Stochastic Homogenisation of Discrete Transport Problems

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GRADIENT FLOWS FACE-TO-FACE 3

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Summary of today's talk.

(1) A general class of **dynamical transport problems** on $\mathbb{R}^d$.

(2) **Multivariate flow-based problems** on euclidean spaces.

(3) **Flow-based problems on random graphs**: discretisation of continuous problems.

(4) **Stochastic homogenisation** of transport problems on graphs.

$(1/4)$ Dynamical transport problems in $\mathbb{R}^d$

Dynamical transport problems in $\mathcal{M}_+(\mathbb{R}^d)$.

For a given measurable, lsc function $f : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we are interested in

$$C_f(\mu_0, \mu_1) := \inf_{(\mu_t, \xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t, \xi_t)\, \mathrm{d}x\, \mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla \cdot \xi_t = 0}_{\text{continuity equation}},\ \mu_{t=i} = \mu_i \right\}$$

where $\mu_0,\ \mu_1 \in \mathcal{M}_+(\mathbb{R}^d)$ are given initial and final measures, $\xi_t := \mu_t v_t$ is the **flux**.

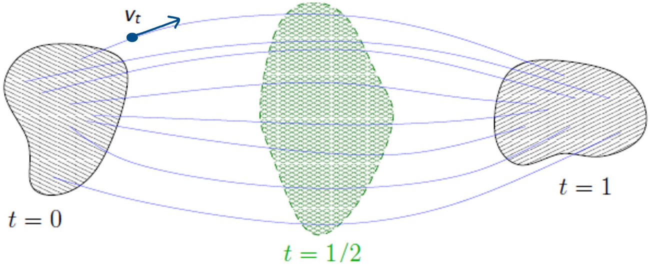

Figure: An evolution $(\mu_t)_t \subset \mathcal{M}_+(\mathbb{R}^d)$ from μ_0 to μ_1 (edited from [Villani, 2009]).

Examples of transport problems (1).

$$C_f(\mu_0,\mu_1) := \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla\cdot\xi_t = 0,\ \mu_{t=i} = \mu_i}_{(\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1)} \right\}$$

- $f(\mu,\xi) = |\xi|^2/\mu$ corresponds to the **(2)-Wasserstein distance** $\mathbb{W}_2$:

$$\mathbb{W}_2(\mu_0,\mu_1)^2 = \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\xi_t|^2}{\mu_t}\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1) \right\}$$

whose dynamical interpretation is due to [Benamou and Brenier, 2000].

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whose dynamical interpretation is due to [Benamou and Brenier, 2000].

- More general: $f(\mu,\xi) = |\xi|^p/m(\mu)^{p-1}$ for $m : \mathbb{R}^+ \to \mathbb{R}^+$ concave mobility:

$$\mathbb{W}_{p,m}(\mu_0,\mu_1)^p := \inf_{(\mu_t,\xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} \frac{|\xi_t|^p}{m(\mu_t)^{p-1}}\,\mathrm{d}x\,\mathrm{d}t \;:\; (\mu_t,\xi_t)_t \in \mathsf{CE}(\mu_0,\mu_1) \right\}$$

are **generalised (p)-Wasserstein distances** [Dolbeault, Nazaret, and Savaré, 2012] .

Examples of transport problems (2).

$$C_f(\mu_0, \mu_1) := \inf_{(\mu_t, \xi_t)_t} \left\{ \int_0^1 \int_{\mathbb{R}^d} f(\mu_t, \xi_t) \, \mathrm{d}x \, \mathrm{d}t \ : \ \underbrace{\partial_t \mu_t + \nabla \cdot \xi_t = 0, \ \mu_{t=i} = \mu_i}_{(\mu_t, \xi_t)_t \in \mathrm{CE}(\mu_0, \mu_1)} \right\}$$

- $f(\mu, \xi) = f(\xi)$ are **flow-based** problems (Beckmann problems). When f is **convex**:

$$\int_0^1 \int_{\mathbb{R}^d} f(\xi_t) \, \mathrm{d}x \, \mathrm{d}t \overset{\text{Jensen}}{\geq} \int_{\mathbb{R}^d} f\left(\underbrace{\int_0^1 \xi_t \, \mathrm{d}t}_{=:\bar{\xi}} \right) \mathrm{d}x = \int_{\mathbb{R}^d} f(\bar{\xi}) \, \mathrm{d}x,$$

