Partitions of the set of nonnegative integers with the same representation functions

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Definitions

Definition

Let $k \geq 2$ *be a fixed integer and* $A = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ *be an infinite set of nonnegative integers. Let* $R_1(A, n, k)$, $R_2(A, n, k)$, $R_3(A, n, k)$ *denote the number of solutions of the equations*

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1}, a_{i_2}, \dots, a_{i_k} \in A,$$

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} < a_{i_2} < \dots < a_{i_k}, \quad a_{i_1}, a_{i_2}, \dots, a_{i_k} \in A,$$

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n, \quad a_{i_1} \leq a_{i_2} \leq \dots \leq a_{i_k}, \quad a_{i_1}, a_{i_2}, \dots, a_{i_k} \in A$$

respectively.

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respectively.

For $k = 2$ we have

$$R_2(A, n, 2) = \left\lfloor \frac{R_1(A, n, 2)}{2} \right\rfloor, \quad R_3(A, n, 2) = \left\lceil \frac{R_1(A, n, 2)}{2} \right\rceil.$$

Motivation

Theorem (Erdős, Turán, 1941)

For an infinite set $A \subset \mathbb{N}$ the representation function $R_1(A, n, 2)$ cannot be a constant from a certain point on.

Theorem (Dirac, Newman, 1951)

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Theorem (Erdős, Fuchs, 1956)

If c is a positive constant, $A \subset \mathbb{N}$ then

$$\sum_{n=1}^{N} R_1(A, n, 2) = cN + o(N^{1/4}(\log N)^{-1/2})$$

cannot hold.

Motivation

Problem (Gauss circle problem)

Consider a circle in $\mathbb{R}^2$ with centre at the origin and radius r. Gauss circle problem asks how many points there are inside this circle of the form (m, n) where m and n are both integers.

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The number of such points is $r^2\pi + E(r)$. It is conjectured that $E(r) = O(r^{1/2+\varepsilon})$. It follows from the above theorem that $E(r) \neq o(r^{1/2}(\log r)^{-1/2})$.

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Sidon asked: Does there exist a set $A \subset \mathbb{N}$ such that $R_1(A, n, 2) > 0$ for $n > n_0$ and for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{R_1(A, n, 2)}{n^\varepsilon} = 0?$$

Motivation

Theorem (Erdős, 1956)

There exists a set $A \subset \mathbb{N}$ so that there are two constans c_1 and c_2 for which for every n

$$c_1 \log n < R_1(A, n, 2) < c_2 \log n.$$

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Conjecture (Erdős, 1956)

There does not exists a set $A \subset \mathbb{N}$ *such that*

$$\lim_{n \to \infty} \frac{R_1(A, n, 2)}{\log n} = c,$$

where $c > 0$*.*

Motivation

Conjecture (Erdős, Turán, 1941)

If $R_1(A, n, 2) > 0$ *from a certain point on, then* $R_1(A, n, 2)$ *cannot be bounded.*

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If $A = \{a_1, a_2, \dots\}$ $(a_1 < a_2 < \dots)$ *is an infinite set of positive integers such that for some* $c > 0$ *and all* $k \in \mathbb{N}$ *we have* $a_k < ck^2$*, then* $R_1(A, n, 2)$ *cannot be bounded.*

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Theorem (Ruzsa, 1990)

There exists an infinite set $A \subset \mathbb{N}$ such that $R_1(A, n, 2) > 0$ for all $n > n_0$ and

$$\limsup_{N\to+\infty} \frac{1}{N}\Bigl(\sum_{n=1}^{N} R_1^2(A, n, 2)\Bigr) < +\infty.$$

Coincide representation functions

Theorem (Nathanson, 1978)

Let A and B be infinite sets of nonnegative integers, $A \neq B$. Then $R_1(A,n,2) = R_1(B,n,2)$ from a certain point on if and only if there exist positive integers n_0, M and finite sets F_A, F_B, T with $F_A \cup F_B \subset [0, Mn_0 - 1]$, $T \subset [0, M-1]$ such that

$$A = F_A \cup \{kM + t : k \geq n_0, t \in T\},$$

$$B = F_B \cup \{kM + t : k \geq n_0, t \in T\},$$

$$(1 - z^M) | (F_A(z) - F_B(z)) T(z).$$

$F_A(z) = \sum_{a \in A} z^a$, $F_B(z) = \sum_{b \in B} z^b$.

Conjecture (Kiss, Sándor, Rozgonyi, 2012)

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If the conditions of the above conjecture hold, then $R_1(A,n,k) = R_1(B,n,k)$.

