Nonlocal particle approximation via Morse potential for the one-dimensional Porous Medium Equation

Marco Di Francesco

University of L'Aquila

Work in collaboration with V. Iorio (L'Aquila) and M. Schmidtchen (TU Dresden)

Gradient Flows Face-To-Face 2023 - Univ. Lyon 1

Table of contents

1. The motivating problem: approximating the porous medium equation via nonlocal interacting particles

2. Main results, key estimates and facts

Table of contents

1 The motivating problem: approximating the porous medium equation via nonlocal interacting particles

2 Main results, key estimates and facts

Quadratic porous medium equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla \rho)\,, \qquad \rho = \rho(x,t) \geq 0\,, \quad x \in \mathbb{R}^d\,,\ t \geq 0\,. \qquad \text{(PME)}$$

Quadratic porous medium equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla \rho)\,, \qquad \rho = \rho(x,t) \geq 0\,, \quad x \in \mathbb{R}^d\,,\ t \geq 0\,. \tag{PME}$$

Special case of

$$\partial_t \rho = \frac{1}{m}\Delta(\rho^m) \quad \text{with } m = 2\,.$$

Quadratic porous medium equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla \rho)\,, \qquad \rho = \rho(x,t) \geq 0\,, \quad x \in \mathbb{R}^d\,,\ t \geq 0\,. \tag{PME}$$

Special case of

$$\partial_t \rho = \frac{1}{m}\Delta(\rho^m) \quad \text{with } m = 2\,.$$

Main applications:

- Filtration in a porous medium.
- Nonlinear heat transfer. Radiation in plasmas.
- Population dynamics, localised repulsive drift.

Quadratic porous medium equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla \rho)\,, \qquad \rho = \rho(x,t) \geq 0\,, \quad x \in \mathbb{R}^d\,,\ t \geq 0\,. \tag{PME}$$

Special case of

$$\partial_t \rho = \frac{1}{m} \Delta(\rho^m) \quad \text{with } m = 2\,.$$

Main applications:

- Filtration in a porous medium.
- Nonlinear heat transfer. Radiation in plasmas.
- Population dynamics, localised repulsive drift.

Selected properties:

- Finite speed of propagation of the support.
- Well posed in L^1_+.
- L^1-L^∞ smoothing effect, consequence of the Aronson - Bénilan estimate (1979).
- Measure initial trace, Aronson-Caffarelli 1983.
- Main ref: the book by J. L. Vázquez (2007).

Nonlocal approximation

Think of (PME) as a continuity equation for a density ρ (e. g. of a population)

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0$$
$$v = -\nabla \rho$$

Drift is opposite to $\nabla \rho$: *local repulsive* movement.

Nonlocal approximation

Think of (PME) as a continuity equation for a density ρ (e. g. of a population)

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0$$
$$v = -\nabla \rho$$

Drift is opposite to $\nabla \rho$: *local repulsive* movement.

Replace v by

$$v = -\nabla W_\varepsilon * \rho$$

$$W_\varepsilon(x) = \varepsilon^{-1} W(\varepsilon^{-1} x)\,, \qquad W(x) \geq 0\,,\ \int_{\mathbb{R}^d} W(x)dx = 1\,.$$

(PME) become the *nonlocal repulsive equation*

$$\partial_t \rho = \mathrm{div}\,(\rho \nabla W_\varepsilon * \rho) \tag{NL}$$

Nonlocal approximation

Think of (PME) as a continuity equation for a density ρ (e. g. of a population)

$$\partial_t \rho + \mathrm{div}\,(\rho v) = 0$$
$$v = -\nabla \rho$$

Drift is opposite to $\nabla\rho$: *local repulsive* movement.

Replace v by

$$v = -\nabla W_\varepsilon * \rho$$
$$W_\varepsilon(x) = \varepsilon^{-1} W(\varepsilon^{-1} x)\,, \qquad W(x) \geq 0\,,\ \int_{\mathbb{R}^d} W(x)dx = 1\,.$$

(PME) become the *nonlocal repulsive equation*

$$\partial_t \rho = \mathrm{div}\,(\rho \nabla W_\varepsilon * \rho) \qquad \text{(NL)}$$

Formally,

$$W_\varepsilon * \rho \rightharpoonup \rho \qquad \text{as } \varepsilon \searrow 0$$

Problem: recover (PME) from (NLR) as $\varepsilon \searrow 0$. This problem may be interpreted as a *small interaction range limit*.