In this case, one has the equivalent **static** formulation:

$$C_f(\mu_0, \mu_1) = \inf_{\bar{\xi}} \left\{ \int_{\mathbb{R}^d} f(\bar{\xi}) \, \mathrm{d}x \ : \ \nabla \cdot \bar{\xi} = \mu_0 - \mu_1 \right\}.$$

This includes $\mathbb{W}_1$ ($f(\bar{\xi}) = |\bar{\xi}|$) and negative Sobolev distance H^{-1} ($f(\bar{\xi}) = |\bar{\xi}|^2$).

Motivations.

(1) **Modeling**: optimal transport, traffic flows, congested transport, ...

(2) Application to PDEs: theory of metric **gradient flows**.

$$\partial_t \mu_t - \nabla \cdot (\mu_t \nabla (\mathsf{DE}(\mu_t))) = 0, \quad \mathsf{E} : \mathcal{M}_+(\mathbb{R}^d) \to [0, +\infty].$$

[Jordan, Kinderlehrer, and Otto, 1998]: heat flow as gradient flow of the entropy

$$\partial_t \mu_t = \Delta \mu_t, \quad \mathsf{E}(\mu) = \int_{\mathbb{R}^d} \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}x}\right) \mathrm{d}\mu.$$

(3) Surprising connections with the **Riemannian geometry** (Lott–Villani–Sturm theory).

(4) [Maas, 2011, Mielke, 2011] : generalisation of these ideas to the **discrete setting**.

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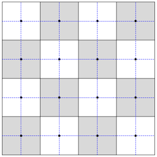
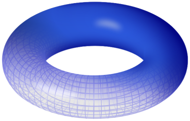
(4) [Maas, 2011, Mielke, 2011] : generalisation of these ideas to the **discrete setting**.

Discrete-to-continuum problem: the study of the convergence of (rescaled) discrete transport problems (and evolutions) towards a continuous one.

Discrete-to-continuum limits of transport problems: some literature.

(1) **First convergence result** [Gigli and Maas, 2013]: transport metrics associated to the cubic mesh on the torus $\mathbb{T}^d$ converge to $\mathbb{W}_2$ in the limit of vanishing mesh size.

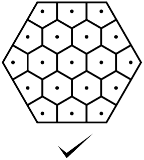


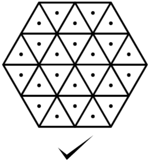


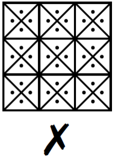
https://en.wikipedia.org/wiki/Torus

(2) **Geometric graphs on point clouds** [García Trillos, 2020]: almost sure convergence of the discrete metrics to $\mathbb{W}_2$, but diverging degree.

(3) **Finite volume partitions** $\mathcal{T}$ in $\mathbb{R}^d$ [Gladbach, Kopfer, and Maas, 2020]: convergence of $\mathcal{W}_{\mathcal{T}}$ to $\mathbb{W}_2$ as size$(\mathcal{T}) \to 0$ is essentially equivalent to an isotropy condition.







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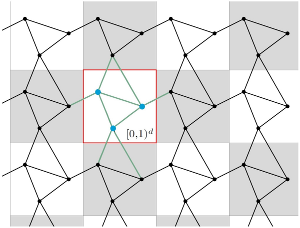
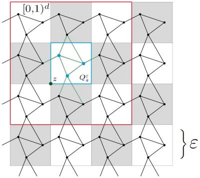
(4) **Periodic homogenisation of transport problems** [Gladbach, Kopfer, Maas, and P., 2020 & 2023]: a complete characterisation of the limit costs in a periodic setting.





(5) **Convergence of the gradient flows I**: convergence of finite-volume discretisation of diffusions [Disser and Liero, 2015], [Forkert, Maas, and P., 2020] (quadratic) ; [Hraivoronska and Tse, 2023], [Hraivoronska, Schlichting, and Tse, 2023] (cosh).

(6) **Convergence of the gradient flows II**: generalised gradient-flow structures associated to jump processes and approximation of nonlocal-interaction equations [Esposito, Patacchini, Schlichting, and Slepčev, 2021], [Esposito, Patacchini, and Schlichting, 2023b], [Esposito, Heinze, and Schlichting, 2023a].

