Applications of the removal lemma

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Stating the problem

Context

Given k and a group G, consider $P \subset G^k$ (set of configurations). Then $S \subset G$ is said to be solution-free if $S^k \cap P = \emptyset$.

Questions

- Maximal size and stability of solution-free sets.
- How many solution-free sets S are there?
- How many solution-free sets S of size t are there?

We consider P to be the solution set of a linear system (or a significant/non-trivial part); we consider sequences of systems and asymptotic results.

Examples

Roth'53 for $k = 3$, Szemerédi'75.

k-term AP-free in dense sets in $[n]$.

Ajtai-Szemerédi'74 corners, Furstenberg-Katznelson'78 any F

F finite and fixed subset of $[n]^m$.
Structures of the type $\{x + aF\}$ in dense sets of $[n]^m$.
(Corners: $\{(x_1, x_2) + a(0,0), (x_1, x_2) + a(1,0), (x_1, x_2) + a(0,1)\}$).

Green'04, Sapozhenko'03, (Cameron-Erdős conj.)

There are $O(2^{n/2})$ sum-free sets in $[1, n]$.

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Arithmetic Removal Lemma. Green'05

If a set does not have many sums, it is "close" to being sum-free.
(removing few elements $\rightarrow$ sum–free).

Removal lemma statements

Theorem (Removal lemma-like statements)

If in *Context–Setting*
there are *Few* *Substructures*
then, by removing *Few elements*
a new *Context–Setting* *with no* *Substructures* *is obtained.*

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Setting	Elements
(1): Graph K	edges

Substructures	Few subs.	Few elem.
Triangles K_3	$o(\|K\|^3)$	$o(\|K\|^2)$

(1): Ruzsa-Szemerédi'78.

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Setting	Elements
(2): Hypergraph K	k-uniform edges

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Hypergraph H	$o(\|K\|^{\|H\|})$	$o(\|K\|^{k})$

(2): Nagel-Rödl-Schacht-Skokan'06, Gowers'07, Tao'06, Elek-Szegedy'12.

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solutions to $x_1 + \cdots + x_k = 0$	$o(\|G\|^{k-1})$	$o(\|G\|)$

(2): several authors. (3): Green'05

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(5): $X_i \subset G$, finite abelian group	elements in the group

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solutions to $x_1 \cdot \dots \cdot x_k = 1$	$o(\lvert G\rvert^{k-1})$	$o(\lvert G\rvert)$
solutions to $Ax = 0$, A $m \times k$ integer matrix, $d_m(A) = 1$	$o(\lvert G\rvert^{k-m})$	$o(\lvert G\rvert)$

(4): KSV. (5): Král'-Serra-V.'13.

Consequences and applications

Distance result

not many substructures $\Rightarrow$ very close to be substructure-free

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several disjoint unavoidable structures $\Rightarrow$ many solutions overall

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several disjoint unavoidable structures $\Rightarrow$ many solutions overall

Also supersaturation

positive proportion above maximal size structure-free set

$$\Downarrow$$

positive proportion of solutions

RL for linear configurations

Theorem (RL homomorphisms in abelian groups)

Context	$X_i \subset G$ *finite abelian group*,
	$A : G^k \to G^m$ *group morphism*,
	$\forall \epsilon > 0 \ \exists \delta(\epsilon, k) > 0$
If Not many	$\lvert\{x_i \in X_i : A(x) = b\}\rvert < \delta \lvert ker_G A \rvert$
then Remv. Few	$\exists X_i' \subset X_i$, $\lvert X_i' \rvert < \epsilon \lvert \pi_i(ker_G A) \rvert$
No Solutions left	$\lvert\{x_i \in X_i \setminus X_i' : A(x) = b\}\rvert = 0$

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- ☺ Includes integer matrices, and more.
  Equations between coordinates in $G = \prod_{i \in I} \mathbb{Z}_i$.
- ☺ Multidimensional Szemerédi (Furstenberg-Katznelson'78).
- $x_1 + 2(x_2 + x_3) = 0$ in $\mathbb{Z}_2^n$ implies $x_1 = 0$.
- Determinantal condition: $2(x_1 + x_2 + x_3) = 0$.

Counting configuration-free sets

Balogh-Morris-Samotij'15

For every positive δ and every positive integers r, k and every $F \subset \mathbb{N}^k$, there exist $n_0(\delta, k, r, F)$ and $C(\delta, r, k, F)$ such that, if

$m \geq Cn^{1-\frac{1}{k-1}}$	$m \geq Cn^{k-\frac{1}{\lvert F\rvert-1}}$	$m \geq Cn^{1-\frac{1}{kr}}$

then there are at most

$\binom{2\delta n}{m}$ m-subsets of $[n]$ that contain no k-term AP.	$\binom{2\delta n^k}{m}$ m-subsets of $[n]^k$ containing no homothetic copy of F.	$\binom{2\delta n}{m}$ m-subsets of $[n]$ that contain no set of the form $\{a, a+d^r, \ldots, a+kd^r, d \in \mathbb{Z}\}$.