Particle approximation

(NL) has a natural *deterministic N-particle* version

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla W_\varepsilon(x_i(t) - x_j(t))\,, \qquad i = 1, \ldots, N\,. \tag{DP}$$

Particle approximation

(NL) has a natural *deterministic N-particle* version

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla W_\varepsilon(x_i(t) - x_j(t)), \qquad i = 1, \ldots, N. \tag{DP}$$

How to recover (DP) from (NL):

- Take an (empirical measure) initial condition

$$\widetilde{\rho}^N(\cdot, 0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(0)}$$

Particle approximation

(NL) has a natural *deterministic N-particle* version

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla W_\varepsilon(x_i(t) - x_j(t)), \qquad i = 1, \dots, N. \tag{DP}$$

How to recover (DP) from (NL):

- Take an (empirical measure) initial condition

$$\widetilde{\rho}^N(\cdot, 0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(0)}$$

- Ansatz: look for a solution ρ to (NL) of the form

$$\widetilde{\rho}^N(\cdot, t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}$$

Particle approximation

(NL) has a natural *deterministic N-particle* version

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N}\nabla W_\varepsilon(x_i(t) - x_j(t)), \qquad i = 1, \dots, N. \tag{DP}$$

How to recover (DP) from (NL):

- Take an (empirical measure) initial condition

$$\widetilde{\rho}^N(\cdot, 0) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(0)}$$

- Ansatz: look for a solution ρ to (NL) of the form

$$\widetilde{\rho}^N(\cdot, t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i(t)}$$

- Fact: if $x_i(t)$ satisfy (DP) for all $i = 1, \dots, N$ then $\widetilde{\rho}^N$ solves (NL).

Particle approximation

(NL) has a natural *deterministic N-particle* version

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{j=1}^{N} \nabla W_\varepsilon(x_i(t) - x_j(t)), \qquad i = 1, \dots, N. \tag{DP}$$

How to recover (DP) from (NL):

- Take an (empirical measure) initial condition

$$\widetilde{\rho}^N(\cdot, 0) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(0)}$$

- Ansatz: look for a solution ρ to (NL) of the form

$$\widetilde{\rho}^N(\cdot, t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}$$

- Fact: if $x_i(t)$ satisfy (DP) for all $i = 1, \dots, N$ then $\widetilde{\rho}^N$ solves (NL).
- This is consequence of

$$-\frac{1}{N}\sum_{j=1}^{N} \nabla W_\varepsilon(x_i(t) - x_j(t)) = \nabla W_\varepsilon * \widetilde{\rho}^N(x_i(t), t)$$

A commutative diagram

Deterministic particles

$$\dot{x}_i(t) = -\frac{1}{N}\sum_{k=1}^{N} \nabla W_\varepsilon(x_i(t) - x_k(t))$$

$N \nearrow +\infty$ $\longrightarrow$

Nonlocal interaction equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla W_\varepsilon * \rho)$$

$\varepsilon \searrow 0$ $\downarrow$

$N \nearrow +\infty$

$\varepsilon \searrow 0$

Porous Medium equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla \rho)$$

Note that one can consider the *joint limit* as $\varepsilon \searrow 0$ and $N \nearrow +\infty$

Some related literature

From (DP) to (NL) as $N \to +\infty$

- Dobrušin (1979) - smooth W
- Ambrosio, Gigli, and Savaré (2008) - Carrillo, DF, Figalli, Laurent, and Slepčev (2011) - λ-convex W, Wasserstein GF theory
- Carrillo, Choi, and Hauray (2014) - Singular W

From (NL) to (PME)

- Lions and Mas-Gallic (2001) - Numerical scheme, bounded domain, periodic data
- Carrillo, Craig, and Patacchini (2019) - Sandier-Serfaty approach, smooth W
- van Meurs (2018) - Γ-convergence approach for singular potentials
- Burger and Esposito (2022) - Relaxing the regularity of W

From (DP) to (PME) - Joint limit

- Oleschläger (1990) - (2001) & Capasso et al. (2005) - Stochastic particles
- Philipowski (2007) & Figalli and Philipowski (2008) - Viscous PME
- Carrillo, Craig, and Patacchini (2019) - Blob method.

