Bounded weak solutions to the thin film Muskat problem

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Outline

A class of degenerate cross-diffusion systems

$$\begin{aligned} \partial_t f &= \operatorname{div}\left[f\, \nabla\left(af + bg\right)\right] && \text{in} \quad (0,\infty)\times\Omega, \\ \partial_t g &= \operatorname{div}\left[g\, \nabla\left(cf + dg\right)\right] && \text{in} \quad (0,\infty)\times\Omega, \end{aligned}$$

where

- $\Omega \subset \mathbb{R}^N$, $N \geq 1$;
- $(a, b, c, d) \in (0,\infty)^4$, $ad - bc > 0$;
- no-flux boundary conditions;
- non-negative and integrable initial conditions (f^{in}, g^{in}).

Degenerate parabolic system with full diffusion matrix

Cross-diffusion system

Thin film Muskat problem

$$\begin{aligned} \partial_t f &= \mathrm{div}\,[f\,\nabla\,(af+bg)] \quad \text{in} \quad (0,\infty)\times\Omega, \\ \partial_t g &= \mathrm{div}\,[g\,\nabla\,(cf+dg)] \quad \text{in} \quad (0,\infty)\times\Omega, \end{aligned}$$

- Reduced model (lubrication approximation) for the motion of two immiscible fluids with different densities $\rho_\pm$ ($\rho_- > \rho_+$) and viscosities $\mu_\pm$ in a porous medium ($N \in \{1,2\}$).
- $(a,b,c,d) = (1+R, R, \mu R, \mu R)$ with

$$R = \frac{\rho_+}{\rho_- - \rho_+}, \qquad \mu = \frac{\mu_-}{\mu_+}.$$

Escher, Matioc & Matioc (2012), Jazar & Monneau (2014), Woods & Mason (2000)

Interacting biological species

$$
\begin{aligned}
\partial_t f &= \operatorname{div}\left[f\,\nabla\left(af+bg\right)\right] \quad \text{in} \quad (0,\infty)\times\Omega,\\
\partial_t g &= \operatorname{div}\left[g\,\nabla\left(cf+dg\right)\right] \quad \text{in} \quad (0,\infty)\times\Omega,
\end{aligned}
$$

- Two interacting biological species for which only dispersal is taken into account. The dispersal of each species is driven by a weighted sum of the densities of the densities f and g.
- $(a,b,c,d)\in(0,\infty)^4$, $ad-bc>0$

  Bertsch, Gurtin, Hilhorst & Peletier (1985), Galiano & Selgas (2014)
- $(a,b,c,d)\in(0,\infty)^4$, $ad-bc=0$ (proportional velocity dispersal)

  Bertsch, Gurtin & Hilhorst (1987), Burger, Di Francesco, Fagioli & Stevens (2018), Carrillo, Huang & Schmidtchen (2018)

Limit case

$$\begin{aligned}\partial_t f &= \mathrm{div}\,[f\,\nabla\,(af+bg)] \quad \text{in} \quad (0,\infty)\times\Omega,\\ \partial_t g &= \mathrm{div}\,[g\,\nabla\,(cf+dg)] \quad \text{in} \quad (0,\infty)\times\Omega.\end{aligned}$$

If $a=1$ and $(b,c,d)\to 0$ (corresponding to the limit $R\to 0$ in the thin film Muskat problem), then reduction to the porous medium equation (PME)

$$\partial_t f = \mathrm{div}\,(f\,\nabla f) \quad \text{in} \quad (0,\infty)\times\Omega.$$

Two-phase generalization of the PME

Outline

1. A class of degenerate cross-diffusion systems
2. Properties
3. Bounded weak solutions

Conservative properties and energy

- $f \geq 0$ and $g \geq 0$,
- $\|f(t)\|_1 = \|f^{in}\|_1$ and $\|g(t)\|_1 = \|g^{in}\|_1$,
- Energy functional:

$$\mathcal{E}_2(f,g) := \int_\Omega \left(\frac{a}{2} f^2 + bfg + \frac{bd}{2c} g^2 \right) \, \mathrm{d}x$$
$$= \frac{ad - bc}{2} \|f\|_2^2 + \frac{b}{2cd} \|cf + dg\|_2^2$$

with

$$\frac{d}{dt}\mathcal{E}_2(f,g) = - \left\| \sqrt{f} \, \nabla \left(af + bg \right) \right\|_2^2 - \frac{b}{c} \|\sqrt{g} \, \nabla \left(cf + dg \right)\|_2^2 \leq 0.$$