(2/4) Multivariate flow-based problems

Vector-valued flow-based problems in $\mathbb{R}^d$.

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- We fix a bounded, open, Lipschitz domain $U \subset \mathbb{R}^d$.
- We study variational problems which are **divergence-constrained**: for $\mu \in \mathcal{M}(\overline{U}; V)$ with $\mu(\overline{U}) = 0$ (e.g. $\mu = \mu_1 - \mu_0$), we consider vector fields

$$\xi \in \mathcal{M}(\overline{U}; V \otimes \mathbb{R}^d) \quad \text{so that} \quad \nabla \cdot \xi = \mu\,.$$

- For a given measurable, lsc function $f : V \otimes \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, we define

$$\mathbb{F}_f^\mu(\xi) := \begin{cases} \displaystyle\int_{\overline{U}} f\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) \mathrm{d}\mathscr{L}^d + \int_{\overline{U}} f^\infty\left(\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}\right) \mathrm{d}|\xi|^s\,, & \text{if } \nabla \cdot \xi = \mu\,, \\ +\infty\,, & \text{otherwise,} \end{cases}$$

for $\xi \in \mathcal{M}(\overline{U}; V \otimes \mathbb{R}^d)$, where $f^\infty(\xi) = \lim_{t \to \infty} \dfrac{f(t\xi)}{t}$ is the **recess** of f.

Vector-valued flow-based problems in $\mathbb{R}^d$.

Subject of study: time-independent flow problems with target space $V = \mathbb{R}^n$.

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Our goal/result: almost sure convergence of random, discretisation of $\mathbb{F}_f^\mu$ on graphs.

(3/4) **Flow-based problems on random graphs**

Discrete flow problems on random graphs.

Discrete framework: we study flow-based problems with **random energy density** on a **random graph** under the assumption of **stationarity**. We consider:

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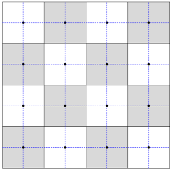
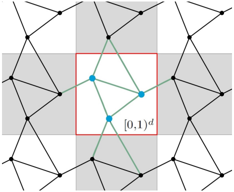
(1) a reference probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

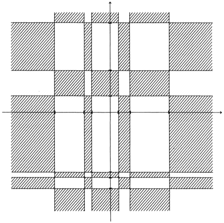
(2) a **measure-preserving** group action $\{\tau_z : \Omega \to \Omega\}_{z \in \mathbb{Z}_d}$ on Ω, i.e. $(\tau_z)_\# \mathbb{P} = \mathbb{P}$.

(3) a **stationary** random graph, meaning $\omega \in \Omega \mapsto (\mathcal{X}_\omega, \mathcal{E}_\omega)$ (vertices, edges) satisfying

$$(\mathcal{X}_{\tau_z(\omega)}, \mathcal{E}_{\tau_z(\omega)}) = (\mathcal{X}_\omega + z, \mathcal{E}_\omega + z)\,, \quad \forall z \in \mathbb{Z}_d \quad (\text{aka } \textit{periodic in law})\,.$$







Discrete flow problems on random graphs.

Discrete energies: consider a random, local, discrete energy functional

$$\omega \in \Omega \mapsto \mathsf{F}_\omega : V_a^{\mathcal{E}} \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}\,,$$

where $V_a^{\mathcal{E}} := \{J : \mathcal{E} \to V \;\;:\;\; J(x,y) = -J(y,x)\}$ (discrete flows), $\mathcal{B}(\mathbb{R}^n)$:=Borel sets.

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Example: for given random conductances $(\omega_{xy})_{x,y} \subset \mathbb{R}_+$, consider the discrete energies

$$\mathsf{F}_\omega(J,A) := \sum_{[x,y]\cap A \neq \emptyset} \omega_{xy} \|J(x,y)\|_V^p\,, \quad J \in V_a^{\mathcal{E}}\,, \quad A \in \mathcal{B}(R^n)\,, \quad p \geq 1\,.$$

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(i) We assume **locality** in the first variable and **σ-additivity** in the second, i.e.

$$\mathsf{F}_\omega(J,A) = \sum_{i=1}^{\infty} \mathsf{F}_\omega(J,A_i) \,, \quad A = \bigcup_{i\in\mathbb{N}} A_i \,, \quad \{A_i\}_{i\in\mathbb{N}} \text{ disjoint} \,, \quad J \in V_a^{\mathcal{E}} \,.$$