An interesting case: the 1d Morse interaction potential

$$W(x) = \frac{1}{2}e^{-|x|}, \qquad W_\varepsilon(x) = \varepsilon^{-1}W(\varepsilon^{-1}x)$$

An interesting case: the $1d$ Morse interaction potential

$$W(x) = \frac{1}{2}e^{-|x|}, \qquad W_\varepsilon(x) = \varepsilon^{-1}W(\varepsilon^{-1}x)$$

Properties:

- Repulsive potential
- $\int_{\mathbb{R}} W_\varepsilon(x)dx = 1$
- $\varepsilon^2 W_\varepsilon'' = W_\varepsilon - \delta_0$
- Lack of regularity at $x = 0$.

Previous results in the literature do not include Morse due to the repulsive singularity at the origin.

An interesting case: the 1d Morse interaction potential

$$W(x) = \frac{1}{2} e^{-|x|}, \qquad W_\varepsilon(x) = \varepsilon^{-1} W(\varepsilon^{-1} x)$$

Properties:

- Repulsive potential
- $\int_{\mathbb{R}} W_\varepsilon(x) dx = 1$
- $\varepsilon^2 W_\varepsilon'' = W_\varepsilon - \delta_0$
- Lack of regularity at $x = 0$.

Previous results in the literature do not include Morse due to the repulsive singularity at the origin.

Particles diffusion: a smoothing effect

- Smooth potential W implies $\rho(\cdot, 0) = \delta_0$ is a stationary solution.
- Morse type W implies particles *do not remain particles*.
- Evidence suggests that initial delta measures instantaneously regularise to L^∞ in (NL).

Existing results related to the repulsive Morse potential

- We observe

$$W(x) = \frac{1}{2}e^{-|x|} = N(x) + S(x)$$
$$N(x) = \frac{1}{2}(1 - |x|)\,, \qquad S''(x) = W(x) \geq 0$$

Existing results related to the repulsive Morse potential

- We observe

$$W(x) = \frac{1}{2}e^{-|x|} = N(x) + S(x)$$
$$N(x) = \frac{1}{2}(1 - |x|)\,, \qquad S''(x) = W(x) \geq 0$$

The (DP) to (NL) limit with $W(x) = |x|$ was studied in Bonaschi-Carrillo-DF-Peletier with initial data in the space of probability measures.

Existing results related to the repulsive Morse potential

- We observe

$$W(x) = \frac{1}{2}e^{-|x|} = N(x) + S(x)$$
$$N(x) = \frac{1}{2}(1 - |x|)\,, \qquad S''(x) = W(x) \geq 0$$

  The (DP) to (NL) limit with $W(x) = |x|$ was studied in Bonaschi-Carrillo-DF-Peletier with initial data in the space of probability measures.

- Carrillo-Ferreira-Precioso covered the case of $1d$ interaction potentials which are convex on $(0, +\infty)$ (which includes Morse) for the existence of gradient flow solutions.

Existing results related to the repulsive Morse potential

- We observe

$$W(x) = \frac{1}{2}e^{-|x|} = N(x) + S(x)$$
$$N(x) = \frac{1}{2}(1 - |x|)\,, \qquad S''(x) = W(x) \geq 0$$

  The (DP) to (NL) limit with $W(x) = |x|$ was studied in Bonaschi-Carrillo-DF-Peletier with initial data in the space of probability measures.

- Carrillo-Ferreira-Precioso covered the case of $1d$ interaction potentials which are convex on $(0, +\infty)$ (which includes Morse) for the existence of gradient flow solutions.

- Recent results involving system with two species, where cross-diffusion is obtained as limit of a nonlocal tissue growth mode, see David, Dębiec, Mandal, and Schmidtchen (2023) and refs. therein. A direct (NL) to (PME) result could be obtained with the techniques in David et al.

Existing results related to the repulsive Morse potential

- We observe

$$W(x) = \frac{1}{2}e^{-|x|} = N(x) + S(x)$$
$$N(x) = \frac{1}{2}(1 - |x|)\,, \qquad S''(x) = W(x) \geq 0$$

  The (DP) to (NL) limit with $W(x) = |x|$ was studied in Bonaschi-Carrillo-DF-Peletier with initial data in the space of probability measures.
- Carrillo-Ferreira-Precioso covered the case of $1d$ interaction potentials which are convex on $(0, +\infty)$ (which includes Morse) for the existence of gradient flow solutions.
- Recent results involving system with two species, where cross-diffusion is obtained as limit of a nonlocal tissue growth mode, see David, Dębiec, Mandal, and Schmidtchen (2023) and refs. therein. A direct (NL) to (PME) result could be obtained with the techniques in David et al.
- A direct (DP) to (PME) result with Morse potential seems to be missing in the literature.