Properties

- Entropy functional:

$$\mathcal{E}_1(f,g) := \|f\ln f - f + 1\|_1 + \frac{b^2}{ad}\ \|g\ln g - g + 1\|_1$$

with

$$\frac{d}{dt}\mathcal{E}_1(f,g) = -\frac{1}{a}\left\|\nabla\left(af + \frac{b(ad+bc)}{2ad}g\right)\right\|_2^2 - \frac{b^2(ad-bc)(3ad+bc)}{4a^3d^2}\|\nabla g\|_2^2 \leq 0.$$

Variational structure

Energy: $\mathcal{E}_2(f,g) := \frac{ad-bc}{2}\|f\|_2^2 + \frac{b}{2cd}\|cf+dg\|_2^2$

$$\begin{aligned} \partial_t f &= \operatorname{div}\left[f\,\nabla\left(\frac{\delta\mathcal{E}_2}{\delta f}(f,g)\right)\right] \quad \text{in} \quad (0,\infty)\times\Omega, \\ \frac{c}{b}\,\partial_t g &= \operatorname{div}\left[g\,\nabla\left(\frac{\delta\mathcal{E}_2}{\delta g}(f,g)\right)\right] \quad \text{in} \quad (0,\infty)\times\Omega, \end{aligned}$$

supplemented with no-flux boundary conditions and non-negative initial conditions $(f^{in}, g^{in}) \in L^1(\Omega, \mathbb{R}^2)$, $\|f^{in}\|_1 = \|g^{in}\|_1 = 1$.

Gradient flow of $\mathcal{E}_2$ with respect to the 2-Wasserstein distance W_2 in $\mathcal{P}_2(\Omega, \mathbb{R}^2)$

Existence

Given $(f^{in}, g^{in}) \in L^1(\Omega;\mathbb{R}^2) \cap \mathcal{P}_2(\Omega;\mathbb{R}^2)$ and

$$(a,b,c,d) \in (0,\infty)^4\,, \qquad ad - bc > 0\,,$$

there is a weak solution (f,g) satisfying

1. $(f,g) \in L^\infty(0,\infty; L^2(\Omega;\mathbb{R}^2))$, $(f,g) \in L^2(0,t; H^1(\Omega;\mathbb{R}^2))$;
2. $(f,g) \in C([0,\infty); H^{-3}(\Omega;\mathbb{R}^2))$ with $(f,g)(0) = (f^{in}, g^{in})$;
3. $\|f(t)\|_1 = \|f^{in}\|_1$ and $\|g(t)\|_1 = \|g^{in}\|_1$ for $t \ge 0$;
4. Energy and entropy inequalities.

L & Matioc (2013): $N = 1$, Aït Hammou Oulhaj, Cancès, Chainais-Hillairet & L (2019): $N = 2$

- The regularity of f and g do not ensure that the quadratic terms $f\nabla f$, $f\nabla g$, $g\nabla f$, $g\nabla g$ belong to $L^2(\Omega)$: not an H^1-weak solution. Not enough to show finite speed of propagation.
- Formal derivation of an estimate in $L^3(\Omega)$.

Other existence results: weak solutions

- $(a, b, c, d) = (1 + R, R, \mu R, \mu R)$, $N = 1$, $\Omega = (0, L)$: compactness method

  Escher, L & Matioc (2011)

- $(a, b, c, d) \in (0, \infty)^4$, $N \in \{1, 2, 3\}$, Ω bounded: compactness method when $4ad - (b + c)^2 > 0$

  Galiano & Selgas (2014)

- Strong ellipticity condition on (a, b, c, d), $N \geq 1$, $\Omega = \mathbb{T}^N$: compactness method

  Alkhayal, Issa, Jazar & Monneau (2018)

Other existence results

1. Classical solutions: $(a, b, c, d) = (1 + R, R, \mu R, \mu R)$, $N = 1$, $\Omega = (0, L)$: local well-posedness of classical solutions

   Escher, Matioc & Matioc (2012)

2. Strong solutions: $(a, b, c, d) = (1 + R, R, \mu R, \mu R)$, $N = 1$, $\Omega = \mathbb{T}$: weak solutions with components in

$$L^\infty((0, T) \times \mathbb{T}) \times L^2((0, T), W^{1,\infty}(\mathbb{T})) \cap L^1((0, T), C^{1+\alpha}(\mathbb{T}))$$

   for all $T > 0$ and $\alpha \in [0, 1/2)$, provided the initial conditions are suitably small, and conditional uniqueness

   Bruell & Granero-Belinchón (2019)

