# q-Racah polynomials from scalar products of Bethe states

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**based on arXiv:2211.14727 with Pascal Baseilhac**

# Plan

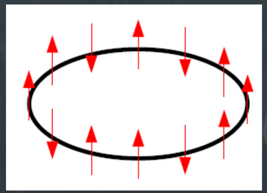
- **Bethe ansatz for XXZ: few facts**
- **Askey-Wilson algebra and reflection equation**
- **Solution of HAW operator**
- **q-Racah polynomials and scalar products**

# XXZ chain

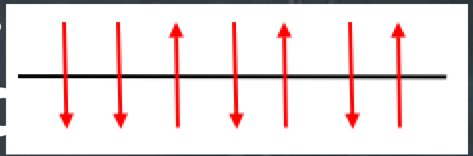
$$H = \sum_{k=1}^{N-1} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \right) + \boxed{\text{boundary conditions}}$$

$$\mathcal{H} = \otimes_{i=1}^{N} \mathbb{C}^2$$

- ☑ case Δ=1, pbc, solved long ago: Bethe 1931

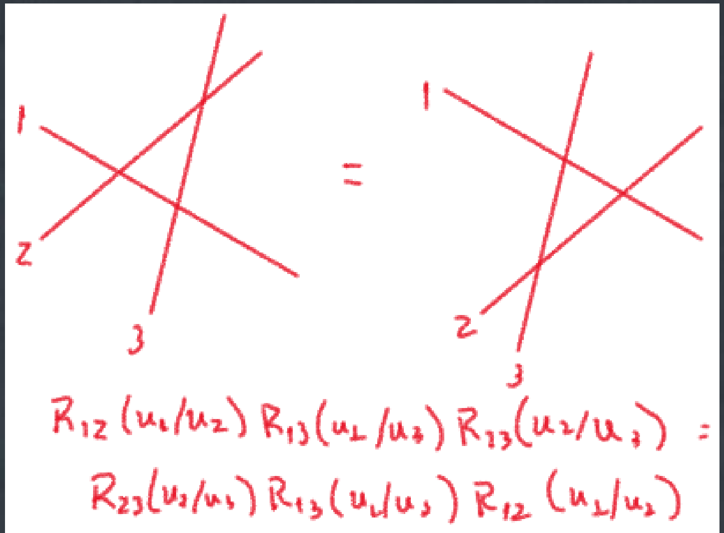


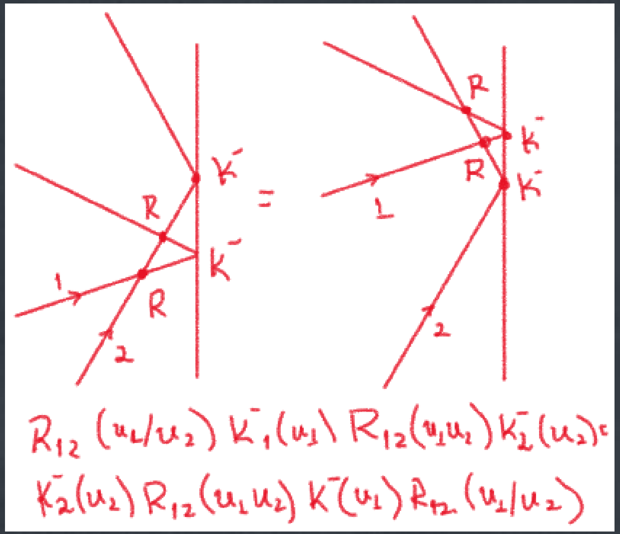
- ☑ fundamental model in Mathematical-Physics
- ☑ paralell boundary fields, solved in the 80' : Alcaraz et. al. (coordinate Bethe ansatz) and Sklyanin (reflection algebra).



# XXZ chain

- Integrability follows from the Yang-Baxter equation (bulk) and reflection equation (boundary).




- From R and K,one can build the so-called transfer matrix (polynomial of conserved charges), which can (hopefully) be diagonalized with Bethe ansatz.

# XXZ chain

- ☑ **Bethe ansatz with longitudinal fields:**

creation operator

$$\Psi = \mathcal{B} \ldots \mathcal{B}\ \Psi_0$$ → rep. theory

$$H\ \Psi = E\ \Psi$$

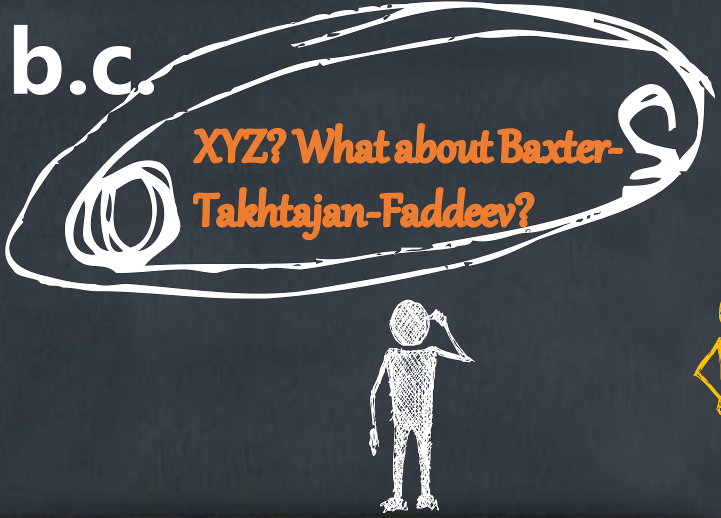
$$\Lambda\ Q = \Lambda_1\ Q^+ + \Lambda_2\ Q^-$$ → Baxter TQ equation ↔ Bethe ansatz equations

# XXZ chain

- ☑ **But for important models we cannot easily find $\Psi_0$ !**

# XXZ chain

- But for important models we cannot easily find $\Psi_0$ !
  - spin chains with generic boundaries
  - XXZ chain with anti-periodic b.c. 
  - XYZ chain with periodic b.c.




# XXZ chain

$$H = \sum_{k=1}^{N-1} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta \sigma_k^z \sigma_{k+1}^z \right) + \epsilon\, \sigma_1^z + \kappa^\pm\, \sigma_1^\pm + \nu\, \sigma_N^z + \tau^\pm\, \sigma_N^\pm$$

**Preserve integrability!**
**But $[H,S^z]\neq 0$**

- ☑ **Thanks to the breaking of the U(1) symmetry, the Bethe ansatz solution of this model remained elusive for quite a while.**

# XXZ chain

Symmetry, Integrability and Geometry: Methods and Applications — SIGMA 9 (2013), 072, 12 pages

**Heisenberg XXX Model with General Boundaries: Eigenvectors from Algebraic Bethe Ansatz**

*Samuel BELLIARD †‡ and Nicolas CRAMPÉ †‡*

☑ **Modified Bethe Ansatz** Belliard-Crampé 13, Belliard-P 15,...

modified creation operator

$$\Psi = \tilde{\mathcal{B}} \ldots \tilde{\mathcal{B}}\, \Omega$$

→ modified rep. theory

$$H\ \Psi = E\ \Psi$$

$$\Lambda\, Q = \Lambda_1\, Q^+ + \Lambda_2\, Q^- + F$$

→ Inhomogeneous TQ (Cao-Yang-Shi-Wang, 13)

# XXZ chain

## ☑ Open problems

IOP Publishing — Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **54** (2021) 344001 (15pp) — https://doi.org/10.1088/1751-8121/ac1482

**Scalar product for the XXZ spin chain with general integrable boundaries***

Samuel Belliard[1], Rodrigo A Pimenta[2,3,**] and Nikita A Slavnov[4]

- Scalar products: (Ψ,Ψ) ☑
- Form factors: (Ψ,σ,Ψ)=?
- k-points: (Ψ,σ...σ,Ψ)=?

$$\Psi = \tilde{\mathcal{B}} \dots \tilde{\mathcal{B}}\ \Omega$$

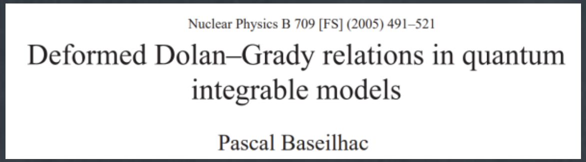
$$H\ \Psi = E\ \Psi$$

$$\Lambda\ Q = \Lambda_1\ Q^+ + \Lambda_2\ Q^- + F$$

→ Classification of Bethe roots?