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If the conditions of the above conjecture hold, then $R_1(A, n, k) = R_1(B, n, k)$.

Theorem (Sándor, Rozgonyi 2014)

The above conjecture holds, when $k = p^s$, where $s \geq 1$ and p is a prime.

Sárközy asked: there exist two sets A and B of positive integers with infinite symmetric difference, i.e, $|(A \cup B) \setminus (A \cap B)| = \infty$ and having $R_i(A, n, 2) = R_i(B, n, 2)$ for all sufficiently large n and $i = 1, 2, 3$.

Partitions and their representation functions

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Theorem (Dombi, 2002)

The set of nonnegative integers can be partitioned into two subsets A and B such that $R_2(A, n, 2) = R_2(B, n, 2)$ for all nonnegative integer n.

Theorem (Chen, Wang, 2003)

The set of positive integers can be partitioned into two subsets A and B such that $R_3(A, n, 2) = R_3(B, n, 2)$ for all positive integer n.

Partitions and their representation functions

Theorem (Lev, Sándor, 2004)

Let N be a positive integer. The equality $R_3(A, n, 2) = R_3(\mathbb{N} \setminus A, n, 2)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A$ if and only if $m \notin A$, $2m + 1 \in A$ if and only if $m \in A$ for $m \geq N$.

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Problem

Characterize all the sets of nonnegative integers A and B such that $R_2(A,n,2) = R_2(B,n,2)$.

Partitions and their representation functions

Definition

Let X be an additive semigroup and $A_1, \ldots, A_h$ are nonempty subsets of X. Let $R_{A_1+\ldots+A_h}(x)$ denote the number of solutions of the equation

$$a_1 + \ldots + a_h = x,$$

where $a_1 \in A_1, \ldots, a_h \in A_h$.

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Theorem (Kiss, Sándor, Rozgonyi, 2014)

The equality $R_{A+B}(n) = R_{\mathbb{N}\setminus A+\mathbb{N}\setminus B}(n)$ holds from a certain point on if and only if $|\mathbb{N} \setminus (A \cup B)| = |A \cap B| < \infty$.

Partitions and their representation functions

Theorem (Chen, Yang, 2012)

The equality $R_1(A, n, 2) = R_1(\mathbb{Z}_m \setminus A, n, 2)$ *holds for all* $n \in \mathbb{Z}_m$ *if and only if* m *is even and* $|A| = m/2$.

Theorem (Chen, Yang, 2012)

For $i \in \{2, 3\}$, *the equality* $R_i(A, n, 2) = R_i(\mathbb{Z}_m \setminus A, n, 2)$ *holds for all* $n \in \mathbb{Z}_m$ *if and only if* m *is even and* $t \in A$ *if and only if* $t + m/2 \notin A$ *for* $t = 0, 1, \ldots, m/2 - 1$.

Partitions and their representation functions

Theorem (Chen, Yang, 2012)

The equality $R_1(A,n,2)=R_1(\mathbb{Z}_m\setminus A,n,2)$ holds for all $n\in\mathbb{Z}_m$ if and only if m is even and $|A|=m/2$.

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Theorem (Kiss, Sándor, Rozgonyi, 2014)

Let G be a finite group, A, $B\subset G$. Then

(i) *If there exists a $g\in G$ for which the equality $R_{A+B}(g)=R_{G\setminus A+G\setminus B}(g)$ holds, then $|A|+|B|=|G|$.*

(ii) *If $|A|+|B|=|G|$, then the equality $R_{A+B}(g)=R_{G\setminus A+G\setminus B}(g)$ holds for all $g\in G$.*

Partitions and their representation functions

Theorem (Kiss, Sándor, Rozgonyi, 2014)

Let $X = G$ be a finite group, $A \subset G$ and $h \geq 2$ a fixed integer.

(i) *If the equality $R_1(A, g, h) = R_1(G \setminus A, g, h)$ holds for all $g \in G$, then $|G|$ is even and $|A| = |G|/2$.*

(ii) *If h is even and $|A| = |G|/2$ then $R_1(A, g, h) = R_1(G \setminus A, g, h)$ holds for all $g \in G$.*

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Problem

Let $h > 1$ be a fixed odd positive integer. Let G be an Abelian group and $A \subset G$ be a nonempty subset. Does there exist a $g \in G$ such that $R_1(A, g, h) \neq R_1(G \setminus A, g, h)$?