(ii) **Stationarity**: we assume that $\mathrm{Law}(\tau_z F) = \mathrm{Law}(F)$ for every $z \in \mathbb{Z}^d$, i.e.

$$(\tau_z \mathsf{F})_\omega = \mathsf{F}_{\tau_z \omega} \,, \quad \text{where} \quad (\tau_z \mathsf{F})_\omega(J,A) := \mathsf{F}_\omega(J(\cdot + z), A - z) \,.$$

(4/4) Stochastic homogenisation of discrete transport problems

Main result: stochastic homogenisation in the linear growth case.

We can prove the sought approximation result under **linear growth**, namely

$$\mathsf{F}_\omega(J,A) \geq c \sum_{[x,y]\cap A\neq\emptyset} \|J(x,y)\|_V - c|A|\,,$$

$$\big|\mathsf{F}_\omega(J,A) - \mathsf{F}_\omega(J',A)\big| \leq C \sum_{[x,y]\subset B_R(A)} \|J(x,y) - J'(x,y)\|_V\,.$$

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Rescaling: for $\varepsilon > 0$, we define $\mathcal{E}_\varepsilon := \varepsilon\mathcal{E}$ and rescaled energies $F_{\omega,\varepsilon} : V_a^{\mathcal{E}_\varepsilon} \times \mathcal{B}(\mathbb{R}^n)$ as

$$F_{\omega,\varepsilon}(J,A) := \varepsilon^d \mathsf{F}_\omega\left(\frac{J(\varepsilon\cdot,\varepsilon\cdot)}{\varepsilon^{d-1}}, \frac{1}{\varepsilon}A\right)\,, \qquad J \in V_a^{\mathcal{E}_\varepsilon}\,,\ A \in \mathcal{B}(\mathbb{R}^n)\,.$$

Constrained functionals: in the same spirit as in the continuous setting, we set

$$\mathsf{F}^m_{\omega,\varepsilon}(J) := \begin{cases} \mathsf{F}_{\omega,\varepsilon}(J,\overline{U})\,, & \text{if } \operatorname{Div} J = m\,, \\ +\infty\,, & \text{otherwise,} \end{cases}$$

for any given $m \in \mathcal{M}_0(\mathbb{Z}^d_\varepsilon \cap \overline{U})$ (i.e. m has zero mass in $\overline{U}$). Here: $\operatorname{Div} J(x) = \sum_{y\sim x} J(x,y)$.

Statement of the main result.

Theorem (Gladbach, Maas, and P., 2023+ ; for simplicity: $\mathcal{X} = \mathbb{Z}^d$)

Assume that $m_\varepsilon \to \mu \in \mathcal{M}(\overline{U}; V)$ $\mathbb{P}$*-almost-surely. Then,* $\mathbb{P}$*-almost surely, under the assumptions mentioned above, the discrete constrained functionals* $F^{m_\varepsilon}_{\omega,\varepsilon}$ **Γ-convergence** *as* $\varepsilon \to 0$ *(wrt the weak topology) to* $\mathbb{F}_{\omega,\mathrm{hom}}$*, where*

$$\mathbb{F}_{\omega,\mathrm{hom}}(\xi) := \begin{cases} \displaystyle\int_{\overline{U}} f_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) \mathrm{d}\mathscr{L}^d + \int_{\overline{U}} f^\infty_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\xi}{\mathrm{d}|\xi|}\right) \mathrm{d}|\xi|^s\,, & \textit{if } \nabla\cdot\xi = \mu\,, \\ +\infty\,, & \textit{otherwise}, \end{cases}$$

where $f_{\omega,\mathrm{hom}} : V \otimes \mathbb{R}^d \to \mathbb{R}$ *is lower semicontinuous, with linear growth, and* **div-quasiconvex***. Moreover, if in addition one assumes* **ergodicity***, then the* $f_{\omega,\mathrm{hom}} = f_{\mathrm{hom}}$ *does not depend on* ω *(the limit is deterministic).*

