2.$$

Theorem (M.B. Nathanson, 2008)

If A is a finite set of integers and r, s are positive coprime integers, at least one of which is ≥ 3, then

$$|s * A + r * A| \geq \frac{7}{2}|A| - 3.$$

Dilates

Theorem (A. Balog, G. Shakan)

For any relatively prime integers $1 \leq p < q$ *and for any finite set A of integers, one has*

$$|p * A + q * A| \geq (p+q)|A| - (pq)^{(p+q-3)(p+q)+1}.$$

A. Balog, G. Shakan, On the sum of dilatations of a set, *Acta Arith.* **2360** (2014), 153-162.

Dilates

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

If $|A + 2 * A| = 3|A| - 2$, *then A must be an arithmetic progression.*

Question

What is the structure of the set A if $|A + 2 * A| < 4|A| - 4$?

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

$$\text{If} \quad |A + 2 * A| < 4|A| - 4 \ , \ |A| \geq 3,$$

then A is a **subset** *of an arithmetic progression of size* $\leq 2|A| - 3$.

Useful results

Write $[m, n] = \{x \in \mathbb{Z} \mid m \leq x \leq n\}$ and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$.
Let A be a finite subset of $\mathbb{Z}$.
Let $A = \{a_0 < a_1 < ... < a_{k-1}\}$ be a finite increasing set of k integers.
By the *length* $\ell(A)$ of A we mean the difference

$$\ell(A) := \max(A) - \min(A) = a_{k-1} - a_0$$

between its maximal and minimal elements and

$$h_A := \ell(A) + 1 - |A|$$

denotes the number of *holes* in A, that is $h_A = |\ [a_0, a_{k-1}] \setminus A\ |$.
Finally, if $k \geq 2$, then we denote

$$d(A) := g.c.d.(a_1 - a_0, a_2 - a_0, ..., a_{k-1} - a_0).$$

Useful results

Theorem (V.F. Lev - P.Y. Smelianski and Y.V. Stanchescu)

Let A and B be finite subsets of $\mathbb{N}$ such that $0 \in A \cap B$. Define

$$\delta_{A,B} = \begin{cases} 1, & \text{if } \ell(A) = \ell(B), \\ 0, & \text{if } \ell(A) \neq \ell(B). \end{cases}$$

Then the following statements hold:

(i) *If* $\ell(A) = \max(\ell(A), \ell(B)) \geq |A| + |B| - 1 - \delta_{A,B}$ *and* $d(A) = 1$, *then*

$$|A + B| \geq |A| + 2|B| - 2 - \delta_{A,B}.$$

(ii) *If* $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \delta_{A,B}$, *then*

$$|A+B| \geq (|A|+|B|-1)+\max(h_A, h_B) = \max(\ell(A)+|B|, \ell(B)+|A|).$$

Dilates

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If $|A + 2 * A| < 4|A| - 4$,

then A *is a* **subset** *of an arithmetic progression of size* $\leq 2|A| - 3$.

Dilates

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let $A = \{a_0 < a_1 < a_2 < \cdots < a_{k-1}\} \subset \mathbb{Z}$ be a finite set of integers of size $k = |A| \geq 1$. Then the following statements hold.

(a) *If $1 \leq k \leq 2$, then $|A + 2 * A| = 3k - 2$ and A* **is** *an arithmetic progression of size k.*

(b) *If $k \geq 3$, assume that* $|A + 2 * A| = (3k-2) + h < 4k - 4.$

Then $h \geq 0, \quad |A + 2 * A| \geq 3k - 2$

and the set A is a **subset** *of an arithmetic progression*

$$P = \{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (t-1)d\}$$

of size $|P|$ bounded by $|P| \leq k + h = |A + 2 * A| - 2k + 2 \leq 2k - 3$.

(c) *If $k \geq 1$ and $|A + 2 * A| = 3k - 2$, then A* **is** *an arithmetic progression*

$$A = \{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (k-1)d\}.$$

Proof of the Theorem - sketch

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let $A = \{a_0 < a_1 < a_2 < \cdots < a_{k-1}\} \subset \mathbb{Z}$ *be a finite set of integers of size* $k = |A| \geq 1$. *Then the following statements hold.*

(b) *If* $k \geq 3$, *assume that* $|A + 2 * A| = (3k-2) + h < 4k - 4.$

Then $h \geq 0$, $|A + 2 * A| \geq 3k - 2$

and the set A *is a* **subset** *of an arithmetic progression*

$$P = \{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (t-1)d\}$$

of size $|P|$ *bounded by* $|P| \leq k + h = |A + 2 * A| - 2k + 2 \leq 2k - 3.$

Sketch of the Proof **(b)** Suppose, first, that A is **normal**, i.e. $\min(A) = a_0 = 0$ and $d = d(A) = gcd(A) = 1$. Thus $\ell(A) = a_{k-1}$.