# Askey-Wilson algebra

- ☑ **Modified Bethe ansatz can be used to solve the spectral problem of operators that appear in the q-Onsager framework** **Baseilhac 04**


Nuclear Physics B 709 [FS] (2005) 491–521

Deformed Dolan–Grady relations in quantum integrable models

Pascal Baseilhac

- ☑ **Here we will be interested in the Askey-Wilson algebra, which can be viewed as a certain quotient of the q-Onsager algebra.**

# Askey-Wilson algebra

$$[\mathsf{A}, [\mathsf{A}, \mathsf{A}^*]_q]_{q^{-1}} = \rho\, \mathsf{A}^* + \omega\, \mathsf{A} + \eta \mathcal{I}\ ,$$
$$[\mathsf{A}^*, [\mathsf{A}^*, \mathsf{A}]_q]_{q^{-1}} = \rho\, \mathsf{A} + \omega\, \mathsf{A}^* + \eta^* \mathcal{I}$$

Zhedanov 91

$$[X, Y]_q = qXY - q^{-1}YX$$

☑ **AW provides a solution of the RE:**

# Askey-Wilson & Reflection equation

$$R(u/v)\ (K(u)\otimes \mathbb{I})\ R(uv)\ (\mathbb{I}\otimes K(v))\ =\ (\mathbb{I}\otimes K(v))\ R(uv)\ (K(u)\otimes \mathbb{I})\ R(u/v)$$

$$R(u)=\begin{pmatrix} uq-u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u-u^{-1} & q-q^{-1} & 0 \\ 0 & q-q^{-1} & u-u^{-1} & 0 \\ 0 & 0 & 0 & uq-u^{-1}q^{-1} \end{pmatrix}$$

$$K(u)=\begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}$$

$$\mathcal{A}(u)=(u^2-u^{-2})\left(qu\,A-q^{-1}u^{-1}A^*\right)-(q+q^{-1})\rho^{-1}\left(\eta u+\eta^* u^{-1}\right)\ ,$$

$$\mathcal{D}(u)=(u^2-u^{-2})\left(qu\,A^*-q^{-1}u^{-1}A\right)-(q+q^{-1})\rho^{-1}\left(\eta^* u+\eta u^{-1}\right)\ ,$$

$$\mathcal{B}(u)=\chi(u^2-u^{-2})\left(\rho^{-1}\left([A^*,A]_q+\frac{\omega}{q-q^{-1}}\right)+\frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right)\ ,$$

$$\mathcal{C}(u)=\rho\chi^{-1}(u^2-u^{-2})\left(\rho^{-1}\left([A,A^*]_q+\frac{\omega}{q-q^{-1}}\right)+\frac{qu^2+q^{-1}u^{-2}}{q^2-q^{-2}}\right)\ .$$

# Transfer matrix

- To build the transfer matrix we consider the most general scalar solution of the dual reflection equation.

$$K^+(u) = \begin{pmatrix} qu\kappa + q^{-1}u^{-1}\kappa^* & \kappa_+(q^2u^2 - q^{-2}u^{-2}) \\ \kappa_-\rho(q^2u^2 - q^{-2}u^{-2}) & qu\kappa^* + q^{-1}u^{-1}\kappa \end{pmatrix}$$

- Transfer matrix:

$$t(u) = \mathrm{tr}\left(K^+(u)K(u)\right)$$

# Transfer matrix

$$t(u) = (q^2u^2 - q^{-2}u^{-2})(u^2 - u^{-2})\left(\kappa\,\mathsf{A} + \kappa^*\mathsf{A}^* + \kappa_+\chi^{-1}\,[\mathsf{A},\mathsf{A}^*]_q + \kappa_-\chi\,[\mathsf{A}^*,\mathsf{A}]_q\right) + \mathcal{F}_0(u)$$

Ann. Henri Poincaré 20 (2019), 3091–3112

1424-0637/19/093091-22
*published online* July 2, 2019
https://doi.org/10.1007/s00023-019-00821-3

**The Heun–Askey–Wilson Algebra and the Heun Operator of Askey–Wilson Type**

Pascal Baseilhac, Satoshi Tsujimoto, Luc Vinet and Alexei Zhedanov

**Heun-Askey-Wilson operator**

**Baseilhac-Tsujimoto-Vinet-Zhedanov 18**

- ✔ **The spectral problem of the HAW operator is the same as the spectral problem of the transfer matrix.**

- ✔ **Modified Bethe ansatz and Leonard pairs**

Nuclear Physics B 949 (2019) 114824

Diagonalization of the Heun-Askey-Wilson operator, Leonard pairs and the algebraic Bethe ansatz

Pascal Baseilhac [a,*], Rodrigo A. Pimenta [a,b,c]

# Leonard pairs

- **From the theory of Leonard pairs, we know how A and A* act on finite dim representations.**

Terwilliger+Vidunas 03

- $\{|\theta_0\rangle, |\theta_1\rangle, \ldots, |\theta_{2s}\rangle\}$ $\{|\theta^*_1\rangle, |\theta^*_1\rangle, \ldots, |\theta^*_{2s}\rangle\}$ **dim(V)=2s+1**

- $$\bar{\pi}(\mathsf{A})|\theta_M\rangle \;=\; \theta_M|\theta_M\rangle\ , \quad \bar{\pi}(\mathsf{A}^*)|\theta_M\rangle = a_{M,M+1}|\theta_{M+1}\rangle + a_{M,M}|\theta_M\rangle + a_{M,M-1}|\theta_{M-1}\rangle$$

- $$\bar{\pi}(\mathsf{A}^*)|\theta^*_M\rangle \;=\; \theta^*_M|\theta^*_M\rangle\ , \quad \bar{\pi}(\mathsf{A})|\theta^*_M\rangle = a^*_{M,M+1}|\theta^*_{M+1}\rangle + a^*_{M,M}|\theta^*_M\rangle + a^*_{M,M-1}|\theta^*_{M-1}\rangle$$

# Leonard pairs

- ☑ **Determined according to rep chose** $\{a^*_{M,M\pm1}\}, \{a^*_{M,M}\}$ $\{a_{M,M\pm1}\}, \{a_{M,M}\}$
- ☑ **Parametrization of the eigenvalues:**

$$\theta_M = bq^{2M} + cq^{-2M} \quad , \quad \theta^*_M = b^* q^{2M} + c^* q^{-2M}$$

- ☑ **Recall the transfer matrix:**

$$t(u) = a(u)\mathcal{A}(u) + b(u)\mathcal{B}(u) + c(u)\mathcal{C}(u) + d(u)\mathcal{D}(u)$$

- ☑ **Two-problems: B,C in t, and cannot find C(u)|0>=0.**

# Gauge transformation

- Manipulate the transfer matrix (Cao-Yang-Shi-Wang, 03)