Theorem (Kiss, Sándor, Rozgonyi, 2014)

Let $X = \mathbb{Z}_m$ and $h > 2$ be a fixed odd integer. If $A \subset \mathbb{Z}_m$ such that $|A| = m/2$ then there exists a $g \in \mathbb{Z}_m$ such that $R_1(A, g, h) \neq R_1(\mathbb{Z}_m \setminus A, g, h)$.

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Problem

Let G be an Abelian group and $h \geq 2$. Characterize all the partitions of G into pairwise disjoint sets $A_1, A_2, \ldots, A_h$ such that for every $g \in G$ and for every $1 \leq i, j \leq h$, $R_1(A_i, g, h) = R_1(A_j, g, h)$.

Theorem (Z. Qu, 2015)

Let G be an Abelian group and $h \geq 3$ an odd integer. Then it is not possible to partition G into h disjoint sets $A_1, A_2, \ldots, A_h$ such that for every $g \in G$ and for every $1 \leq i, j \leq h$, $R_1(A_i, g, h) = R_1(A_j, g, h)$.

Partitions and their representation functions

Let A be the set of those nonnegative integers which contains even number of 1 binary digits in its binary representation and let B be the complement of A. Put $A_l = A \cap [0, 2^l - 1]$ and $B_l = B \cap [0, 2^l - 1]$.

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Theorem (Kiss, Sándor, 2016)

Let C and D be sets of nonnegative integers such that $C \cup D = \mathbb{N}$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_2(C, n, 2) = R_2(D, n, 2)$ if and only if $C = A$ and $D = B$.

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Theorem (Kiss, Sándor, 2016)

Let C and D be sets of nonnegative integers such that $C \cup D = [0, m]$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_2(C, n, 2) = R_2(D, n, 2)$ if and only if there exists an l natural number such that $C = A_l$ and $D = B_l$.

Theorem (Tang, Yu, 2012)

If $C \cup D = \mathbb{N}$ *and* $C \cap D = \{4k : k \in \mathbb{N}\}$, *then* $R_2(C, n, 2) = R_2(D, n, 2)$ *cannot hold for all sufficiently large* n.

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Conjecture (Tang, Yu, 2012)

Let $m \in \mathbb{N}$ and $R \subset \{0, 1, \ldots, m-1\}$. If $C \cup D = \mathbb{N}$ and $C \cap D = \{r + km : k \in \mathbb{N}, r \in R\}$, then $R_2(C, n, 2) = R_2(D, n, 2)$ cannot hold for all sufficiently large n.

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Theorem (Chen - Lev, 2015)

Let l be a positive integer. There exist sets C, $D \subset \mathbb{N}$ such that $C \cup D = \mathbb{N}$, $C \cap D = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}$ and $R_2(C, n, 2) = R_2(D, n, 2)$.

Problem (Chen - Lev, 2015)

Let C and D be sets of nonnegative integers such that $C \cup D = [0, m-1]$ and $C \cap D = \{r\}$, where $r \geq 0$, $m \geq 2$ and $R_2(C, n, 2) = R_2(D, n, 2)$. Does there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$, $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$?

Partitions and their representation functions

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Theorem (Kiss, Sándor, 2016)

Let C and D be sets of nonnegative integers such that $C \cup D = [0, m-1]$ and $C \cap D = \{r\}$, $0 \in C$. Then $R_2(C, n, 2) = R_2(D, n, 2)$ if and only if there exists an l natural number such that $C = A_{2l} \cup (2^{2l} - 1 + B_{2l})$ and $D = B_{2l} \cup (2^{2l} - 1 + A_{2l})$.

Problem (Kiss, Sándor, 2016)

Let C and D be sets of nonnegative integers such that $C \cup D = [0, m-1]$ and $C \cap D = \{r + n\mathbb{N}\}$, where $r \geq 0$, $m \geq 2$ integers and $R_2(C, n, 2) = R_2(D, n, 2)$. Does there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$?

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Let C and D be sets of nonnegative integers such that $C \cup D = [0, m-1]$ and $C \cap D = \{r + n\mathbb{N}\}$, where $r \geq 0$, $m \geq 2$ integers and $R_2(C, n, 2) = R_2(D, n, 2)$. Does there exists an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$?

Theorem (Kiss, Sándor, 2016)

Let $m \geq 2$ be an even positive integer and let A and B be sets of nonnegative integers such that $A \cup B = \mathbb{N}$ and $A \cap B = m\mathbb{N}$. Then there exist infinitely many positive integer n such that $R_A(n) \neq R_B(n)$.

Thank you for your attention!