Table of contents

1 The motivating problem: approximating the porous medium equation via nonlocal interacting particles

2 Main results, key estimates and facts

A modified particle scheme

Let $\overline{\rho}$ be a probability measure on $\mathbb{R}$ with finite second moment.

A modified particle scheme

Let $\overline{\rho}$ be a probability measure on $\mathbb{R}$ with finite second moment.
For a given $N \in \mathbb{N}$, let $(\overline{x}_0, \ldots, \overline{x}_N)$ be a suitable atomization of $\overline{\rho}$ satisfying

$$\overline{\rho}([\overline{x}_i, \overline{x}_{i+1})]) = \frac{1}{N} \qquad \text{for all } i = 0, \ldots, N-1$$

A modified particle scheme

Let $\overline{\rho}$ be a probability measure on $\mathbb{R}$ with finite second moment.
For a given $N \in \mathbb{N}$, let $(\overline{x}_0, \ldots, \overline{x}_N)$ be a suitable atomization of $\overline{\rho}$ satisfying

$$\overline{\rho}([\overline{x}_i, \overline{x}_{i+1})]) = \frac{1}{N} \qquad \text{for all } i = 0, \ldots, N-1$$

We replace the usual scheme

$$\dot{x}_i(t) = -\frac{1}{N} \sum_{k=0}^{N-1} W_\varepsilon'(x_i(t) - x_k(t))$$

by

$$\boxed{\dot{x}_i(t) = -\frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{W_\varepsilon(x_i(t) - x_{k+1}(t)) - W_\varepsilon(x_i(t) - x_k(t))}{x_k(t) - x_{k+1}(t)} \right)} \qquad \text{(Scheme)}$$

A modified particle scheme

Let $\overline{\rho}$ be a probability measure on $\mathbb{R}$ with finite second moment.
For a given $N \in \mathbb{N}$, let $(\overline{x}_0, \ldots, \overline{x}_N)$ be a suitable atomization of $\overline{\rho}$ satisfying

$$\overline{\rho}([\overline{x}_i, \overline{x}_{i+1})]) = \frac{1}{N} \qquad \text{for all } i = 0, \ldots, N-1$$

We replace the usual scheme

$$\dot{x}_i(t) = -\frac{1}{N} \sum_{k=0}^{N-1} W_\varepsilon'(x_i(t) - x_k(t))$$

by

$$\boxed{\dot{x}_i(t) = -\frac{1}{N} \sum_{k=0}^{N-1} \left(\frac{W_\varepsilon(x_i(t) - x_{k+1}(t)) - W_\varepsilon(x_i(t) - x_k(t))}{x_k(t) - x_{k+1}(t)} \right)} \qquad \text{(Scheme)}$$

Setting

$$d_i(t) = x_{i+1}(t) - x_i(t)\,, \qquad R_i(t) = \frac{1}{N d_i(t)}\,, \qquad \rho_\varepsilon^N(x,t) = \sum_{i=0}^{N-1} R_i(t) \mathbf{1}_{[x_i(t), x_{i+1}(t))}(x)$$

(Scheme) becomes

$$\boxed{\dot{x}_i(t) = -W_\varepsilon' * \rho_\varepsilon^N(x_i(t), t)}$$

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

$$\dot{x}_i(t) = -W'_\varepsilon * \widetilde{\rho}^N(x_i(t), t) \quad \xrightarrow{\text{replaced by}} \quad \dot{x}_i(t) = -W'_\varepsilon * \rho^N_\varepsilon(x_i(t), t)$$

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

$$\dot{x}_i(t) = -W_\varepsilon' * \widetilde{\rho}^N(x_i(t), t) \quad \overset{\text{replaced by}}{\longrightarrow} \quad \dot{x}_i(t) = -W_\varepsilon' * \rho_\varepsilon^N(x_i(t), t)$$

- Due to the singularity of W_ε we need weak L^p compactness to give sense to weak solutions in the limit (see below, weak solutions should be absolutely continuous w.r.t. Lebesgue in order to be uniquely determined). As we shall see, L^p estimate come almost for free with this scheme.