$\epsilon = \pm 1$, $\alpha$, $\beta$ be generic complex parameters and $m$ be an integer

$$|X^{\epsilon}(u,m)\rangle = \begin{pmatrix} \alpha q^{\epsilon m} u^{\epsilon} \\ 1 \end{pmatrix}, \quad |Y^{\epsilon}(u,m)\rangle = \begin{pmatrix} \beta q^{-\epsilon m} u^{\epsilon} \\ 1 \end{pmatrix}$$

$$\langle \tilde{X}^{\epsilon}(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma^{\epsilon}(1, m-1)} \begin{pmatrix} -1 & \alpha q^{\epsilon m} u^{\epsilon} \end{pmatrix}, \quad \langle \tilde{Y}^{\epsilon}(u,m)| = -\epsilon \frac{q^{-\epsilon} u^{-\epsilon}}{\gamma^{\epsilon}(1, m+1)} \begin{pmatrix} 1 & -\beta q^{-\epsilon m} u^{\epsilon} \end{pmatrix}$$

$$\gamma^{\epsilon}(u,m) = \alpha^{\frac{1-\epsilon}{2}} \beta^{\frac{\epsilon+1}{2}} q^{-m} u - \alpha^{\frac{\epsilon+1}{2}} \beta^{\frac{1-\epsilon}{2}} q^{m} u^{-1}$$

# Gauge transformation

## ☑ Dynamical operators

$$\mathscr{A}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m-2)|K(u)|X^\epsilon(u^{-1},m)\rangle,$$
$$\mathscr{B}^\epsilon(u,m) = \langle \tilde{Y}^\epsilon(u,m)|K(u)|Y^\epsilon(u^{-1},m)\rangle,$$
$$\mathscr{C}^\epsilon(u,m) = \langle \tilde{X}^\epsilon(u,m)|K(u)|X^\epsilon(u^{-1},m)\rangle,$$
$$\mathscr{D}^\epsilon(u,m) = \frac{\gamma^\epsilon(1,m+1)}{\gamma^\epsilon(1,m)}\langle \tilde{X}^\epsilon(u,m+2)|K(u)|Y^\epsilon(u^{-1},m)\rangle - \frac{(q-q^{-1})\gamma^\epsilon(u^{-2},m+1)}{(qu^2-q^{-1}u^{-2})\gamma^\epsilon(1,m)}\mathscr{A}^\epsilon(u,m)$$

## ☑ Transfer matrix

$$t(u) = (q^2u^2 - q^{-2}u^{-2})\big(a(u,m)\mathscr{A}^\epsilon(u,m) + d(u,m)\mathscr{D}^\epsilon(u,m) + b(u,m)\mathscr{B}^\epsilon(u,m) + c(u,m)\mathscr{C}^\epsilon(u,m)\big)$$

# Gauge transformation

$$a(u,m) = \frac{\alpha u^{2\epsilon}\left(\kappa u + \kappa^* u^{-1}\right) + u^{\epsilon}(u^2 - u^{-2})q^{-(m+1)\epsilon}(\kappa_+ - \alpha\beta\kappa_-\rho) - \beta q^{-(2m+2)\epsilon}\left(\kappa^* u + \kappa u^{-1}\right)}{(\alpha - \beta q^{-\epsilon(2m+2)})(qu^2 - q^{-1}u^{-2})},$$

$$d(u,m) = \frac{-\beta u^{2\epsilon} q^{-(2m+1)\epsilon}\left(q\kappa u + q^{-1}\kappa^* u^{-1}\right) - u^{\epsilon} q^{-(m+1)\epsilon}\left(q^2u^2 - q^{-2}u^{-2}\right)\left(\kappa_+ - \alpha\beta\kappa_-\rho\right) + \alpha q^{-\epsilon}\left(q\kappa^* u + q^{-1}\kappa u^{-1}\right)}{(\alpha - \beta q^{-\epsilon(2m+2)})(q^2u^2 - q^{-2}u^{-2})}$$

$$b(u,m) = \frac{u^{\epsilon}\left(\alpha^2\kappa_-\rho q^{(m+2)\epsilon} - \alpha\epsilon q^{\epsilon}\kappa^{\frac{\epsilon+1}{2}}\kappa^{*\,\frac{1-\epsilon}{2}} - \kappa_+ q^{-m\epsilon}\right)}{\alpha - \beta q^{-2m\epsilon}},$$

$$c(u,m) = \frac{u^{\epsilon}\left(-\beta^2\kappa_-\rho q^{(2-3m)\epsilon} + \beta\epsilon q^{(1-2m)\epsilon}\kappa^{\frac{\epsilon+1}{2}}\kappa^{*\,\frac{1-\epsilon}{2}} + \kappa_+ q^{-m\epsilon}\right)}{\alpha - \beta q^{-2m\epsilon}}.$$

# Gauge transformation

- **Example:**

$$\mathscr{B}^-(u,m) = \frac{\beta b(u^2)q^{2m+1}}{\alpha q^{-2} - \beta q^{2m}}\Big(\frac{u\chi q^{-m-1}}{\beta\rho}[\mathsf{A}^*,\mathsf{A}]_q - \frac{\beta u q^{m-1}}{\chi}[\mathsf{A},\mathsf{A}^*]_q + \frac{(q^2u^4+1)}{q^2u}\mathsf{A} - \frac{(q^2+1)u}{q^2}\mathsf{A}^*$$

$$-\frac{q^{-m-4}}{\beta\rho\chi b(q^2)}(\rho q^2(q^2u^3+u^{-1})(\beta^2\rho q^{2m}-\chi^2) + (q^2+1)u(\omega(\beta^2\rho q^{2m+2} - q^2\chi^2) + \beta(q^4-1)\eta\chi q^m))\Big)$$

# Commutation relations: dynamical operators

## Commutation relations

$$\begin{aligned}
\mathscr{B}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= \mathscr{B}^{\epsilon}(v,m+2)\mathscr{B}^{\epsilon}(u,m), \\
\mathscr{A}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= f(u,v)\mathscr{B}^{\epsilon}(v,m)\mathscr{A}^{\epsilon}(u,m) \\
&\quad +g(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{A}^{\epsilon}(v,m)+w(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{D}^{\epsilon}(v,m) \\
\mathscr{D}^{\epsilon}(u,m+2)\mathscr{B}^{\epsilon}(v,m) &= h(u,v)\mathscr{B}^{\epsilon}(v,m)\mathscr{D}^{\epsilon}(u,m), \\
&\quad +k(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{D}^{\epsilon}(v,m)+n(u,v,m)\mathscr{B}^{\epsilon}(u,m)\mathscr{A}^{\epsilon}(v,m)
\end{aligned}$$

## Define the string of creation operators

$$\begin{aligned}
B^{\epsilon}(\bar{u},m,M) &= \mathscr{B}^{\epsilon}(u_1,m+2(M-1))\cdots\mathscr{B}^{\epsilon}(u_M,m)\,, \\
B^{\epsilon}(\{u,\bar{u}_i\},m,M) &= \mathscr{B}^{\epsilon}(u_1,m+2(M-1))\cdots\mathscr{B}^{\epsilon}(u,m+2(M-i))\ldots\mathscr{B}^{\epsilon}(u_M,m)
\end{aligned}$$

# Commutation relations: dynamical operators

## Commutation relations

$$
\begin{aligned}
\mathscr{C}^{\epsilon}(u,m-2)\mathscr{C}^{\epsilon}(v,m) &= \mathscr{C}^{\epsilon}(v,m-2)\mathscr{C}^{\epsilon}(u,m),\\
\mathscr{C}^{\epsilon}(v,m+2)\mathscr{A}^{\epsilon}(u,m+2) &= f(u,v)\mathscr{A}^{\epsilon}(u,m)\mathscr{C}^{\epsilon}(v,m+2)\\
&\quad +g(u,v,m)\mathscr{A}^{\epsilon}(v,m)\mathscr{C}^{\epsilon}(u,m+2)+w(v,u,m)\mathscr{D}^{\epsilon}(v,m)\mathscr{C}^{\epsilon}(u,m+2),\\
\mathscr{C}^{\epsilon}(v,m+2)\mathscr{D}^{\epsilon}(u,m+2) &= h(u,v)\mathscr{D}^{\epsilon}(u,m)\mathscr{C}^{\epsilon}(v,m+2)\\
&\quad +k(u,v,m)\mathscr{D}^{\epsilon}(v,m)\mathscr{C}^{\epsilon}(u,m+2)+n(u,v,m)\mathscr{A}^{\epsilon}(v,m)\mathscr{C}^{\epsilon}(u,m+2)
\end{aligned}
$$