Outline

Additional estimates: Liapunov functionals

$$(a,b,c,d) \in (0,\infty)^4 , \qquad ad - bc > 0 ,$$

For each integer $n \ge 3$, there is $\Phi_n \in \mathbb{R}_n[X,Y]$ such that

- Φ_n is convex and non-negative on $(0,\infty)^2$ and

$$\mathcal{E}_n(F,G) := \int_\Omega \Phi_n(F(x),G(x))\,\mathrm{d}x \in \left[c_n\|F+G\|_n^n, C_n\|F+G\|_n^n\right]$$

  for some $0 < c_n < C_n$;
- Consider $(f^{in},g^{in}) \in L^n(\Omega;\mathbb{R}^2)$. Then (formally)

$$\mathcal{E}_n(f(t),g(t)) \le \mathcal{E}_n(f^{in},g^{in}), \qquad t \ge 0,$$

  and $\{(f(t),g(t)) \ : \ t \ge 0\}$ is bounded in $L^n(\Omega;\mathbb{R}^2)$.

Additional estimates: $n \to \infty$

$$(a,b,c,d) \in (0,\infty)^4\,, \qquad ad - bc > 0\,,$$

- There are $0 < c_\infty < C_\infty$ such that

$$c_\infty \|F+G\|_\infty \le \liminf_{n\to\infty} \mathcal{E}_n(F,G)^{1/n}$$
$$\limsup_{n\to\infty} \mathcal{E}_n(F,G)^{1/n} \le C_\infty \|F+G\|_\infty$$

  for $(F,G) \in L^\infty(\Omega;\mathbb{R}^2)$;
- Then (formally) $\{(f(t),g(t)) \ : \ t \ge 0\}$ is bounded in $L^\infty(\Omega;\mathbb{R}^2)$.

Construction of Φ_n, $n \geq 3$

Set

$$u = (f, g) \quad \text{and} \quad M(u) = \begin{pmatrix} af & bf \\ cg & dg \end{pmatrix}$$

so that

$$\partial_t u = \mathrm{div}\,(M(u)\nabla u) \quad \text{in} \quad (0,\infty)\times\Omega\,.$$

If $\Phi \in C^2([0,\infty)^2)$ is a convex function, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi(u)\,\mathrm{d}x + \sum_{i=1}^N \int_\Omega \langle D^2\Phi(u)M(u)\partial_i u, \partial_i u\rangle\,\mathrm{d}x = 0\,,$$

and we are left with looking for Φ such that the matrix $D^2\Phi(u)M(u)$ is symmetric and definite positive.

Construction of Φ_n, $n \geq 3$

Let $n \geq 3$.

$$\Phi_n(X_1, X_2) = \sum_{j=0}^{n} a_{j,n} X_1^j X_2^{n-j} ,$$

with $a_{0,n} = 1$ and, for $1 \leq j \leq n$,

$$a_{j,n} = \prod_{k=0}^{j-1} \frac{(n-k)[ak + c(n-k-1)]}{(k+1)[bk + d(n-k-1)]} = \binom{n}{j} \prod_{k=0}^{j-1} \frac{ak + c(n-k-1)}{bk + d(n-k-1)}$$

Bounded weak solutions

Let $(f^{in}, g^{in}) \in L^1(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$, $f^{in} \geq 0$, $g^{in} \geq 0$.

There exists a weak solution (f, g):

- $(f, g) \in L^\infty((0, \infty); L^1(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2))$;
- $(f, g) \in L^2(0, t; H^1(\Omega; \mathbb{R}^2))$, $t > 0$;
- $(f, g) \in C([0, \infty); H^{-1}(\Omega; \mathbb{R}^2))$ with $(f, g)(0) = (f_0, g_0)$;
- $\|f(t)\|_1 = \|f_0\|_1$ and $\|g(t)\|_1 = \|g_0\|_1$, $t \geq 0$;
- Entropy estimate;
- Let $n \in \mathbb{N}$, $n \geq 2$. Then

$$\mathcal{E}_n(f(t), g(t)) \leq \mathcal{E}_n(f^{in}, g^{in}), \qquad t \geq 0.$$

L & Matioc (2022, 2023)

Bounded weak solutions: proof

- implicit time scheme;
- approximation by truncature complying with the *a priori* estimates;
- compactness method;

Observe that all quadratic terms $f\nabla f$, $f\nabla g$, $g\nabla f$, and $g\nabla g$ now belong to $L^2((0,T)\times\Omega)$, so that (f,g) is an H^1-weak solution.