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

$$\dot{x}_i(t) = -W'_\varepsilon * \widetilde{\rho}^N(x_i(t), t) \quad \xrightarrow{\text{replaced by}} \quad \dot{x}_i(t) = -W'_\varepsilon * \rho^N_\varepsilon(x_i(t), t)$$

- Due to the singularity of W_ε we need weak L^p compactness to give sense to weak solutions in the limit (see below, weak solutions should be absolutely continuous w.r.t. Lebesgue in order to be uniquely determined). As we shall see, L^p estimate come almost for free with this scheme.
- This scheme works better with our (piecewise constant) approximated density ρ^N_ε, in that is eases the consistency in the limit of the particle scheme.

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

$$\dot{x}_i(t) = -W_\varepsilon' * \widetilde{\rho}^N(x_i(t), t) \quad \xrightarrow{\text{replaced by}} \quad \dot{x}_i(t) = -W_\varepsilon' * \rho_\varepsilon^N(x_i(t), t)$$

- Due to the singularity of W_ε we need weak L^p compactness to give sense to weak solutions in the limit (see below, weak solutions should be absolutely continuous w.r.t. Lebesgue in order to be uniquely determined). As we shall see, L^p estimate come almost for free with this scheme.
- This scheme works better with our (piecewise constant) approximated density ρ_ε^N, in that is eases the consistency in the limit of the particle scheme.
- This scheme is not totally new, see the recent paper by Radici and Stra (2022) for nonlocal interaction equations with nonlinear mobility.

Why the modified scheme?

- Recalling the empirical measure $\widetilde{\rho}^N$

$$\dot{x}_i(t) = -W'_\varepsilon * \widetilde{\rho}^N(x_i(t), t) \quad \xrightarrow{\text{replaced by}} \quad \dot{x}_i(t) = -W'_\varepsilon * \rho^N_\varepsilon(x_i(t), t)$$

- Due to the singularity of W_ε we need weak L^p compactness to give sense to weak solutions in the limit (see below, weak solutions should be absolutely continuous w.r.t. Lebesgue in order to be uniquely determined). As we shall see, L^p estimate come almost for free with this scheme.
- This scheme works better with our (piecewise constant) approximated density ρ^N_ε, in that is eases the consistency in the limit of the particle scheme.
- This scheme is not totally new, see the recent paper by Radici and Stra (2022) for nonlocal interaction equations with nonlinear mobility.
- The use of the piecewise constant approximating density is reminiscent of DF-Rosini (2015) for scalar conservation laws.

Main results (DF, Iorio, Schmidtchen - in preparation)

Theorem (Many particle limit for fixed $\varepsilon > 0$)

Let $\overline{\rho}$ be a probability measure with finite second moment. Then, for fixed $\varepsilon > 0$

$$\rho_\varepsilon^N \to \rho_\varepsilon$$

where ρ_ε is the unique weak solution to the Cauchy problem

$$\begin{cases} \partial_t \rho - \partial_x \left(\rho W_\varepsilon' * \rho \right) = 0 \\ \rho(\cdot, 0) = \overline{\rho} \end{cases}$$

Main results (DF, Iorio, Schmidtchen - in preparation)

Theorem (Many particle limit for fixed $\varepsilon > 0$)

Let $\overline{\rho}$ be a probability measure with finite second moment. Then, for fixed $\varepsilon > 0$

$$\rho_\varepsilon^N \to \rho_\varepsilon$$

where ρ_ε is the unique weak solution to the Cauchy problem

$$\begin{cases} \partial_t \rho - \partial_x \left(\rho W_\varepsilon' * \rho\right) = 0 \\ \rho(\cdot, 0) = \overline{\rho} \end{cases}$$

Theorem (Joint $\varepsilon \to 0$ / $N \to +\infty$ limit)

Let $\overline{\rho} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be nonnegative, with finite first moment and such that $\overline{\rho} \log \overline{\rho} \in L^1(\mathbb{R})$. Let $\varepsilon = \varepsilon_N$ be a sequence such that $\frac{1}{\varepsilon^3 N} \leq C$ for some $C \geq 0$. Then,

$$\rho_{\varepsilon_N}^N \to \rho_0$$

where ρ_0 is the unique weak solution to the Cauchy problem

$$\begin{cases} \partial_t \rho = \partial_x (\rho \nabla \rho) \\ \rho(\cdot, 0) = \overline{\rho} \end{cases}$$

Some comments on interaction scales

This is a *multiscale problem*:

- Interaction range $\varepsilon \to 0$
- Number of particles $N \to +\infty$

Some comments on interaction scales

This is a *multiscale problem*:

- Interaction range $\varepsilon \to 0$
- Number of particles $N \to +\infty$

Our results can be framed in the notation introduced by Oleschläger and collaborators:

- In the second Theorem, condition $\frac{1}{\varepsilon^3 N} \leq C$ can be interpreted as a *moderate interaction regime*: particles interact with those present in a small segment with length degenerating not faster than $N^{-1/3}$, which implies interaction with as many as $\mathcal{O}\left(N^{2/3}\right)$ particles, which grows to $+\infty$ as $N \to +\infty$.