Saxton-Thomason'16+

Let $\mathbb{F}$ be a finite field and A be a $m \times k$ linear system over $\mathbb{F}$ with $\sum A_i = 0$ over $\mathbb{F}$ and $\text{rank}(A \setminus \{A_i, A_j\}) = \text{rank}(A) = k - m$ for each $i, j \in [k]$, then there are at most $2^{\text{ex}(\mathbb{F}, A, b) + o(\lvert F\rvert)}$ solution-free sets.

Tools used

Both previous results use:

- *a hypergraph container result*: each independent set of a hypergraph H is contained in some container $C \subset V(H)$. Not many containers. Each C contains few edges.

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II *a strong counting/supersaturation/Varnavides-type result*: for every δ there exist a γ for which, given any $S \subset G$ with $|S| \geq \delta|G|$, then $|S^k \cap P| \geq \gamma P$.

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Our goal: extend this strategy to linear configurations in finite abelian groups, or other supersaturated contexts.

Counting configurations II

Density condition provided by an arithmetic removal lemma:

Green'05

G finite abelian group,
$\epsilon_i \in \{-1, 1\}$ with $\sum_{i=1}^{k} \epsilon_i = 0$,
$P = \{(x_i) | \epsilon_1 x_1 + \cdots + \epsilon_k x_k = 0\}$ satisfies supersaturation.

Kral'-Serra-V.'09

G finite group,
$\epsilon_i \in \{-1, 1\}$ with $\sum_{i=1}^{k} \epsilon_i = 0$,
$P = \{(x_i) | x_1^{\epsilon_1} \cdots x_k^{\epsilon_k} = 1\}$ satisfies supersaturation.

V.'16+

G finite abelian group,
$A : G^k \to G^m$ group morphism, $A(g, \dots, g) = 0$ for each $g \in G$,
then $P = A^{-1}(0)$ satisfies supersaturation.

Counting configurations I

Theorem: Counting configuration-free sets

Let $k > 0$ be an integer and $1/40 > \delta > 0$. Let (A, G) be a "supersaturated system", largest configuration-free set has size $< \delta/2|G|$,

$$t \geq C(k, \delta, A)|G| \max_{\ell \in [2,k]} \left\{ \left(\frac{\alpha_\ell^k}{\alpha_1^k} \right)^{\frac{1}{\ell - 1}} \right\}$$

there are at most

$$\binom{2\delta|G|}{t}$$

sets of size t with no solution in $S^k(A, G)$.

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α_i^k: i-th degree of freedom.
Measure concentrations over partial solutions.
Generalization of m_A (Rödl-Ruciński'97).

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Used with sequence $\{(A_i, G_i)\}_{i \in \mathbb{N}}$ where supersaturation $\gamma = \gamma(\delta, \text{whole sequence})$.

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Proof based on hypergraph containers: Balogh-Morris-Samotij'15, and Saxton-Thomason'16+.

Examples

• Equations in non-abelian groups.

• (Linear) point configurations in finite abelian groups which include:

- Integer linear systems, such as k-APs, in abelian groups.
- homothetic configurations with one or multiple degrees of freedom in $[n]^k$, such as homothetic copies of simplices (multidimensional Szemerédi).
- More involved configurations involving different subgroups.

Some of these results were known (k-AP in the integers, configurations in $[n]^l$ with one degree of freedom, linear systems in finite fields).

Some are new.

Some can be obtained using other supersaturation results (Tao, Sidorenko,. . .).

Comments

Difficulties and considerations:

- We should consider solutions where all the variables are different.
- Restrictions imposed by subconfigurations on the threshold of application t (balanced graphs, Rödl-Ruziński's m_A) $\to \alpha_i^k$.
- Some thresholds for t (lower bound) can be too large to say something interesting (such as for Sidon sets) $\to$ Shapira: for most integer linear systems, any polynomial bound on t is meaningful.

Example

Theorem (Rué-Serra-V. Rectangles in abelian groups)

$\{G_i\}_{i\in\mathbb{N}}$ *finite abelian groups,* $H_i, K_i \subset G_i$, $|H_i|, |K_i|, |G_i| \to \infty$.