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where $f_{\omega,\mathrm{hom}} : V \otimes \mathbb{R}^d \to \mathbb{R}$ is lower semicontinuous, with linear growth, and **div-quasiconvex**. *Moreover, if in addition one assumes* **ergodicity**, *then the $f_{\omega,\mathrm{hom}} = f_{\mathrm{hom}}$ does not depend on ω (the limit is deterministic).*

A function $f : V \otimes \mathbb{R}^d \to \mathbb{R}$ is said to be **div-quasiconvex** if for every cube $Q \subset \mathbb{R}^d$,

$$f(\xi) \le \fint_Q f(\xi + h(x))\,\mathrm{d}x \;:\; \forall h \in C_c^\infty(Q) \quad \text{with} \quad \nabla \cdot h = 0 \,.$$

Generalisation of **quasiconvexity** by Morrey [1952] (weaker than convexity if $n > 1$).

Multi-cell formula in the stochastic setting: computing $f_{\omega,\mathrm{hom}}$.

The limit density $f_{\omega,\mathrm{hom}}$ can be computed as limit of **cell problems** on **on large cubes**. In particular, for every $\xi \in V \otimes \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, we have define the cell problem

$$f_\omega(\xi, A) = \inf \{\mathrm{F}_\omega(J, A) \ : \ J \in \mathrm{Rep}(\xi, A)\} \ ,$$

where the set of representatives of ξ on A is given by

$$\mathrm{Rep}(\xi, A) := \left\{ J \in V_a^{\mathcal{E}} \ : \ \mathrm{Div} J = 0 \quad \text{and} \quad "J = \xi" \text{ on } \partial A \right\} .$$

Then the **homogenized energy density** can be computed by taking the limit

$$f_{\omega,\mathrm{hom}}(\xi) := \lim_{N \to \infty} \frac{f_\omega(\xi, NQ)}{|NQ|} \, . \tag{1}$$

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The existence ($\mathbb{P}$-almost surely) of the limit in (1) follows by stationarity, as application of the **subadditive ergodic theorem** [Akcoglu-Krengel '81; Dal-Maso Modica '86].

$$f_\omega(\xi, A) \leq \sum_{i \in \mathbb{N}} f_\omega(\xi, A_i) , \quad A = \bigcup_{i \in \mathbb{N}} , \quad \{A_i\}_{i \in \mathbb{N}} \text{ disjoint} , \quad \xi \in V \otimes \mathbb{R}^d .$$

Open problems/future directions.

- Full generality: beyond the linear growth and the flow-based (i.e. $f = f(\mu, \xi)$).
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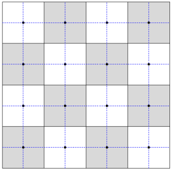

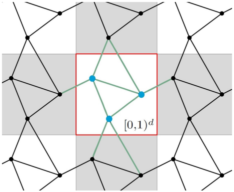

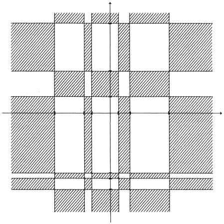

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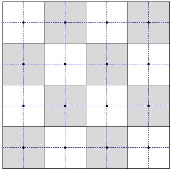
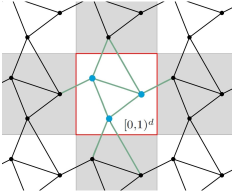
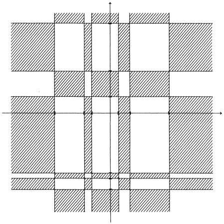






Sketch of the proof

Liminf: for $J_\varepsilon \to \nu$, $\mathrm{Div} J_\varepsilon = m_\varepsilon \to \mu$, we must show

$$\infty > \liminf_{\varepsilon\to 0} F_{\omega,\varepsilon}(J_\varepsilon, \overline{U}) \geq \mathbb{F}_{\omega,\mathrm{hom}}(\nu, \overline{U}) = \int_{\overline{U}} f_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right) \mathrm{d}\mathscr{L}^d + \int_{\overline{U}} f^\infty_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right) \mathrm{d}|\nu|^s .$$

The key tool is the **blow-up technique** á la Fonseca–Müller: define the measures

$$\sigma_\varepsilon := F_\varepsilon(J_\varepsilon, \cdot) \to \sigma \in \mathcal{M}_+(\overline{U}) \quad \Longrightarrow \quad \sigma(\overline{U}) = \lim_{\varepsilon\to 0} \sigma_\varepsilon(\overline{U}) = \liminf_{\varepsilon\to 0} F_{\omega,\varepsilon}(J_\varepsilon, \overline{U}) .$$