We split the set A into a disjoint union $A = A_0 \cup A_1$, where $A_0 \subseteq 2\mathbb{Z}$ and $A_1 \subseteq 2\mathbb{Z} + 1$. Since $0 = a_0 \in A_0$ and $d(A) = 1$, it follows that $A_0 \neq \emptyset$ and $A_1 \neq \emptyset$. Therefore

$$m := |A_0| \geq 1, n := |A_1| \geq 1 \text{ and } k = m + n. \ . \ . \ .$$

Proof of the Theorem - sketch

. . . It follows that Theorem (b) holds for **normal** sets A satisfying the hypothesis.

Let now A be an **arbitrary** finite set of $k = |A| \geq 3$ integers satisfying the hypothesis. We define

$$B = \frac{1}{d(A)}(A - a_0) = \{\frac{1}{d(A)}(x - a_0) : x \in A\}.$$

...

//

Small doubling property

Let G be a (semi)group and S a finite subset of G.
Let $S^2 = \{s_1 s_2 \mid s_1, s_2 \in S\}$.

Problem

What if the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta,$$

for some small $\alpha \geq 1$ and small $|\beta|$?

Definition

The subset S of G is said to satisfy the *small doubling property* if

$$|S^2| \leq \alpha|S| + \beta,$$

where α and β denote real numbers, $\alpha \geq 1$.

The groups $\mathcal{BS}(m,n)$

For integers m and n, the general Baumslag-Solitar group $\mathcal{BS}(m,n)$ is a group with two generators a, b and one defining relation $b^{-1}a^m b = a^n$:

$$\mathcal{BS}(m,n) = \langle a, b \mid \; a^m b = ba^n \rangle.$$

"The Baumslag-Solitar groups are a particular class of two-generator one-relator groups which have played a surprisingly useful role in combinatorial and, more recently (the 1990s), geometric group theory. In a number of situations they have provided examples which mark boundaries between different classes of groups and they often provide a testbed for theories and techniques."

Encyclopedia of Mathematics

The groups $\mathcal{BS}(m,n) = \langle a, b \mid a^m b = b a^n \rangle$

$$\mathcal{BS}(m,n) = \langle a, b \mid a^m b = b a^n \rangle$$

1933-2014

These groups were introduced by **Gilbert Baumslag** and **Donald Solitar** in 1962 in order to provide some simple examples of non-Hopfian groups.

1932-2008

("Some two generator one-relator non-Hopfian groups", *Bull. Amer. Math. Soc.*, **689** (1962), 199-201).

A group is called *Hopfian* (or nowadays *Hopf*) if every epimorphism from the group to itself is an isomorphism.

Hopfian groups

The name is derived from the topologist *Heinz Hopf* and is thought to reflect the fact that whether fundamental groups of manifolds are *Hopfian* is of interest.

1874-1971

In the early 30's *Heinz Hopf* asked whether a finitely generated group can be isomorphic to a proper factor group of itself (*i.e. whether a finitely generated non-Hopfian group exists).*

Hopfian groups

In 1944 *Reinhold Baer* published an example of a non-Hopfian 2-generator group but then he discovered a mistake.

1902-1979

1909-2002

B.H. Neumann in 1950 found an example of a 2-generator infinitely related non-Hopfian group.

("A two-generator group isomorphic to a proper facotor group, *J. London Math. Soc.*, **25** (1950), 247-248)

Hopfian groups

One year after *Graham Higman* exhibited an example of a finitely presented non-Hopfian group; more precisesely, this group was 3-generator and with 2 defining relations.

1917-2008

("A finitely related group with an isomorphic proper factor group, *J. London Math. Soc.*, **26** (1951), 59-61).

In his paper he quoted *Bernhard* and *Hanna Neumann* for a proof that one-relator groups had to be Hopfian, but they were only trying to show this, unsuccessfully.

Finally, in 1962, *Gilbert Baumslag* and *Donald Solitar* showed that the group

$$\mathcal{BS}(2,3) = \langle a, b \mid \ a^2 b = b a^3 \rangle$$

is non-Hopfian.