## Define the string of creation operators

$$
C^{\epsilon}(\bar{v},m,N) = \mathscr{C}^{\epsilon}(v_1,m+2)\cdots\mathscr{C}^{\epsilon}(v_N,m+2N)
$$

$$
C^{\epsilon}(\{v,\bar{v}_i\},m,N) = \mathscr{C}^{\epsilon}(v_1,m+2)\cdots\mathscr{C}^{\epsilon}(v,m+2i)\cdots\mathscr{C}^{\epsilon}(v_N,m+2N)
$$

# Commutation relations: dynamical operators

$$f(u,v) = \frac{b(qv/u)b(uv)}{b(v/u)b(quv)}\,, \quad h(u,v) = \frac{b(q^2uv)b(qu/v)}{b(quv)b(u/v)}$$

$$b(x) = x - x^{-1}$$

# Commutation relations: dynamical operators

## Multiple actions

$$
\begin{aligned}
\mathscr{A}^{\epsilon}(u,m+2M)B^{\epsilon}(\bar{u},m,M) &= \prod_{i=1}^{M} f(u,u_i)B^{\epsilon}(\bar{u},m,M)\mathscr{A}^{\epsilon}(u,m) \\
&+ \sum_{i=1}^{M} g(u,u_i,m+2(M-1)) \prod_{j=1,j\neq i}^{M} f(u_i,u_j)B^{\epsilon}(\{u,\bar{u}_i\},m,M)\mathscr{A}^{\epsilon}(u_i,m) \\
&+ \sum_{i=1}^{M} w(u,u_i,m+2(M-1)) \prod_{j=1,j\neq i}^{M} h(u_i,u_j)B^{\epsilon}(\{u,\bar{u}_i\},m,M)\mathscr{D}^{\epsilon}(u_i,m)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}^{\epsilon}(u,m+2M)B^{\epsilon}(\bar{u},m,M) &= \prod_{i=1}^{M} h(u,u_i)B^{\epsilon}(\bar{u},m,M)\mathscr{D}^{\epsilon}(u,m) \\
&+ \sum_{i=1}^{M} k(u,u_i,m+2(M-1)) \prod_{j=1,j\neq i}^{M} h(u_i,u_j)B^{\epsilon}(\{u,\bar{u}_i\},m,M)\mathscr{D}^{\epsilon}(u_i,m) \\
&+ \sum_{i=1}^{M} n(u,u_i,m+2(M-1)) \prod_{j=1,j\neq i}^{M} f(u_i,u_j)B^{\epsilon}(\{u,\bar{u}_i\},m,M)\mathscr{A}^{\epsilon}(u_i,m)
\end{aligned}
$$

# Commutation relations: dynamical operators

## ☑ Multiple actions

$$
\begin{aligned}
C^{\epsilon}(\bar{v},m,N)\mathscr{A}^{\epsilon}(v,m+2N) &= f(v,\bar{v})\mathscr{A}^{\epsilon}(v,m)C^{\epsilon}(\bar{v},m,N) \\
&+ \sum_{i=1}^{N} g(v,v_i,m+2(N-1))f(v_i,\bar{v}_i)\mathscr{A}^{\epsilon}(v_i,m)C^{\epsilon}(\{v,\bar{v}_i\},m,N) \\
&+ \sum_{i=1}^{N} w(v,v_i,m+2(N-1))h(v_i,\bar{v}_i)\mathscr{D}^{\epsilon}(v_i,m)C^{\epsilon}(\{v,\bar{v}_i\},m,N)
\end{aligned}
$$

$$
\begin{aligned}
C^{\epsilon}(\bar{v},m,N)\mathscr{D}^{\epsilon}(v,m+2N) &= h(v,\bar{v})\mathscr{D}^{\epsilon}(v,m)C^{\epsilon}(\bar{v},m,N) \\
&+ \sum_{i=1}^{N} k(v,v_i,m+2(N-1))h(v_i,\bar{v}_i)\mathscr{D}^{\epsilon}(v_i,m)C^{\epsilon}(\{v,\bar{v}_i\},m,N) \\
&+ \sum_{i=1}^{N} n(v,v_i,m+2(N-1))f(v_i,\bar{v}_i)\mathscr{A}^{\epsilon}(v_i,m)C^{\epsilon}(\{v,\bar{v}_i\},m,N).
\end{aligned}
$$

# Reference state

**Lemma 3.1.** *If the parameter $\alpha$ is such that:*

$$(q^2 - q^{-2})\chi^{-1}\alpha \mathsf{c}^* q^{m_0} = 1 \qquad (\textit{resp.}\ (q^2 - q^{-2})\chi^{-1}\alpha \mathsf{b} q^{-m_0} = -1) \tag{3.1}$$

*then*

$$\pi(\mathscr{C}^+(u, m_0))|\theta_0^*\rangle = 0 \qquad (\textit{resp.}\ \pi(\mathscr{C}^-(u, m_0))|\theta_0\rangle = 0\ ). \tag{3.2}$$

**Lemma 3.2.** *If the parameter $\beta$ is such that:*

$$(q^2 - q^{-2})\chi^{-1}\beta \mathsf{b}^* q^{-m_0+2} = 1 \qquad (\textit{resp.}\ (q^2 - q^{-2})\chi^{-1}\beta \mathsf{c} q^{m_0-2} = -1) \tag{3.3}$$

*then*

$$\langle\theta_0^*|\pi(\mathscr{B}^+(u, m_0 - 2)) = 0 \qquad (\textit{resp.}\ \langle\theta_0|\pi(\mathscr{B}^-(u, m_0 - 2)) = 0\ ). \tag{3.4}$$

$$|\theta_0\rangle = |\Omega^-\rangle\ , \quad |\theta_0^*\rangle = |\Omega^+\rangle\ , \qquad \langle\theta_0| = \langle\Omega^-|\ , \quad \langle\theta_0^*| = \langle\Omega^+|$$

**Lemma 3.4.** *Let $\alpha, \beta$ be fixed according to Lemmas 3.1, 3.2. Then, the dynamical operators act as:*

$$\pi(\mathscr{A}^\pm(u, m_0))|\Omega^\pm\rangle = \Lambda_1^\pm(u)|\Omega^\pm\rangle \quad \textit{and} \quad \pi(\mathscr{D}^\pm(u, m_0))|\Omega^-\rangle = \Lambda_2^\pm(u)|\Omega^\pm\rangle\ , \tag{3.9}$$

$$\langle\Omega^\pm|\pi(\mathscr{A}^\pm(v, m_0)) = \langle\Omega^\pm|\Lambda_1^\pm(v) \quad \textit{and} \quad \langle\Omega^\pm|\pi(\mathscr{D}^\pm(v, m_0)) = \langle\Omega^\pm|\Lambda_2^\pm(v)\ , \tag{3.10}$$

# Bethe states – 2 families

$$|\Psi_-^M(\bar{u}, m_0)\rangle = \pi(B^-(\bar{u}, m_0, M))|\Omega^-\rangle \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha \mathsf{b} q^{-m_0} = -1 \quad \text{and} \quad \beta = 0$$

$$|\Psi_+^M(\bar{w}, m_0)\rangle = \pi(B^+(\bar{w}, m_0, M))|\Omega^+\rangle \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\alpha \mathsf{c}^* q^{m_0} = 1 \quad \text{and} \quad \beta = 0\,,$$

$$\langle\Psi_-^N(\bar{v}, m_0)| = \langle\Omega^-|\pi(C^-(\bar{v}, m_0, N)) \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta \mathsf{c} q^{m_0-2} = -1 \quad \text{and} \quad \alpha = 0$$

$$\langle\Psi_+^N(\bar{y}, m_0)| = \langle\Omega^+|\pi(C^+(\bar{y}, m_0, N)) \quad \text{for} \quad (q^2 - q^{-2})\chi^{-1}\beta \mathsf{b}^* q^{-m_0+2} = 1 \quad \text{and} \quad \alpha = 0$$