Writing the Radon–Nykodym decomposition of σ and ν, the liminf reduces to show

$$f_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}x}\right) \leq \frac{\mathrm{d}\sigma}{\mathrm{d}x} \qquad \mathscr{L}^d - \text{a.e.}\,, \qquad \textbf{(AC)}$$

$$f^\infty_{\omega,\mathrm{hom}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}|\nu|}\right) \leq \frac{\mathrm{d}\sigma}{\mathrm{d}|\sigma|} \qquad |\nu|^s - \text{a.e.}\,. \qquad \textbf{(S)}$$

For example, in the **(AC)** case, one observe that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x}(x_0) = \lim_{\delta\to 0} \frac{\sigma(Q_\delta)}{|Q_\delta|} = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \frac{F_\omega(\tilde{J}_\varepsilon, Q_{\delta/\varepsilon})}{|Q_{\delta/\varepsilon}|}\,, \quad \text{where } \tilde{J}_\varepsilon = \frac{J_\varepsilon(\varepsilon\cdot)}{\varepsilon^{d-1}} .$$

In this case, $\tilde{J}_\varepsilon \approx \frac{\mathrm{d}\nu}{\mathrm{d}x}(x_0)\mathscr{L}^d$ (**tangent measure**) and $\mathrm{Div}\tilde{J}_\varepsilon \approx 0$ — need **correction**.

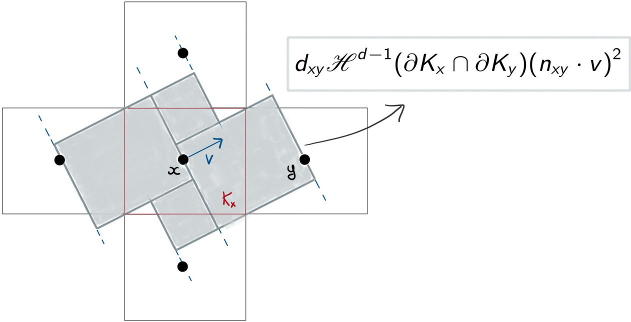
The role of isotropy in the periodic setting

Theorem (multidimensional): $\mathcal{W}_\theta$ converges as $\varepsilon \to 0$ to $\mathbb{W}_{\mathrm{hom}}$, where

$$\mathbb{W}^2_{\mathrm{hom}}(\mu_0,\mu_1) = \left\{ \int_0^1 \int_{\mathbb{T}^d} f_{\mathrm{hom}}(\mu_t,\xi_t)\,\mathrm{d}x\,\mathrm{d}t \ : \ (\mu_t,\xi_t)_t \in \mathrm{CE}(\mu_0,\mu_1) \right\}, \quad \text{where}$$

- $\mathbb{W}_{\mathrm{hom}} = \mathbb{W}_2$ if and only if the mesh is isotropic: in the periodic setting, it reads

$$\frac{1}{2}\sum_{y\sim x} d_{xy}\mathscr{H}^{d-1}(\partial K_x \cap \partial K_y) n_{xy} \otimes n_{xy} = |K_x|\mathrm{id}, \quad \forall x \in \mathcal{X}.$$



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- $f_{\text{hom}}(\mu, \xi) = \dfrac{\|\xi\|^2_{\text{hom}}}{\mu} \leq \dfrac{|\xi|^2}{\mu}$, where $\|\cdot\|_{\text{hom}}$ is a norm (possibly not Riemannianian!)

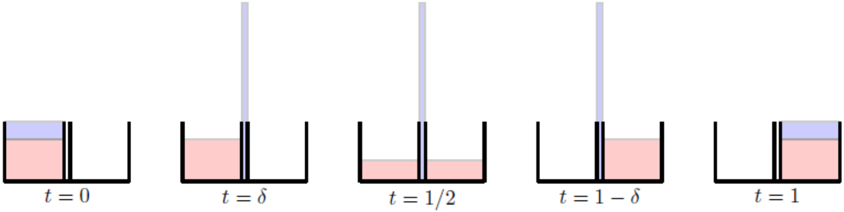

Figure: Strongly oscillating measures on the graph scale can be cheaper.