(Gradient Flows Face-To-Face)$_3$
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Optimal control problems for nonlocal interaction equations

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joint with S. Fagioli & A. E. Kaufmann

Summary

1. Transport and Optimal Control Problems
2. Main results
3. Interesting control functionals and applications
4. Gradient Flow structure
5. Open Problems

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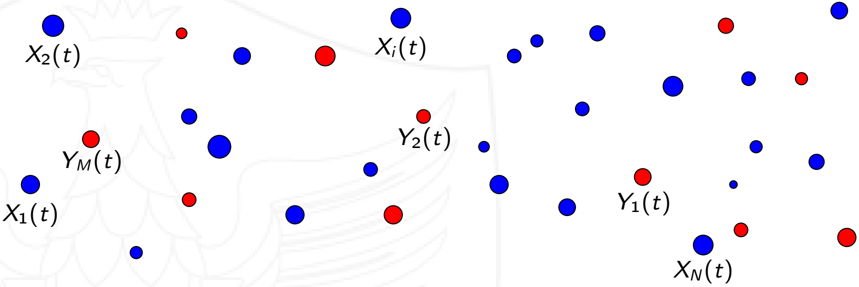
Microscopic Description



$X_1(t)\dots X_N(t)$ positions in $\mathbb{R}^d$ of individuals at time t with masses $n_1\dots n_N$

$Y_1(t)\dots Y_M(t)$ positions in $\mathbb{R}^d$ of control agents at time t with masses $m_1\dots m_M$

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$$\dot{X}_i(t) = -\sum_{j=1}^{N} n_j \mathcal{K}\big(X_i(t) - X_j(t)\big) + u(t) \quad i = 1 \dots N$$

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$$\begin{cases} \dot{X}_i(t) = -\sum_{j=1}^{N} n_j \mathcal{K}\big(X_i(t) - X_j(t)\big) - \sum_{k=1}^{M(t)} m_k \mathcal{H}\big(X_i(t) - Y_k(t)\big) \\ (X, Y) = \arg\min_{(Z,U)} \mathcal{J}(Z, U) \quad \text{where} \quad \begin{matrix} U & \text{admissible control vectors} \\ Z & \text{solves the ODEs with } U \end{matrix} \end{cases}$$

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Problem: curse of dimensionality

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Continuum Description

This dimensionality problem can be bypassed by introducing an optimal control strategy independent on the number of agents but depending on their distributions:

$$\rho(t,\cdot) \text{ represents the distribution of the population of individuals } X_1 \dots X_N \text{ at time } t$$
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while the role of ν is designed by an optimal control problem

$$\inf \mathcal{J}(\rho, \nu) \quad \text{s.t.} \quad \partial_t \rho(t, x) + \nabla \cdot \big(\rho(t, x)(\mathcal{K} * \rho\,(t, x) + \mathcal{H} * \nu\,(t, x))\big) = 0$$

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The ambient spaces

$$Lip_{L,2}(0,T;\mathscr{P}_2(\mathbb{R}^d)) = \{\mu : [0,T] \to \mathscr{P}_2(\mathbb{R}^d) : W_2(\mu(t),\mu(s)) \leq L|t-s|\}$$

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The case with smooth potential W was considered in [Bongini,Buttazzo 2017]

Theorem (Fagioli,Kaufmann,R. 2023)

Given $\rho_0 \in \mathscr{P}_2(\mathbb{R}^d)$, $\nu \in Lip_{L',d}(0,T;\mathcal{M}_M^R(\mathbb{R}^d))$ *and* W, V *as in(Self), (Cross) respectively, there exists* $\rho \in Lip_{L,2}(0,T;\mathscr{P}_2(\mathbb{R}^d))$ *for some* $L = L(M, LipV, LipW)$ *such that* $\partial^0 W * \rho \in L^1(0,T;L^2(\rho(t)))$ *and for every* $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^d)$ *it holds*

$$\begin{cases} \int_0^T \int_{\mathbb{R}^d} \Big(\partial_t\varphi + (\partial^0 W * \rho + \nabla V * \nu)\cdot\nabla\varphi\Big) d\rho(t,x) = 0 \quad (0,T)\times\mathbb{R}^d \\ \rho(0,\cdot) = \rho_0. \end{cases} \tag{TE}$$

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Theorem (Fagioli,Kaufmann,R. 2023)

Given $\rho_0 \in \mathscr{P}_2(\mathbb{R}^d)$, W, V *and* $\mathfrak{J}$ *as in (Self), (Cross) and (Contr) respectively, then the variational problem*

$$\min_{\mathfrak{S}} \{\mathfrak{J}(\rho,\nu) : \rho,\nu \ \textit{satisfy} \ \text{(TE)}\}$$

admits a solution.

Applications

- driving a mass of pedestrian to (or out of) a certain location using a small number of stewards;

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- computing the minimal amount of manually controlled units such that a swarm of drones performs a given task (as, for instance, wind harvesting or the recognition of a given area).

