# On Multiplicative Bases and some Related Problems

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# Connection

$a_1, \ldots, a_k, b_1, \ldots, b_k$ are distinct
$a_1 a_2 \ldots a_k = b_1 b_2 \ldots b_k \implies a_1 \ldots a_k b_1 \ldots b_k = x^2$

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$x^2 = a_1 \ldots a_{2k}$ has no solution in $A \implies a_1 a_2 \ldots a_k = b_1 b_2 \ldots b_k$ has no solution in $A$

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## element $\longrightarrow$ edge

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## How to choose $g(n)$?

condition (3) $\longrightarrow g(n) = e^{c \log n / \log \log n}$

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- $ex(u, v, C_6) < 2u + v^2/2$

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**Theorem (P.)**

$$\pi(n) + \pi(n/2) + cn^{2/3}(\log n)^{-4/3} \leq \max |A| \leq \pi(n) + \pi(n/2) + cn^{2/3}\frac{\log n}{\log\log n}$$

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# $a_0 \nmid a_1 a_2 \ldots a_k$ and multiplicative bases

**Lemma**

If $B$ is a multiplicative basis of order $k$ for $\{1, 2, \ldots, n\}$, then $F_k(n) \leq |B|$.

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# Multiplicative bases (of order $k$)

## Lemma

Let $B_0 = \{\text{primes} \le n\} \cup \left\{x : x \le \frac{n^{\frac{2}{k+1}}}{(\log n)^2}\right\}$.

If $a \le n$ is not in $B_0^k$, then

$$a = p_1 p_2 \dots p_{k+1} a',$$

where $p_1 \ge p_2 \ge \dots \ge p_{k+1}$ are primes such that $p_k p_{k+1} > \frac{n^{\frac{2}{k+1}}}{(\log n)^2}$.

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## $a_0 \nmid a_1 \dots a_k$-problem

The sets $\{p_1, p_2, \dots, p_{k+1}\}$ intersect each other in at most one element.

# Infinite multiplicative bases (of order $k$)

## Raikov (1938)

$B$ is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2$ $\exists$ a MB of order $k$ such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < \infty$.

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## Theorem (P., Sándor)

$B$ is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \frac{\sqrt{6}}{e\pi}$.

$\exists C > 0$ : For every $k \geq 2$ $\exists$ a MB of order $k$ such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < C$.

# Infinite multiplicative bases (of order $k$)

## Raikov (1938)

$B$ is a MB of order $k \implies \limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} \geq \Gamma\left(\frac{1}{k}\right)^{-1}$.

For every $k \geq 2$ $\exists$ a MB of order $k$ such that $\limsup_{n\to\infty} \frac{|B(n)|}{n/(\log n)^{\frac{k-1}{k}}} < \infty$.

## Theorem (P., Sándor)

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## Theorem (P., Sándor)

$B$ is a MB of order $k \implies \liminf_{n\to\infty} \frac{|B(n)|}{\frac{n}{\log n}} > 1$.

But it can be $< 1 + \varepsilon$.

# $a_0 \nmid a_1 \dots a_k$-problem, infinite case

**Theorem (P., Sándor)**

$\forall k \geq 2 \ \exists\ A \subseteq \mathbb{Z}^+$ such that $\limsup_{n\to\infty} \frac{|A(n)|-\pi(n)}{\frac{n^{2/(k+1)}}{(\log n)^2}} > 0.$

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**Theorem (P., Sándor)**

$\forall \varepsilon > 0 \; \liminf_{n\to\infty} \frac{|A(n)| - \pi(n)}{n^{\varepsilon}} < \infty$.
But $\exists c > 0$ such that $\forall k \geq 2 \; \exists A \subseteq \mathbb{Z}^+$ such that
$|A(n)| \geq \pi(n) + \exp\left\{ (\log n)^{1 - \frac{c\sqrt{\log k}}{\sqrt{\log \log n}}} \right\}$ holds for every $n \geq 10$.

# $a_0 \nmid a_1 \dots a_k$-problem, infinite case, liminf

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## Upper bound

Lemma: Let $Q$ be a subset of the prime numbers satisfying $|Q(n)| \ll n^c$ for some constant $c > 0$. Then for every $\varepsilon > 0$ there exists some integer $N_0 = N_0(\varepsilon, Q)$ such that for every $n \geq N_0$ we have

$$|\{k : k \leq n \text{ and every prime divisor of } k \text{ is in } Q\}| \leq n^{c+\varepsilon}.$$

# $a_0 \nmid a_1 \dots a_k$-problem, infinite case, liminf

## Construction

# $a_0 \nmid a_1 \dots a_k$-problem, infinite case, liminf

## Construction

- Take 3 sequences:
  $l_n = k^n$
  $f_n$ satisfies the recurrence formula $f_{n+1} = \left(\frac{f_n}{2l_n k}\right)^{l_n}$
  $g_n$ satisfies the recurrence formula $g_{n+1} = g_n^{kl_n}$

# $a_0 \nmid a_1 \ldots a_k$-problem, infinite case, liminf

## Construction

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  $g_n$ satisfies the recurrence formula $g_{n+1} = g_n^{kl_n}$
- Modify the set $P$ of primes in the following way:
  $P_m := \{\text{first } f_m \text{ primes after } g_m\}$
  $B_m$: a set of $kl_m - k + 1$-factor products from $P_m$ such that none of them divides the product of $k$ others
  $B_m$ can be chosen such that $|B_m| \geq \left(\frac{f_m}{2(kl_m - k + 1)}\right)^{l_m}$.
  All elements of $B_m$ are less than $g_{m+1}$.
  $A = \left(P \setminus \bigcup_{n=1}^{\infty} P_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$

# $a_0 \nmid a_1 \dots a_k$-problem, infinite case, liminf

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# $a_0 \nmid a_1 \dots a_k$-problem, infinite case, liminf

## Construction

- $A = \left(P \setminus \bigcup_{n=1}^{\infty} P_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n\right)$
- Then for $g_{m+1} \leq n < g_{m+2}$ we have
  $|A(n)| \geq \pi(n) - \left(\sum_{i=1}^{m+1} |P_i|\right) + |B_m|.$
  $\dots \implies |A(n)| > \pi(n) + \exp\left\{(\log n)^{1-\frac{c\sqrt{\log k}}{\sqrt{\log\log n}}}\right\}$

# Logarithmic density

**Theorem (P., Sándor)**

$B$ is a MB of order $k \implies \liminf_{n\to\infty} \frac{\sum\limits_{b\in B, a\le n} \frac{1}{b}}{k\sqrt[k]{\log n}} \ge \frac{\sqrt{6}}{e\pi}$.

$\exists C\ \forall k \ge 2\ \exists B$ MB of order $k$ such that $\limsup_{n\to\infty} \frac{\sum\limits_{b\in B, a\le n} \frac{1}{b}}{k\sqrt[k]{\log n}} < C$.

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**$a_0 \nmid a_1 \dots a_k$-problem, logarithmic density**

$\log\log n + O(1)$