$S(A, G_i) = \{(x, x+a, x+b, x+a+b)$ *with* $x \in G_i, a \in H_i, b \in K_i\}$

Assume $\max\{|H_i|, |K_i|\} \leq (|S^k(A, G_i)|/|G_i|)^{2/3}$. *For each* $1/40 > \delta > 0$ *there exist a* $C(\delta)$ *and an* $i_0(\delta, family)$*, such that, For each* $i \geq i_0$ *the number of sets free of configurations in* $S^k(A, G)$ *of size* t, $t > \frac{C}{\delta}\left(\frac{|G_i|^4}{|S^k(A,G_i)|}\right)^{1/3}$, *is at most*

$$\binom{2\delta|G_i|}{t}.$$

If $G = \mathbb{Z}_n^2$, $H = \mathbb{Z}_n \times 0$, $K = 0 \times \mathbb{Z}_n$: count number of C_4-free graphs

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If $G = \mathbb{Z}_n^2$, $H = \mathbb{Z}_n \times 0$, $K = 0 \times \mathbb{Z}_n$: count number of C_4-free graphs (not trivial as $t \ge n^{4/3} \ll n^{3/2}$, but not good upper bound).

Random sparse analogues

Consider sequence of systems $\{(A_i, G_i)\}$, $1 > \delta > 0$, for probability p consider $[G_i]_p$ (pick elements from G_i uniformly and random with probability p).
Does there exist a p' such that (?)

$$\lim_{n\to\infty} \mathbb{P}(\text{each } S \subset [G]_p, |S| > \delta|[G]_p|, \text{ contains a } \; s \in S^k(A_i, G_i)) = \begin{cases} 0, & p \ll p' \\ 1, & p \gg p' \end{cases} \tag{1}$$

Breakthrough by Conlon-Gowers, Schacht: showed upper bound matches lower bound given by the alteration method.
Containers gives correct upper bound (Balogh-Morris-Samotij, Saxton-Thomason).

Example sparse random

Theorem (Rué-Serra-V.'16+)

For every positive $1 > \delta > 0$ *and finite group with* $|G| \geq n_0(\delta)$ *and* $|G|$ *odd, then for the binomial random set* G_p *of* G *we have*

$$\lim_{n\to\infty} \mathbb{P}(\textit{each } S \subset G_p, |S| \geq \delta|G_p|, \ \textit{has } x,y,z \in S \textit{ with } xy = z^2)$$
$$= \begin{cases} 0, & p < c_1(\delta)|G|^{-1/2} \\ 1, & p > c_2(\delta)|G|^{-1/2} \end{cases}$$

Continuous Setting

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If Not many	$\lvert\{x_i \in X_i : A(x) = b\}\rvert < \delta \lvert ker_G A\rvert$
then Remv. Few	$\exists X_i' \subset X_i,\ \lvert X_i'\rvert < \epsilon \lvert \pi_i(ker_G A)\rvert$
No Solutions left	$\lvert\{x_i \in X_i \setminus X_i' : A(x) = b\}\rvert = 0$

Theorem (RL CAG. Candela-Szegedy-V'16+)

Context	*Borel* $X_i \subset G$ *Hausdorff cmpct. ab. group*,
	$A : G^k \to G^m$ *integer matrix*, $d_m(A) = 1$,
	$\forall \epsilon > 0\ \exists \delta(\epsilon, A) > 0$
If Not many	$\mu_{ker_G A}(\prod_1^k X_i \cap ker_G A) < \delta$
then Remv. Few	$\exists X_i' \subset X_i$, *Borel* $\mu_G(X_i') < \epsilon$
No Solutions left	$[ker_G A] \cap \prod_1^k X_i \setminus X_i' = \emptyset$

Sketch of the proof, compact abelian

Hypergraph representation

- Removal lemma for measurable hypergraphs. RL with symmetry preserving.
- Adequate representation of the system by a measurable hypergraph. Second-countable compact abelian groups.
- Retrieve information in original setting for second-countable compact abelian groups.
- Extend the result to all compact abelian groups.

Adequate representation:

- Giving solution, every edge has unique extension: deal with $x + 2y + 2z = 0$.
- Circular matrices.

Applications RL compact abelian groups

Szemerédi Theorem for compact abelian groups

For every ϵ, k there exist a c such that

$$\int_G \int_G 1_A(x) 1_A(x+r) \cdots 1_A(x+(k-1)r) \, \mathrm{d}\mu_G(r) \geq c$$

for every measurable set A, $\mu(A) \geq \epsilon$, in any Hausdorff compact abelian group.

Questions

- Delete the $d_k(A)$ condition.
- Homomorphisms?
- Small proportion with respect projection?

Merci pour votre attention!

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