

On Multiplicative Bases and some Related Problems

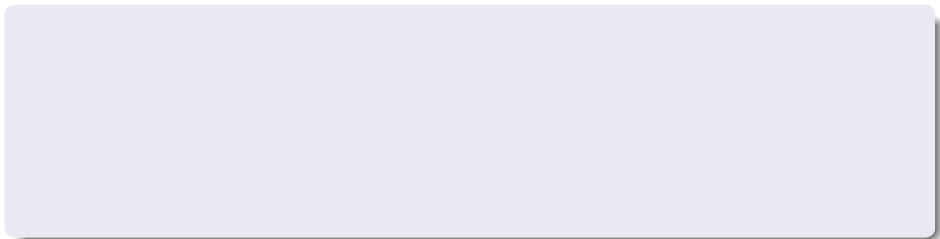
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13 April 2016

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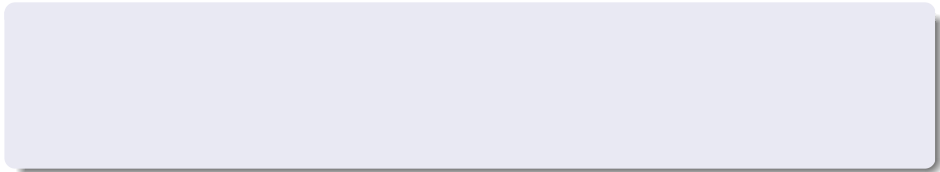
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$$x^2 = a_1 a_2 \dots a_{2k} \quad (a_1, a_2, \dots, a_{2k} \in A)$$

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$a_1, \dots, a_k, b_1, \dots, b_k$ are distinct

$$a_1 a_2 \dots a_k = b_1 b_2 \dots b_k \implies a_1 \dots a_k b_1 \dots b_k = x^2$$

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element \longrightarrow edge

$m = uv \longrightarrow uv$ edge

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How to choose $g(n)$?

condition (3) $\longrightarrow g(n) = e^{c \log n / \log \log n}$

Maximal number of edges of C_6 -free graphs

Füredi, Naor, Verstraëte, Györi

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Theorem (P.)

$$\pi(n) + \pi(n/2) + cn^{2/3}(\log n)^{-4/3} \leq \max |A| \leq \pi(n) + \pi(n/2) + cn^{2/3} \frac{\log n}{\log \log n}$$

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$a_0 \nmid a_1 a_2 \dots a_k$ and multiplicative bases

Lemma

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Multiplicative bases (of order k)

Lemma

Let $B_0 = \{\text{primes} \leq n\} \cup \left\{x : x \leq \frac{n^{\frac{2}{k+1}}}{(\log n)^2}\right\}$.

If $a \leq n$ is not in B_0^k , then

$$a = p_1 p_2 \dots p_{k+1} a',$$

where $p_1 \geq p_2 \geq \dots \geq p_{k+1}$ are primes such that $p_k p_{k+1} > \frac{n^{\frac{2}{k+1}}}{(\log n)^2}$.

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We need at least one edge in each $\{p_1, p_2, \dots, p_{k+1}\}$.

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$a_0 \nmid a_1 \dots a_k$ -problem

The sets $\{p_1, p_2, \dots, p_{k+1}\}$ intersect each other in at most one element.

Infinite multiplicative bases (of order k)

Raikov (1938)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2 \exists$ a MB of order k such that $\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < \infty$.

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Theorem (P., Sándor)

B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2 \exists$ a MB of order k such that

$\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < C$.

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B is a MB of order $k \implies \limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$: For every $k \geq 2 \exists$ a MB of order k such that

$\limsup_{n \rightarrow \infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < C$.

Theorem (P., Sándor)

B is a MB of order $k \implies \liminf_{n \rightarrow \infty} \frac{|B(n)|}{\log n} > 1$.

But it can be $< 1 + \varepsilon$.

Theorem (P., Sándor)

$\forall k \geq 2 \exists A \subseteq \mathbb{Z}^+$ such that $\limsup_{n \rightarrow \infty} \frac{|A(n)| - \pi(n)}{\frac{n^{2/(k+1)}}{(\log n)^2}} > 0$.

$a_0 \nmid a_1 \dots a_k$ -problem, infinite case

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Theorem (P., Sándor)

$$\forall \varepsilon > 0 \liminf_{n \rightarrow \infty} \frac{|A(n)| - \pi(n)}{n^\varepsilon} < \infty.$$

But $\exists c > 0$ such that $\forall k \geq 2 \exists A \subseteq \mathbb{Z}^+$ such that

$$|A(n)| \geq \pi(n) + \exp \left\{ (\log n)^{1 - \frac{c\sqrt{\log k}}{\sqrt{\log \log n}}} \right\} \text{ holds for every } n \geq 10.$$

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Upper bound

Lemma: Let Q be a subset of the prime numbers satisfying $|Q(n)| \ll n^c$ for some constant $c > 0$. Then for every $\varepsilon > 0$ there exists some integer $N_0 = N_0(\varepsilon, Q)$ such that for every $n \geq N_0$ we have

$$|\{k : k \leq n \text{ and every prime divisor of } k \text{ is in } Q\}| \leq n^{c+\varepsilon}.$$

$a_0 \nmid a_1 \dots a_k$ -problem, infinite case, \liminf

Construction

Construction

- Take 3 sequences:

$$l_n = k^n$$

$$f_n \text{ satisfies the recurrence formula } f_{n+1} = \left(\frac{f_n}{2l_n k} \right)^{l_n}$$

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- Modify the set P of primes in the following way:

$P_m := \{\text{first } f_m \text{ primes after } g_m\}$

B_m : a set of $kl_m - k + 1$ -factor products from P_m such that none of them divides the product of k others

B_m can be chosen such that $|B_m| \geq \left(\frac{f_m}{2(kl_m - k + 1)}\right)^{l_m}$.

All elements of B_m are less than g_{m+1} .

$$A = \left(P \setminus \bigcup_{n=1}^{\infty} P_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$$

$a_0 \nmid a_1 \dots a_k$ -problem, infinite case, \liminf

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- Then for $g_{m+1} \leq n < g_{m+2}$ we have
$$|A(n)| \geq \pi(n) - \left(\sum_{i=1}^{m+1} |P_i| \right) + |B_m|.$$
$$\dots \implies |A(n)| > \pi(n) + \exp \left\{ (\log n)^{1 - \frac{c\sqrt{\log k}}{\sqrt{\log \log n}}} \right\}$$

Theorem (P., Sándor)

$$B \text{ is a MB of order } k \implies \liminf_{n \rightarrow \infty} \frac{\sum_{b \in B, a \leq n} \frac{1}{b}}{k \sqrt[k]{\log n}} \geq \frac{\sqrt{6}}{e\pi}.$$

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Logarithmic density

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$a_0 \nmid a_1 \dots a_k$ -problem, logarithmic density

$$\log \log n + O(1)$$