# Solution

$$\mathsf{I}(\kappa, \kappa^*, \kappa_+, \kappa_-) = \kappa\,\mathsf{A} + \kappa^* \mathsf{A}^* + \kappa_+ \chi^{-1} [\mathsf{A}, \mathsf{A}^*]_q + \kappa_- \chi\, [\mathsf{A}^*, \mathsf{A}]_q$$

- We know how the dynamical operators act on a string of creation operators. All we have to do is to express $\mathsf{A}, \mathsf{A}^*$ in terms of them.
- Special case:

$$\mathsf{I}(\kappa, 0, 0, 0) = \kappa\,\mathsf{A} \qquad \text{or} \qquad \mathsf{I}(0, \kappa^*, 0, 0) = \kappa^*\,\mathsf{A}^*$$

- Diagonal case:

$$\mathsf{I}(\kappa, \kappa^*, 0, 0) = \kappa\,\mathsf{A} + \kappa^*\,\mathsf{A}^*$$

- Generic case

# Special case

$$\mathsf{A} = \mathbb{A}^-(u,m) + \frac{(q\,u\,\bar\eta(u) + q^{-1}u^{-1}\bar\eta(u^{-1}))}{(u^2-u^{-2})(q^2u^2-q^{-2}u^{-2})}\,,$$

$$\mathsf{A}^* = \mathbb{A}^+(u,m) + \frac{(q\,u\,\bar\eta(u^{-1}) + q^{-1}u^{-1}\bar\eta(u))}{(u^2-u^{-2})(q^2u^2-q^{-2}u^{-2})} \quad \text{with} \quad \bar\eta(u) = (q+q^{-1})\rho^{-1}\left(\eta u + \eta^* u^{-1}\right)$$

$$\mathbb{A}^-(u,m) = \frac{u^{-1}}{(u^2-u^{-2})}\left(\frac{1}{(qu^2-q^{-1}u^{-2})}\mathscr{A}^-(u,m) + \frac{1}{(q^2u^2-q^{-2}u^{-2})}\mathscr{D}^-(u,m)\right)$$

$$\mathbb{A}^+(u,m) = \frac{u}{(u^2-u^{-2})}\left(\frac{1}{(qu^2-q^{-1}u^{-2})}\mathscr{A}^+(u,m) + \frac{1}{(q^2u^2-q^{-2}u^{-2})}\mathscr{D}^+(u,m)\right)$$

# Special case

**Proposition 3.1.** *Define*

$$|\Psi^M_{sp,-}(\bar{u}, m_0)\rangle = \bar{\pi}(B^-(\bar{u}, m_0, M))|\Omega^-\rangle\,. \tag{3.57}$$

*One has:*

$$\bar{\pi}\left(I(\kappa,0,0,0)\right)|\Psi^M_{sp,-}(\bar{u}, m_0)\rangle = \frac{\kappa}{2} q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu}q^{-2s+2M} + e^{\mu}q^{2s-2M}\right)|\Psi^M_{sp,-}(\bar{u}, m_0)\rangle \tag{3.58}$$

*where the set* $\bar{u}$ *satisfies the Bethe equations:*

$$\prod_{j=1, j\neq i}^{M}\left(\frac{b(u_i/(qu_j))b(u_iu_j)}{b(qu_i/u_j)b(q^2u_iu_j)}\right) = \frac{\left(qe^{\mu'}u_i + q^{-1}e^{\mu}u_i^{-1}\right)\left(qe^{-\mu}u_i + q^{-1}e^{\mu'}u_i^{-1}\right)b\left(q^{\frac{1}{2}-s}vu_i\right)b\left(q^{\frac{1}{2}-s}v^{-1}u_i\right)}{\left(e^{\mu'}u_i + e^{-\mu}u_i^{-1}\right)\left(e^{\mu}u_i + e^{\mu'}u_i^{-1}\right)b\left(q^{s+\frac{1}{2}}vu_i\right)b\left(q^{s+\frac{1}{2}}v^{-1}u_i\right)}$$

*for* $i = 1, \ldots, M$.

# Special case

**Proposition 3.2.** *Define*

(3.64) $$|\Psi^M_{sp,+}(\bar{u}, m_0)\rangle = \bar{\pi}(B^+(\bar{u}, m_0, M))|\Omega^+\rangle\,.$$

*One has:*

(3.65) $$\bar{\pi}\left(I(0,\kappa^*,0,0)\right)|\Psi^M_{sp,+}(\bar{u}, m_0)\rangle = \frac{\kappa^*}{2}q^{\frac{1}{2}(\nu+\nu')}\left(e^{-\mu'}q^{2s-2M} + e^{\mu'}q^{-2s+2M}\right)|\Psi^M_{sp,+}(\bar{u}, m_0)\rangle$$

*where the set* $\bar{u}$ *satisfies the Bethe equations:*

$$\prod_{j=1, j\neq i}^{M}\left(\frac{b(u_i/(qu_j))b(u_iu_j)}{b(qu_i/u_j)b(q^2u_iu_j)}\right) = \frac{\left(qe^{-\mu}u_i + q^{-1}e^{-\mu'}u_i^{-1}\right)\left(qe^{\mu'}u_i + q^{-1}e^{-\mu}u_i^{-1}\right)b\left(q^{\frac{1}{2}-s}vu_i\right)b\left(q^{\frac{1}{2}-s}v^{-1}u_i\right)}{\left(e^{-\mu}u_i + e^{\mu'}u_i^{-1}\right)\left(e^{-\mu'}u_i + e^{-\mu}u_i^{-1}\right)b\left(q^{s+\frac{1}{2}}vu_i\right)b\left(q^{s+\frac{1}{2}}v^{-1}u_i\right)}$$

*for* $i = 1, \ldots, M$.

# Diagonal case

$$\mathsf{I}(\kappa,\kappa^*,0,0) = \kappa\,\mathsf{A} + \kappa^*\,\mathsf{A}^*$$

$$\mathsf{A} \;=\; \mathbb{A}^-(u,m) + \frac{(q\,u\,\bar\eta(u) + q^{-1}u^{-1}\bar\eta(u^{-1}))}{(u^2-u^{-2})(q^2u^2-q^{-2}u^{-2})}$$

$$\mathbb{A}^-(u,m) = \frac{u^{-1}}{(u^2-u^{-2})}\left(\frac{1}{(qu^2-q^{-1}u^{-2})}\mathscr{A}^-(u,m) + \frac{1}{(q^2u^2-q^{-2}u^{-2})}\mathscr{D}^-(u,m)\right)$$

$$\mathsf{A}^* = \tilde{\mathbb{A}}^-(u,m) + \frac{(q\,u\,\bar\eta(u^{-1}) + q^{-1}u^{-1}\bar\eta(u))}{(u^2-u^{-2})(q^2u^2-q^{-2}u^{-2})}$$

$$\begin{aligned}\tilde{\mathbb{A}}^-(u,m) \;=\; & \frac{u^{-1}}{(u^2-u^{-2})}\left(\frac{\gamma^-(q^{-1}u^{-2},m)}{(qu^2-q^{-1}u^{-2})\gamma^-(1,m+1)}\mathscr{A}^-(u,m) + \frac{\gamma^-(qu^2,m)}{(q^2u^2-q^{-2}u^{-2})\gamma^-(1,m+1)}\mathscr{D}^-(u,m)\right.\\ & \left. + \frac{\alpha q^{-m-1}}{\gamma^-(1,m)}\mathscr{B}^-(u,m) - \frac{\beta q^{m-1}}{\gamma^-(1,m)}\mathscr{C}^-(u,m)\right)\end{aligned}$$