Some comments on interaction scales

This is a *multiscale problem*:

- Interaction range $\varepsilon \to 0$
- Number of particles $N \to +\infty$

Our results can be framed in the notation introduced by Oleschläger and collaborators:

- In the second Theorem, condition $\frac{1}{\varepsilon^3 N} \leq C$ can be interpreted as a *moderate interaction regime*: particles interact with those present in a small segment with length degenerating not faster than $N^{-1/3}$, which implies interaction with as many as $\mathcal{O}\left(N^{2/3}\right)$ particles, which grows to $+\infty$ as $N \to +\infty$.
- The result in the first theorem is instead framed in the co called *macroscale regime* (or McKean-Vlasov regime), in which all particles keep interacting with $\mathcal{O}(N)$ particles in the $N \to +\infty$ limit.

Some comments on interaction scales

This is a *multiscale problem*:

- Interaction range $\varepsilon \to 0$
- Number of particles $N \to +\infty$

Our results can be framed in the notation introduced by Oleschläger and collaborators:

- In the second Theorem, condition $\frac{1}{\varepsilon^3 N} \leq C$ can be interpreted as a *moderate interaction regime*: particles interact with those present in a small segment with length degenerating not faster than $N^{-1/3}$, which implies interaction with as many as $\mathcal{O}\left(N^{2/3}\right)$ particles, which grows to $+\infty$ as $N \to +\infty$.
- The result in the first theorem is instead framed in the co called *macroscale regime* (or McKean-Vlasov regime), in which all particles keep interacting with $\mathcal{O}(N)$ particles in the $N \to +\infty$ limit.
- Other results consider *hydrodynamic interactions*, in which the range of interaction is of order $1/N$, e.g. nearest neighbour interaction with $\mathcal{O}(1)$ particles, see Gosse-Toscani (2006) extending the first pioneering paper by G. Russo (1990), various papers by D. Matthes and collaborators, and a recent paper by Daneri, Radici and Runa (2022).

The discrete particle setting

- Unlike the classical (DP), our (Scheme) is not a gradient flow. However, using the $W = N + S$ decomposition we can write it as Lipschitz perturbation of a gradient flow.

The discrete particle setting

- Unlike the classical (DP), our (Scheme) is not a gradient flow. However, using the $W = N + S$ decomposition we can write it as Lipschitz perturbation of a gradient flow.
- This allows to obtain an existence and uniqueness theorem with initial particles that are possibly overlapping.

The discrete particle setting

- Unlike the classical (DP), our (Scheme) is not a gradient flow. However, using the $W = N + S$ decomposition we can write it as Lipschitz perturbation of a gradient flow.
- This allows to obtain an existence and uniqueness theorem with initial particles that are possibly overlapping.
- We prove the estimate

$$d_i(t) \geq d_i(0)e^{-\frac{t}{2\varepsilon^3}} + \frac{2\varepsilon}{N}\left[1 - e^{-\frac{t}{2\varepsilon^3}}\right]$$

which shows that particles *detach instantaneously* as $t > 0$.

The discrete particle setting

- Unlike the classical (DP), our (Scheme) is not a gradient flow. However, using the $W = N + S$ decomposition we can write it as Lipschitz perturbation of a gradient flow.
- This allows to obtain an existence and uniqueness theorem with initial particles that are possibly overlapping.
- We prove the estimate

$$d_i(t) \geq d_i(0)e^{-\frac{t}{2\varepsilon^3}} + \frac{2\varepsilon}{N}\left[1 - e^{-\frac{t}{2\varepsilon^3}}\right]$$

  which shows that particles *detach instantaneously* as $t > 0$.
- Consequence: L^∞ bound for positive times

$$\|\rho_\varepsilon^N(t)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2\varepsilon}\left(1 - e^{-\frac{t}{\varepsilon^3}}\right)^{-1}$$

  Estimate is *uniform* for initial data in the space of probability measures: *measure-to-L^∞ smoothing effect*