When $\mathcal{BS}(m, n)$ is a Hopfian group

More generally:

$$\mathcal{BS}(m, n) = \langle a, b \mid \; a^m b = b a^n \rangle$$

is Hopfian if and only if :

(i) $|m| = |n|$ or

(ii) $|m| = 1$ or

(iii) $|n| = 1$ or

(iv) $\pi(m) = \pi(n)$ where $\pi(m)$ denotes the set of prime divisors of m.

We shall concentrate on the Baumslag-Solitar groups

$$\mathcal{BS}(1, n) = \langle a, b \mid \; ab = ba^n \rangle.$$

They are extensions of a copy of the additive group of n-adic rational numbers by an infinite cyclic group. They are orderable groups.

The groups $\mathcal{BS}(1, n) = \langle a, b \mid \ ab = ba^n \rangle$

Let S be a finite subset of $\mathcal{BS}(1, n)$ of size k contained in the coset $b^r < a >$ for some $\mathbf{r} \geq \mathbf{0}$. Then

$$S = \{b^r a^{x_0}, b^r a^{x_1}, \ldots, b^r a^{x_{k-1}}\},$$

where $A = \{x_0, x_1, \ldots, x_{k_1 - 1}\}$ is a subset of $\mathbb{Z}$. We introduce now the notation

$$S = \{b^r a^x : x \in A\} =: b^r a^A.$$

Thus $|S| = |A|$.

The groups $\mathcal{BS}(1,n) = \langle a, b \mid ab = ba^n \rangle$

Let S be a finite subset of $\mathcal{BS}(1,n)$ of size k contained in the coset $b^r < a >$ for some $r \in \mathbb{N}$ and let T be a finite subset of $\mathcal{BS}(1,n)$ of size h contained in the coset $b^s < a >$ for some $s \in \mathbb{N}$.

Then

$$S = b^r a^A \quad , \quad T = b^s a^B$$

for some subsets $A = \{x_0, x_1, \ldots, x_{k-1}\}$ and $B = \{y_0, y_1, \ldots, y_{h-1}\}$ of $\mathbb{Z}$.

From $a^x b = ba^{nx}$ for each $x \in \mathbb{Z}$ it follows

$$(b^r a^x)(b^s a^y) = b^r(a^x b^s)a^y = b^r(b^s a^{n^s x})a^y = b^{r+s} a^{n^s x + y}.$$

Therefore if

$$S = b^r a^A \quad , \quad T = b^s a^B$$

where $A = \{x_0, x_1, \ldots, x_{k-1}\}$ and $B = \{y_0, y_1, \ldots, y_{h-1}\}$ are subsets of $\mathbb{Z}$, from $\big(b^r a^x\big)\big(b^s a^y\big) = b^{r+s} a^{n^s x + y}$ it follows

$$ST = b^{r+s} a^{n^s * A + B} \quad \text{and} \quad |ST| = |n^s * A + B|.$$

In particular

$$S^2 = b^{2r} a^{n^r * A + A} \quad \text{and} \quad |S^2| = |n^r * A + A|.$$

The groups $\mathcal{BS}(1,n) = \langle a, b \mid ab = ba^n \rangle$

Theorem

Suppose that $S = b^r a^A \subseteq \mathcal{BS}(1,n)$, $T = b^s a^B \subseteq \mathcal{BS}(1,n)$, *where* $r, s \in \mathbb{Z}$, $r, s \geq 0$ *and* A, B *are finite subsets of* $\mathbb{Z}$. *Then*

$$ST = b^{r+s} a^{n^s * A + B}$$

and

$$|ST| = |n^s * A + B|.$$

In particular,

$$S^2 = b^{2r} a^{n^r * A + A}$$

and

$$|S^2| = |n^r * A + A| = |A + n^r * A|.$$

The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

If A is a finite set of integers, then $|A + 2 * A| \geq 3|A| - 2$ and $|A + 2 * A| = 3|A| - 2$ if and only if A is an arithmetic progression.

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If $S = ba^A \subseteq \mathcal{BS}(1,2)$, where A is a finite subset of $\mathbb{Z}$, then

$$|S^2| \geq 3|S| - 2$$

and if $|S^2| = 3|S| - 2$, then A is an arithmetic progression and S is a geometric progression.