# Diagonal case

- ☑ **Combining the possibilities, we may write:**

$$
\begin{aligned}
\kappa\,\mathsf{A}+\kappa^*\,\mathsf{A}^* = {} & \frac{u^\epsilon\left(\alpha u^\epsilon\left(\kappa u+\kappa^* u^{-1}\right)-\beta q^{-(2m+2)\epsilon}u^{-\epsilon}\left(\kappa^* u+\kappa u^{-1}\right)\right)}{\left(u^2-u^{-2}\right)\left(qu^2-q^{-1}u^{-2}\right)\left(\alpha-\beta q^{-(2m+2)\epsilon}\right)}\mathscr{A}^\epsilon(u,m) \\
& +\frac{q^{-\epsilon}u^\epsilon\left(\alpha u^{-\epsilon}\left(q\kappa^* u+q^{-1}\kappa u^{-1}\right)-\beta q^{-2m\epsilon}u^{\epsilon}\left(q\kappa u+q^{-1}\kappa^* u^{-1}\right)\right)}{\left(u^2-u^{-2}\right)\left(q^2u^2-q^{-2}u^{-2}\right)\left(\alpha-\beta q^{-(2m+2)\epsilon}\right)}\mathscr{D}^\epsilon(u,m) \\
& -\frac{\epsilon\,\alpha q^\epsilon\kappa^{\frac{\epsilon+1}{2}}\kappa^{*\,\frac{1-\epsilon}{2}}u^\epsilon}{\left(u^2-u^{-2}\right)\left(\alpha-\beta q^{-2m\epsilon}\right)}\mathscr{B}^\epsilon(u,m)+\frac{\epsilon\,\beta\kappa^{\frac{\epsilon+1}{2}}\kappa^{*\,\frac{1-\epsilon}{2}}q^{-(2m-1)\epsilon}u^\epsilon}{\left(u^2-u^{-2}\right)\left(\alpha-\beta q^{-2m\epsilon}\right)}\mathscr{C}^\epsilon(u,m) \\
& +\frac{\left(q+q^{-1}\right)^2}{\rho\left(u^2-u^{-2}\right)\left(q^2u^2-q^{-2}u^{-2}\right)}\left(\eta\kappa^*+\eta^*\kappa+\left(\eta\kappa+\eta^*\kappa^*\right)\left(\frac{qu^2+q^{-1}u^{-2}}{q+q^{-1}}\right)\right)
\end{aligned}
$$

- ☑ **Using the gauge freedom, we set $\beta=0$ .**

# Diagonal case

- **Bethe vector:**

$$
\begin{aligned}
|\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle &= \bar{\pi}(B^{\epsilon}(\bar{u}, m_0, 2s))|\Omega^{\epsilon}\rangle\,, \\
|\Psi^{2s}_{d,\epsilon}(\{u, \bar{u}_i\}, m_0)\rangle &= \bar{\pi}(B^{\epsilon}(\{u, \bar{u}_i\}, m_0, 2s))|\Omega^{\epsilon}\rangle
\end{aligned}
$$

# Diagonal case

- **Bethe vector:**

**Lemma 3.5.** *For $M = 2s$ and generic $\{u, u_i\}$, one has:*

(3.70)
$$\bar{\pi}(\mathscr{B}^{\epsilon}(u, m_0 + 4s))|\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle =$$
$$\delta_d \frac{u^{-\epsilon} b(u^2) \prod_{k=0}^{2s} b(q^{1/2+k-s} v u) b(q^{1/2+k-s} v^{-1} u)}{\prod_{i=1}^{2s} b(u u_i^{-1}) b(q^{-1} u^{-1} u_i^{-1})} |\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle$$
$$-\delta_d \sum_{i=1}^{2s} \frac{u_i^{-\epsilon} b(u_i^2) \prod_{k=0}^{2s} b(q^{1/2+k-s} v u_i) b(q^{1/2+k-s} v^{-1} u_i)}{b(u u_i^{-1}) b(q^{-1} u^{-1} u_i^{-1}) \prod_{j=1, j\neq i}^{2s} b(u_i u_j^{-1}) b(q^{-1} u_i^{-1} u_j^{-1})} |\Psi^{2s}_{d,\epsilon}(\{u, \bar{u}_i\}, m_0)\rangle$$

*where we denote*

(3.71)
$$\delta_d = -\frac{\epsilon(-1)^{2s+1}}{2} e^{-\mu(1-\epsilon)/2 - \mu'(1+\epsilon)/2} q^{(\nu+\nu')/2 - \epsilon(2s+2)} .$$

# Diagonal case

## ☑ Solution

**Proposition 3.3.** *For $\epsilon = \pm 1$, one has:*

$$\bar{\pi}\left(I(\kappa, \kappa^*, 0, 0)\right)|\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle = \Lambda^{2s}_{d,\epsilon}|\Psi^{2s}_{d,\epsilon}(\bar{u}, m_0)\rangle \tag{3.72}$$

*with*

$$\Lambda^{2s}_{d,+} = \kappa^*\theta^*_{2s} + \kappa e^{\mu-\mu'} b\Big((v^2 + v^{-2})[2s]_q + 2e^{\mu'}\cosh(\mu) - q\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2})\Big), \tag{3.73}$$

$$\Lambda^{2s}_{d,-} = \kappa\theta_{2s} + \kappa^* e^{\mu'-\mu} c^*\Big((v^2 + v^{-2})[2s]_q + 2e^{\mu}\cosh(\mu') - q^{-1}\sum_{j=1}^{2s}(qu_j^2 + q^{-1}u_j^{-2})\Big) \tag{3.74}$$

*where the set $\bar{u}$ satisfies the (inhomogeneous) Bethe equations:*

$$\frac{b(u_i^2)}{b(qu_i^2)}(\kappa\, u_i + \kappa^* u_i^{-1})\prod_{j=1, j\neq i}^{2s} f(u_i, u_j)\Lambda_1^{\epsilon}(u_i) - q^{-\epsilon}u_i^{-2\epsilon}(q\kappa^* u_i + q^{-1}\kappa u_i^{-1})\prod_{j=1, j\neq i}^{2s} h(u_i, u_j)\Lambda_2^{\epsilon}(u_i)$$
$$+(-1)^{2s}\ \epsilon(q - q^{-1})^{-1} q^{\epsilon}\kappa^{(1+\epsilon)/2}\kappa^{*(1-\epsilon)/2}\delta_d \frac{u_i^{-2\epsilon} b(u_i^2)\prod_{k=0}^{2s} b(q^{1/2+k-s}vu_i)b(q^{1/2+k-s}v^{-1}u_i)}{\prod_{j=1, j\neq i}^{2s} b(u_i u_j^{-1})b(qu_iu_j)} = 0 \tag{3.75}$$

*for $i = 1, \ldots, 2s$.*

# Generic case

$$
\begin{aligned}
[\mathsf{A}^*, \mathsf{A}]_q = & -\frac{\alpha\beta\rho\chi^{-1}q^{-\epsilon(m+1)}u^{\epsilon}}{\alpha - q^{-2\epsilon(m+1)}\beta}\left(\frac{1}{qu^2 - q^{-1}u^{-2}}\mathscr{A}^{\epsilon}(u,m) - \frac{1}{u^2 - u^{-2}}\mathscr{D}^{\epsilon}(u,m)\right) \\
& + \frac{\rho\chi^{-1}u^{\epsilon}}{(\alpha - q^{-2\epsilon m}\beta)(u^2 - u^{-2})}\left(\alpha^2 q^{\epsilon(m+2)}\mathscr{B}^{\epsilon}(u,m) - \beta^2 q^{\epsilon(-3m+2)}\mathscr{C}^{\epsilon}(u,m)\right) \\
& - \left(\rho\frac{qu^2 + q^{-1}u^{-2}}{q^2 - q^{-2}} + \frac{\omega}{q - q^{-1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
[\mathsf{A}, \mathsf{A}^*]_q = & \frac{\chi q^{-\epsilon(m+1)}u^{\epsilon}}{\alpha - q^{-2\epsilon(m+1)}\beta}\left(\frac{1}{qu^2 - q^{-1}u^{-2}}\mathscr{A}^{\epsilon}(u,m) - \frac{1}{u^2 - u^{-2}}\mathscr{D}^{\epsilon}(u,m)\right) \\
& - \frac{\chi e^{-m\epsilon}u^{\epsilon}}{(\alpha - q^{-2\epsilon m}\beta)(u^2 - u^{-2})}\left(\mathscr{B}^{\epsilon}(u,m) - \mathscr{C}^{\epsilon}(u,m)\right) \\
& - \left(\rho\frac{qu^2 + q^{-1}u^{-2}}{q^2 - q^{-2}} + \frac{\omega}{q - q^{-1}}\right).
\end{aligned}
$$