Uniform estimates

Consider a C^1 function $\varphi : [0, +\infty) \to \mathbb{R}$ let us set

$$\psi(\rho) = \frac{\varphi(\rho)}{\rho}\,.$$

Proposition

ρ_ε^N satisfies the estimate

$$\frac{d}{dt}\int \varphi(\rho_\varepsilon^N(y,t))dy = \frac{1}{\varepsilon^2}\int \rho_\varepsilon^N(y,t)^2\psi'(\rho_\varepsilon^N(y,t))\left[W_\varepsilon * \rho_\varepsilon^N(y,t) - \rho_\varepsilon^N(y,t)\right]dy\,.$$

Uniform estimates

Consider a C^1 function $\varphi : [0,+\infty) \to \mathbb{R}$ let us set

$$\psi(\rho) = \frac{\varphi(\rho)}{\rho}\,.$$

Proposition

ρ_ε^N satisfies the estimate

$$\frac{d}{dt}\int \varphi(\rho_\varepsilon^N(y,t))dy = \frac{1}{\varepsilon^2}\int \rho_\varepsilon^N(y,t)^2\psi'(\rho_\varepsilon^N(y,t))\left[W_\varepsilon * \rho_\varepsilon^N(y,t) - \rho_\varepsilon^N(y,t)\right] dy\,.$$

Corollary

All L^p norms of ρ_ε^N for $p \in [1,+\infty]$ and the functional

$$\int_{\mathbb{R}} \rho_\varepsilon^N(y,t)\log \rho_\varepsilon^N(y,t)dy$$

are contractive in time.

We also observe that our atomisation scheme is contractive in all L^p norms.

More estimates

- While estimates are relatively simple for ρ-depending functionals, things get more complicated when we include space depending test functions in the functional.

More estimates

- While estimates are relatively simple for ρ-depending functionals, things get more complicated when we include space depending test functions in the functional.
- However, we can prove the following

Proposition

Let $\varphi \in \mathrm{Lip}(\mathbb{R})$. Then, ρ_ε^N satisfies

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(x)\rho_\varepsilon^N(x,t)dx = -\int_{\mathbb{R}} \varphi'(x)\rho_\varepsilon^N(x,t)W_\varepsilon' * \rho_\varepsilon^N(x,t)dx + [\varphi]_{\mathrm{Lip}}\mathcal{O}\left(\frac{1}{N\varepsilon^2}\right)$$

More estimates

- While estimates are relatively simple for ρ-depending functionals, things get more complicated when we include space depending test functions in the functional.
- However, we can prove the following

Proposition

Let $\varphi \in \mathrm{Lip}(\mathbb{R})$. Then, ρ_ε^N satisfies

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi(x)\rho_\varepsilon^N(x,t)dx = -\int_{\mathbb{R}} \varphi'(x)\rho_\varepsilon^N(x,t)W_\varepsilon' * \rho_\varepsilon^N(x,t)dx + [\varphi]_{\mathrm{Lip}}\mathcal{O}\left(\frac{1}{N\varepsilon^2}\right)$$

This has several consequences:

- Convergence to weak solutions of (NL) for fixed $\varepsilon > 0$.
- Uniform control of

$$\int_{\mathbb{R}} |x|\rho_\varepsilon^N(x,t)dx$$
$$\frac{1}{2}\int_{\mathbb{R}} \rho_\varepsilon^N(x,t)W_\varepsilon * \rho_\varepsilon^N(x,t)dx$$

assuming $\frac{1}{\varepsilon^3 N} \leq C$.

Strong compactness in the joint limit

- Strong compactness of $W_\varepsilon * \rho_\varepsilon^N$, achieved from
  - Uniform $L^2_{x,t}$ estimate of $W'_\varepsilon * \rho_\varepsilon^N$, using the dissipation of the (contractive) functional $\int \rho_\varepsilon^N \log \rho_\varepsilon^N dx$,
  - H^{-1} estimate of $W_\varepsilon * \partial_t \rho_\varepsilon^N$,
  - Aubin-Lions lemma.

Strong compactness in the joint limit

- Strong compactness of $W_\varepsilon * \rho_\varepsilon^N$, achieved from
  - Uniform $L^2_{x,t}$ estimate of $W_\varepsilon' * \rho_\varepsilon^N$, using the dissipation of the (contractive) functional $\int \rho_\varepsilon^N \log \rho_\varepsilon^N dx$,
  - H^{-1} estimate of $W_\varepsilon * \partial_t \rho_\varepsilon^N$,
  - Aubin-Lions lemma.
- Proving that the strong L^2 limit of $W_\varepsilon * \rho_\varepsilon^N$ concides with the weak limit ρ of ρ_ε^N.