The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If A is a finite set of integers, $|A| \geq 3$ and $|A + 2 * A| < 4|A| - 4$, then A is a **subset** *of an arithmetic progression of size $\leq 2|A| - 3$.*

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If $S = ba^A \subseteq \mathcal{BS}(1,2)$, $|S| \geq 3$ and $|S^2| < 4|S| - 4$, then A is a **subset** *of an arithmetic progression of size $\leq 2|S| - 3$.*

The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If A is a finite set of integers, $r \geq 3$, then $|A + r * A| \geq 4|A| - 4$.

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let $S = b^m a^A \subseteq \mathcal{BS}(1,2)$, where A is a finite set of integers of size $k \geq 2$ and $m \geq 2$. Then

$$|S^2| \geq 4k - 4.$$

The group $\mathcal{BS}(1, n) = \langle a, b \mid ab = ba^n \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

If A is a finite set of integers, $r \geq 3$, then $|A + r * A| \geq 4|A| - 4$.

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let $S \subseteq \mathcal{BS}(1, n)$ be a finite set of size $k = |S| \geq 2$ and suppose that $n \geq 3$ and

$$S = ba^A,$$

where $A \subseteq \mathbb{Z}$ is a finite set of integers.

Then

$$|S^2| = |A + n * A| \geq 4k - 4.$$

The group $\mathcal{BS}(1,2) = \langle a, b \mid \; ab = ba^2 \rangle$

Problem

What is the structure of an arbitrary subset of $\mathcal{BS}(1,2)$, satisfying some small doubling condition?

Very difficult!

Definition

Consider the submonoid

$$\mathcal{BS}^+(1,2) := \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$$

of $\mathcal{BS}(1,2)$.

The monoid $\mathcal{BS}^+(1,2)$

Definition

Consider the submonoid

$$\mathcal{BS}^+(1,2) = \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$$

of $\mathcal{BS}(1,2)$.

Remark

All elements of

$$\mathcal{BS}^+(1,2)$$

can be uniquely represented by a word of the form $b^m a^x$, which is not the case in $\mathcal{BS}(1,2)$.

$\mathcal{BS}^+(1,2) = \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let S be a finite non-abelian subset of $\mathcal{BS}^+(1,2)$ and suppose that

$$|S^2| < \frac{7}{2}|S| - 4.$$

Then

$$S = ba^A,$$

where A is a set of integers of size $|S|$, which is contained in an arithmetic progression of size less than $\frac{3}{2}|S| - 2$.

$\mathcal{BS}^+(1,2) = \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let S be a finite non-abelian subset of $\mathcal{BS}^+(1,2)$ and suppose that $|S^2| < \frac{7}{2}|S| - 4$. Then $S = ba^A$, where A is a set of integers of size $|S|$, which is contained in an arithmetic progression of size less than $\frac{3}{2}|S| - 2$.

Remark

This result is best possible.

In fact, there exist non-abelian subsets S of $\mathcal{BS}^+(1,2)$ satisfying $|S^2| = \frac{7}{2}|S| - 4$, which are not contained in one coset of $\langle a \rangle$ in $\mathcal{BS}^+(1,2)$.

$\mathcal{BS}^+(1,2) = \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$

There exist non-abelian subsets S of $\mathcal{BS}^+(1,2)$ satisfying $|S^2| = \frac{7}{2}|S| - 4$, which are not contained in one coset of $\langle a \rangle$ in $\mathcal{BS}^+(1,2)$.

Example

Let

$$S = a^{A_0} \cup \{b\} \subset \mathcal{BS}^+(1,2),$$

where

$$A_0 = \{0, 1, 2, ..., k-2\} \text{ and } k > 2 \text{ is even.}$$

The set S is clearly non-abelian, and it intersects non-trivially the two distinct cosets $1\langle a \rangle$ and $b\langle a \rangle$ of $\langle a \rangle$ in $\mathcal{BS}^+(1,2)$.
Moreover, $|S^2| = \frac{7}{2}k - 4$.

$S = a^{A_0} \cup \{b\} \subset \mathcal{BS}^+(1,2)$, $A_0 = \{0,1,2,...,k-2\}, k > 2$ *even*

For,

$$S^2 = a^{A_0} a^{A_0} \cup ba^{A_0} \cup a^{A_0} b \cup \{b^2\},$$

and using $a^{A_0} b = ba^{2*A_0}$, we get

$$S^2 = a^{A_0+A_0} \cup (ba^{A_0} \cup ba^{2*A_0}) \cup \{b^2\} = a^{A_0+A_0} \cup ba^{A_0 \cup 2*A_0} \cup \{b^2\}$$