# Leonard pairs from Bethe states

- We have seen, for example, that:

$$\pi(\mathsf{A})|\Psi_-^M(\bar{u}, m_0)\rangle \quad = \quad \theta_M|\Psi_-^M(\bar{u}, m_0)\rangle$$

- As the spectrum of A is non-degenerate, if there is a solution of the BAE associated with $\theta_M$, it follows that $|\Psi_-^M(\bar{u}, m_0)\rangle$ and $|\theta_M\rangle$ must be proportional to each other.
- The proportionality factor can be computed recalling that the B-operator is expressed in terms of A,A* and q[]'s.

# Leonard pairs from Bethe states

**Hypothesis 1.** *For each integer $M$ (resp. $N$) with $0 \leq M, N \leq 2s$, there exists at least one set of non trivial admissible Bethe roots $S_-^{M(h)} = \{u_1, ..., u_M\}$ (resp. $S_+^{*N(h)} = \{w_1, ..., w_N\}$) such that*

(3.18) $$E_-^M(u_i, \bar{u}_i) = 0 \quad \text{for} \quad \bar{u} = S_-^{M(h)} , \qquad (\text{resp.} \quad E_+^N(w_i, \bar{w}_i) = 0 \quad \text{for} \quad \bar{w} = S_+^{*N(h)}) .$$

**Lemma 3.5.** *Assume Hypothesis 1. The following relations hold:*

(3.19) $$|\theta_M\rangle = \mathcal{N}_M(\bar{u})|\Psi_-^M(\bar{u}, m_0)\rangle \quad \text{for} \quad \bar{u} = S_-^{M(h)} ,$$

(3.20) $$|\theta_N^*\rangle = \mathcal{N}_N^*(\bar{w})|\Psi_+^N(\bar{w}, m_0)\rangle \quad \text{for} \quad \bar{w} = S_+^{*N(h)}$$

*with*

(3.21) $$\mathcal{N}_M(\bar{u}) = \prod_{k=1}^{M} \left(q u_k b(u_k^2) A^*_{k,k-1}\right)^{-1} , \qquad \mathcal{N}_N^*(\bar{w}) = \prod_{k=1}^{N} \left(-q^{-1} w_k^{-1} b(w_k^2) A_{k,k-1}\right)^{-1} ,$$

*and* $\mathcal{N}_0(.) = \mathcal{N}_0^*(.) = 1$.

**Lemma 3.6.** *Assume Hypothesis 1. The following relations hold:*

(3.23) $$\langle\theta_M| = \tilde{\mathcal{N}}_M(\bar{v})\langle\Psi_-^M(\bar{v}, m_0)| \quad \text{for} \quad \bar{v} = S_-^{M(h)} ,$$

(3.24) $$\langle\theta_N^*| = \tilde{\mathcal{N}}_N^*(\bar{y})\langle\Psi_+^N(\bar{y}, m_0)| \quad \text{for} \quad \bar{y} = S_+^{*N(h)}$$

*with*

(3.25) $$\tilde{\mathcal{N}}_M(\bar{v}) = \prod_{k=1}^{M} \left(q^{-1} v_k b(v_k^2) \tilde{A}^*_{k,k-1}\right)^{-1} , \qquad \tilde{\mathcal{N}}_N^*(\bar{y}) = \prod_{k=1}^{N} \left(-q y_k^{-1} b(y_k^2) \tilde{A}_{k,k-1}\right)^{-1}$$

*and* $\tilde{\mathcal{N}}_0(.) = \tilde{\mathcal{N}}_0^*(.) = 1$.

# Leonard pairs from Bethe states - inhomogeneous

**Lemma 3.7.** *Assume Hypothesis 2. The following relations hold:*

$$|\theta_M\rangle = \mathcal{N}_M^{(i)}(\bar{u}')|\Psi_+^{2s}(\bar{u}', m_0)\rangle \quad \textit{for} \quad \bar{u}' = S_+^{M(i)} , \tag{3.31}$$

$$|\theta_N^*\rangle = \mathcal{N}_N^{*(i)}(\bar{w}')|\Psi_-^{2s}(\bar{w}', m_0)\rangle \quad \textit{for} \quad \bar{w}' = S_-^{*N(i)} \tag{3.32}$$

*with*

$$\mathcal{N}_M^{(i)}(\bar{u}') = \mathcal{N}_{2s}^*(\bar{u}')(P^{-1})_{2s,M} , \qquad \mathcal{N}_N^{*(i)}(\bar{w}') = \mathcal{N}_{2s}(\bar{w}')P_{2s,N} . \tag{3.33}$$

**Lemma 3.8.** *Assume Hypothesis 3. The following relations hold:*

$$\langle\theta_M| = \tilde{\mathcal{N}}_M^{(i)}(\bar{v}')\langle\Psi_+^{2s}(\bar{v}', m_0)| \quad \textit{for} \quad \bar{v}' = dS_+^{M(i)} , \tag{3.42}$$

$$\langle\theta_N^*| = \tilde{\mathcal{N}}_N^{*(i)}(\bar{y}')\langle\Psi_-^{2s}(\bar{y}', m_0)| \quad \textit{for} \quad \bar{y}' = dS_-^{*N(i)} \tag{3.43}$$

*with*

$$\tilde{\mathcal{N}}_M^{(i)}(\bar{v}') = \tilde{\mathcal{N}}_{2s}^*(\bar{v}')P_{M,2s}\frac{\xi_M}{\xi_{2s}^*} , \qquad \tilde{\mathcal{N}}_N^{*(i)}(\bar{y}') = \tilde{\mathcal{N}}_{2s}(\bar{y}')(P^{-1})_{N,2s}\frac{\xi_N^*}{\xi_{2s}} . \tag{3.44}$$

# Leonard pairs from Bethe states

- Given a Leonard pair, the transition matrix between two eigenbasis is given by, **Zhedanov 91 + Terwilliger 04**

$$|\theta^*_N\rangle = \sum_{M=0}^{2s} P_{MN}|\theta_M\rangle \qquad \text{and} \qquad |\theta_M\rangle = \sum_{N=0}^{2s} (P^{-1})_{NM}|\theta^*_N\rangle$$

$$\langle\theta^*_N| = \sum_{M=0}^{2s} \frac{\xi^*_N}{\xi_M} P^{-1}_{NM}\langle\theta_M| \qquad \text{and} \qquad \langle\theta_M| = \sum_{N=0}^{2s} \frac{\xi_M}{\xi^*_N} P_{MN}\langle\theta^*_N|$$

$$P_{MN} = \langle\theta_M|\theta^*_N\rangle/\langle\theta_M|\theta_M\rangle \qquad \text{and} \qquad (P^{-1})_{NM} = \langle\theta^*_N|\theta_M\rangle/\langle\theta^*_N|\theta^*_N\rangle$$

$$R_M(\theta^*_N) = {}_4\phi_3\left[\begin{matrix} q^{-2M}, \frac{\mathsf{b}}{\mathsf{c}}q^{2M}, q^{-2N}, \frac{\mathsf{b}^*}{\mathsf{c}^*}q^{2N} \\ -\frac{\mathsf{b}}{\mathsf{c}^*}q^{2s+1}\zeta^2, -\frac{\mathsf{b}^*}{\mathsf{c}}q^{2s+1}\zeta^{-2}, q^{-4s} \end{matrix}; q^2, q^2\right]$$

$$R_M(\theta^*_N) = \frac{\langle\theta_M|\theta^*_N\rangle}{\langle\theta_0|\theta^*_N\rangle}\frac{\langle\theta_0|\theta_0\rangle}{\langle\theta_M|\theta_M\rangle} = \frac{\langle\theta^*_N|\theta_M\rangle}{\langle\theta^*_0|\theta_M\rangle}\frac{\langle\theta^*_0|\theta^*_0\rangle}{\langle\theta^*_N|\theta^*_N\rangle}$$

$$P_{MN} = k_N R_M(\theta^*_N) \qquad \text{and} \qquad (P^{-1})_{NM} = \nu_0^{-1}k^*_M R_M(\theta^*_N)$$