Strong compactness in the joint limit

- Strong compactness of $W_\varepsilon * \rho_\varepsilon^N$, achieved from
  - Uniform $L^2_{x,t}$ estimate of $W'_\varepsilon * \rho_\varepsilon^N$, using the dissipation of the (contractive) functional $\int \rho_\varepsilon^N \log \rho_\varepsilon^N dx$,
  - H^{-1} estimate of $W_\varepsilon * \partial_t \rho_\varepsilon^N$,
  - Aubin-Lions lemma.
- Proving that the strong L^2 limit of $W_\varepsilon * \rho_\varepsilon^N$ concides with the weak limit ρ of ρ_ε^N.
- Proving that $W_\varepsilon * \rho_\varepsilon^N - \rho_\varepsilon^N$ is infinitesimal in L^2, using the elliptic equation satisfied by W_ε, trick is courtesy of David et al. [2023].

Uniqueness of (weak) limiting solutions

Nonlocal Interaction Equation, fixed $\varepsilon > 0$

$$\int_0^T \int_{\mathbb{R}} \left(\rho(x,t)\varphi_t(x,t) - \rho(x,t)W_\varepsilon' * \rho(x,t)\varphi_x(x,t)\right) dxdt + \int_{\mathbb{R}} \varphi(x,0)d\rho(x,0) = 0$$

- Due to the discontinuity of W_ε at $x = 0$, possibility of singular solutions.
- Equation for $F(x,t) = \int_{-\infty}^x \rho(y,t)dy$

$$F_t + \frac{1}{2\varepsilon^2}(2F-1)F_x + (S_\varepsilon' * \rho)F_x = 0$$

  Increasing shocks for F do not satisfy *Lax entropy condition*.
- To select the correct solution we use the (gradient flow) pseudo-inverse (or quantile) equation

$$\partial_t X(z,t) = -\int_0^1 W_\varepsilon'(X(z,t) - X(\zeta,t))d\zeta$$

- We prove that having the pseudo-inverse equation satisfied a.e. is equivalent to having ρ in L^∞ for positive times, which we have from the smoothing effect.

Final comments and future work

- The result is not totally satisfactory in that we would like to catch the initial *measure to* L^∞ smoothing effect in the (DP) to (PME) limit. Indeed, weak solutions to (PME) have an initial trace in the weak * measure sense (think of the Barenblatt solution).

Final comments and future work

- The result is not totally satisfactory in that we would like to catch the initial *measure to* L^∞ smoothing effect in the (DP) to (PME) limit. Indeed, weak solutions to (PME) have an initial trace in the weak * measure sense (think of the Barenblatt solution).
- However, we can catch the smoothing effect in the result with fixed ε, thus showing that the Morse potential is a good candidate to approximate (PME) with initial data in the space of probability measures.

Final comments and future work

- The result is not totally satisfactory in that we would like to catch the initial *measure to* L^∞ smoothing effect in the (DP) to (PME) limit. Indeed, weak solutions to (PME) have an initial trace in the weak * measure sense (think of the Barenblatt solution).
- However, we can catch the smoothing effect in the result with fixed ε, thus showing that the Morse potential is a good candidate to approximate (PME) with initial data in the space of probability measures.
- Another argument in favour of the Morse potential for the joint limit is that the corresponding energy functional is 0-convex in the Wasserstein sense for all ε. So, there is room for proving the $\varepsilon \to 0$ limit in the gamma-convergence framework, possibly by catching the case of measure initial conditions.

Final comments and future work

- The result is not totally satisfactory in that we would like to catch the initial *measure to* L^∞ smoothing effect in the (DP) to (PME) limit. Indeed, weak solutions to (PME) have an initial trace in the weak * measure sense (think of the Barenblatt solution).
- However, we can catch the smoothing effect in the result with fixed ε, thus showing that the Morse potential is a good candidate to approximate (PME) with initial data in the space of probability measures.
- Another argument in favour of the Morse potential for the joint limit is that the corresponding energy functional is 0-convex in the Wasserstein sense for all ε. So, there is room for proving the $\varepsilon \to 0$ limit in the gamma-convergence framework, possibly by catching the case of measure initial conditions.
- Extension to two species, in the spirit of DF-Esposito-Schmidtchen (2021) for Newtonian potentials: in progress.

End of the talk

Thank you!