. Since

$$a^{A_0+A_0} \subseteq a^{\mathbb{Z}}, \qquad ba^{A_0 \cup 2*A_0} \subseteq ba^{\mathbb{Z}}, \qquad \{b^2\} \subseteq b^2 a^{\mathbb{Z}},$$

it follows that the three components of S^2 are disjoint in pairs and hence

$$|S^2| = |A_0 + A_0| + |A_0 \cup 2 * A_0| + 1 =$$

$$(2k-3) + (\frac{3}{2}k - 2) + 1 = \frac{7}{2}k - 4.$$

Theorem - sketch of the Proof

Theorem (G.A. Freiman, M. Herzog, P. L., M. Maj, Y.V. Stanchescu)

Let S be a finite non-abelian subset of $\mathcal{BS}^+(1,2)$ and suppose that $|S^2| < \frac{7}{2}|S| - 4$. Then $S = ba^A$, where A is a set of integers of size $|S|$, which is contained in an arithmetic progression of size less than $\frac{3}{2}|S| - 2$.

Write

$$S = S_0 \cup S_1 \cup \ldots \cup S_t,$$

where $t \geq 0$,

$$S_i = b^{m_i} a^{A_i} \subseteq b^{m_i} a^{\mathbb{Z}},$$

$$0 \leq m_0 < m_1 < \ldots < m_t,$$

and

$$k_i = |S_i| = |A_i| \geq 1.$$

Lemma (1)

Let $S \subseteq \mathcal{BS}^+(1,2)$ be a finite set of size $k = |S|$. Suppose that $t \geq 1$ and there is $0 \leq j \leq t$ such that $k_j = |S_j| \geq 2$. Then S generates a non-abelian group.

Proof. If $j = 0$ and $m_0 = 0$, then $k_0 = |S_0| = |A_0| \geq 2$ implies that $S_0 \neq \{1\}$ and $A_0 \neq \{0\}$. Since $t \geq 1$, it follows that there are three integers m, x, z such that $m \geq 1$, $x \neq 0$, $a^x \in S_0$ and $b^m a^z \in S_1$. In this case

$$a^x(b^m a^z) = b^m a^{z+2^m x} \neq (b^m a^z)a^x = b^m a^{z+x}$$

and therefore S generates a non-abelian group.

$t \geq 1$ and there is $0 \leq j \leq t$ such that $k_j = |S_j| \geq 2$

It remains to examine the following two cases:

(i) $j \geq 1$.

(ii) $j = 0$ and $m_0 \geq 1$.

If $j \geq 1$, then $m_j \geq 1$ and $k_j = |S_j| = |b^{m_j} a^{A_j}| \geq 2$ implies that $|A_j| \geq 2$. On the other hand, if $j = 0$ and $m_0 \geq 1$, then $k_0 = |S_0| = |b^{m_0} a^{A_0}| \geq 2$ implies that $|A_0| \geq 2$. In both cases, let $m = m_j$. Then $m \geq 1$ and there are two integers $x \neq y$ such that $\{b^m a^x, b^m a^y\} \subseteq S_j$. We conclude that

$$(b^m a^x)(b^m a^y) = b^{2m} a^{y+2^m x} \neq (b^m a^y)(b^m a^x) = b^{2m} a^{x+2^m y},$$

since $x \neq y$ and $m \geq 1$. The proof of Lemma is complete. //

Let $S \subseteq \mathcal{BS}^+(1,2)$ be a finite set of size $k = |S|$.

Lemma (2)

Suppose that $t = 1$. *Then* $|S^2| \geq \frac{7}{2}|S| - 4$.

Lemma (3)

Suppose that $t \geq 2$. *If* $k_0 = |S_0| \geq 2$ *and* $k_i = |S_i| = 1$ *for every* $1 \leq i \leq t$, *then* $|S^2| \geq 4k - 5 > \frac{7}{2}|S| - 4$ *and the inequality is tight.*

Lemma (4)

Suppose that $t \geq 2$. *If* $k_t = |S_t| \geq 2$ *and* $k_i = |S_i| = 1$ *for every* $0 \leq i \leq t-1$, *then* $|S^2| \geq 4k - 5 > \frac{7}{2}|S| - 4$ *and the inequality is tight.*

Open problems

Question

What about arbitrary finite subsets of $\mathcal{BS}(1,2)$?

Question

What about arbitrary finite subsets of $\mathcal{BS}(1,n)$, $n \neq 2$?

Question

What about arbitrary finite subsets of any $\mathcal{BS}(m,n)$?

Thank you for the attention !

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