# Leonard pairs from Bethe states

- It follows:

$$R_M(\theta_N^*) = \mathcal{N}_M^{(i)}(\bar{u})^{-1} \frac{\langle \Psi_-^M(\bar{v}, m_0) | \Psi_-^{2s}(\bar{w}, m_0) \rangle}{\langle \Omega^- | \Psi_-^{2s}(\bar{w}, m_0) \rangle} \frac{\langle \Omega^- | \Omega^- \rangle}{\langle \Psi_-^M(\bar{v}, m_0) | \Psi_+^{2s}(\bar{u}, m_0) \rangle}$$

$$R_M(\theta_N^*) = \tilde{\mathcal{N}}_N^*(\bar{y}')^{-1} \frac{\langle \Psi_-^{2s}(\bar{y}', m_0) | \Psi_+^{2s}(\bar{u}, m_0) \rangle}{\langle \Omega^+ | \Psi_+^{2s}(\bar{u}, m_0) \rangle} \frac{\langle \Omega^+ | \Omega^+ \rangle}{\langle \Psi_-^{2s}(\bar{y}', m_0) | \Psi_-^{2s}(\bar{y}, m_0) \rangle}$$

- Intriguing connection between orthogonal polynomials and integrable systems!

# Ongoing work and perspectives

- ✔ **Can we express these quantities in a determinant form? This is expected from integrable systems:**

JHEP

**Why scalar products in the algebraic Bethe ansatz have determinant representation**

S. Belliard[a] and N.A. Slavnov[b]

# Ongoing work and perspectives

- ☑ **Generalization to q-Onsager using tridiagonal pairs?**

$$[A,[A,[A,A^*]_q]_{q^{-1}}] = \rho[A,A^*]$$
$$[A^*,[A^*,[A^*,A]_q]_{q^{-1}}] = \rho[A^*,A]$$

Terwilliger 99 + Baseilhac 04

  - **Bethe states can be built! (spin-s XXZ)**

  - **Ratios of scalar products of Bethe states are multivariable analogs of q-Racah?**

# Ongoing work and perspectives

- ☑ **Play with homogeneous/inhomogeneous TQ.**
  - **Homogeneous Q: Askey-Wilson polynomial**
  - **New difference equations?**
  - **What is the inhomogeneous Q-polynomial?**
  - **New families of polynomials?**

# Ongoing work and perspectives

☑ **Applications to free fermions?**

SciPost Physics Submission

**Computation of entanglement entropy in inhomogeneous free fermions chains by algebraic Bethe ansatz**

Pierre-Antoine Bernard[1⋆], Gauvain Carcone[1], Nicolas Crampé[2] and Luc vinet[1,3]

Thank you!

Merci!

# Baxter TQ equation

- **Symmetry of Bethe equations:**

$$u_i \longleftrightarrow \pm q^{-1} u_i^{-1} \ , \quad u_i \longleftrightarrow -u_i$$
$$u_j \longleftrightarrow \pm q^{-1} u_j^{-1} \ , \quad u_j \longleftrightarrow -u_j$$

- **Nice to use big-U :**

$$U_i = \frac{q u_i^2 + q^{-1} u_i^{-2}}{q + q^{-1}} \quad \text{with} \quad i = 1, ..., M,$$

# Baxter TQ equation

- Consider again the special case. Rewrite the solution in terms of a TQ equation:

**Proposition 4.1.** *The eigenvalues* $\Lambda^{*M}_{sp,+}$ *of the Heun-Askey-Wilson operator* $\bar{\pi}(I(0,\kappa^*,0,0))$ *are given by the homogeneous Baxter T-Q relation*

$$((u^2-u^{-2})(q^2u^2-q^{-2}u^{-2}))\,\Lambda^{*M}_{sp,+}Q_M(U) = \kappa^* u\Lambda_2^+(u)T_+Q_M(U)+\kappa^* u\Lambda_1^+(u)\frac{(q^2u^2-q^{-2}u^{-2})}{(qu^2-q^{-1}u^{-2})}T_-Q_M(U)$$
$$+\kappa^*\frac{(q+q^{-1})^2}{\rho}(\eta+\eta^* U)Q_M(U)$$

*with* (3.33), (3.35), (3.36).

$$Q_M(U)=\prod_{j=1}^{M}(U-U_j)$$

$$T_\pm(f(u^2)) = f(q^{\pm 2}u^2)$$

# Baxter TQ equation

- For the diagonal case, we have an inhomogeneous term:

**Proposition 4.3.** *The eigenvalues* $\Lambda^{2s}_{d,+}$ *of the Heun-Askey-Wilson operator* $\overline{\pi}(I(\kappa,\kappa^*,0,0))$ *are given by the inhomogeneous Baxter T-Q relation*

$$((u^2-u^{-2})(q^2u^2-q^{-2}u^{-2}))\,\Lambda^{2s}_{d,+}Q_{2s}(U)= \tag{4.7}$$

$$=u\Delta_d(q^{-1}u^{-1})\Lambda_2^+(u)T_+Q_{2s}(U)+u\Delta_d(u)\Lambda_1^+(u)\frac{(q^2u^2-q^{-2}u^{-2})}{(qu^2-q^{-1}u^{-2})}T_-Q_{2s}(U)$$

$$+\frac{(q+q^{-1})^2}{\rho}(\kappa\eta^*+\kappa^*\eta+(\kappa\eta+\kappa^*\eta^*)U)Q_{2s}(U)\;+\kappa q\delta_d(-1)^{2s+1}\frac{(U^2-1)}{(q+q^{-1})^{2s-2}}H(U)$$

*with (3.33), (3.35), (3.36), (3.71), (3.76) and (D.5).*

$$\prod_{k=0}^{2s} b(q^{1/2+k-s}vu)b(q^{1/2+k-s}v^{-1}u)=H(U)$$

# Baxter TQ equation

- ✓ By using a realization of the AW algebra in terms of q-difference operators, one can identify the Baxter Q-polynomial with the Askey-Wilson polynomial:

**Proposition 4.5.** *For the special case $\kappa = \kappa_\pm = 0$, the Q-polynomial (4.2) of Proposition 4.1 is given by*

$$Q_M(Z) = \frac{(\mathfrak{ab}; q^2)_M (\mathfrak{ac}; q^2)_M (\mathfrak{ad}; q^2)_M (\mathfrak{abcd}q^{-2}; q^2)_M}{(q+q^{-1})^M \mathfrak{a}^M (\mathfrak{abcd}q^{-2}; q^4)_M (\mathfrak{abcd}; q^4)_M} P_M(z+z^{-1}; \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \tag{4.24}$$

*with*

$$\mathfrak{a} = -qe^{-\mu+\mu'}\ , \quad \mathfrak{b} = -qe^{\mu+\mu'}\ , \quad \mathfrak{c} = q^{-2s}v^2\ , \quad \mathfrak{d} = q^{-2s}v^{-2}\ . \tag{4.25}$$

- ✓ rare example of explicit solution of the TQ eq.