Interesting functionals included

- Optimization of the quantity of control agents in accordance with the goal to achieve

$$\mathcal{J}(\rho,\nu) = \int_0^T \int_\Omega f(t,x)d\nu(t,x)$$

where $f : [0,T] \times \Omega \to [0,\infty]$ is a lsc function. A standard choice is $f(t,x) = c(t)|x - x_0|^p$ and x_0 represents a sort of *manpower storage room*

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- Require the dynamics of ρ to satisfy a specific feature, like the collapse of one of its moments or marginals. For example alignment models like Cucker-Smale one, where the goal of the control strategy is to force the alignment of the group

$$\bar{v}(t) = \int_{\mathbb{R}^{2d}} w d\rho(t,x,w) \qquad \mathcal{J}(\rho,\nu) = \int_0^T \int_\Omega \int_{\mathbb{R}^{2d}} |v - \bar{v}(t)|^2 d\rho(t,x,v)$$

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In the first case the dynamics of $\rho(t)$ should be in average close to $\bar{\rho}$ while in the second case $\rho(t)$ has much more freedom as only $\rho(T)$ should be as close as possible to $\bar{\rho}$

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- the dynamics of ν has a structure, for example it conserves mass and solves a transport equation of the form $\partial_t \nu(t, x) + \nabla \cdot (\nu(t, x) u(t, x)) = 0$. Then the admissible ν belong to a precise subset B of the usual ambient space

$$\mathcal{J}(\rho, \nu) = \chi_B(\rho, \nu)$$

Interesting functionals included

- Another way to give more structure to the dynamics of ν is to embed the desired features inside the functional

$$\mathcal{J}(\rho, \nu) = \int_0^T \int_\Omega \int_\Omega Q(x, y) d\nu(t, x) d\nu(t, y) dt$$

which forces ν to self-interact through the kernel Q, for example

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- other interesting functionals are of the form

$$\mathcal{J}(\rho,\nu) = \int_0^T \left(\int_\Omega h(t,\nu^a(t,x))dx + \sum_{x\in\Omega} k(\nu^\sharp(t,x)) \right) dt$$

where ν^a and $\nu^\sharp$ denotes the absolutely continuous part and the atomic part of ν respectively, and
  - $h : [0,\infty) \to [0,\infty)$ convex, $h(0) = 0$, $\lim_{|x|\to\infty} h' = +\infty$
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a famous example is the Mumford-Shah functional

$$h(s) = s^2 \qquad k(s) = \begin{cases} 0 & \text{if } s = 0, \\ 1 & \text{otherwise.} \end{cases}$$

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another famous example is the counting measure

$$h(s) = \begin{cases} 0 & \text{if } s = 0, \\ +\infty & \text{otherwise.} \end{cases} \qquad k(s) = \begin{cases} 0 & \text{if } s = 0, \\ 1 & \text{otherwise.} \end{cases}$$

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Interesting functionals not included

- Penalization of the change of the mass of ν in time,

$$\mathcal{J}(\rho,\nu) = \int_0^T \left| \partial_t \int_\Omega d\nu(t,x) \right| dt$$

this appears in contexts where hiring control agents after the dynamics has started is costlier than doing it before

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- another popular functional in the applications is

$$\mathcal{J}(\rho,\nu) = \int_0^T |\nu'(t)| dt$$

where ν' is the metric derivative of ν at time t

Problem The populations ρ and ν interact trhough the kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{H}_1, \mathcal{H}_2$ and, in addition, the population ν tries to optimize its trajectory in order to let ρ evacuate from the set C while, at the same time, penalizing too high values for the velocity field u

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$$\min \left\{ \int_0^T \left(\int_C \rho(t,x)dx + \int_\Omega |u(t,x)|^p dx \right) dt : (\rho, u) \text{ satisfy } \begin{array}{l} \partial_t \rho + \nabla \cdot \big(\rho(\mathcal{K}_1 * \rho + \mathcal{H}_1 * \nu)\big) = 0 \\ \partial_t \nu + \nabla \cdot \big(\nu(\mathcal{K}_2 * \nu + \mathcal{H}_2 * \rho + u)\big) = 0 \end{array} \right\}$$

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Simulations for a control problem in pedestrian dynamics

Problem The populations ρ and ν interact trhough the kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{H}_1, \mathcal{H}_2$ and, in addition, the population ν tries to optimize its trajectory in order to let ρ evacuate from the set C while, at the same time, penalizing too high values for the velocity field u

$$\min\left\{ \int_0^T \left(\int_C \rho(t,x)dx + \int_\Omega |u(t,x)|^p dx \right) dt : (\rho, u) \text{ satisfy } \begin{array}{l} \partial_t \rho + \nabla \cdot \big(\rho(\mathcal{K}_1 * \rho + \mathcal{H}_1 * \nu)\big) = 0 \\ \partial_t \nu + \nabla \cdot \big(\nu(\mathcal{K}_2 * \nu + \mathcal{H}_2 * \rho + u)\big) = 0 \end{array} \right\}$$

Example in $\mathbb{R}^2$: ν atomic, ρ diffuse, area to be evacuated $C = \mathbb{R}^2 \setminus \{\text{one point}\}$

- The control agents know where the exit is
- The exit becomes visible only to the individuals which are closer than a prescribed range
- The interaction kernels are repulsive at short range since pedestrians cannot overlap in space
- The control agents aim at optimizing their trajectories in order to reach the goal encoded in the cost functional (evacuate ρ from C penalizing too high values of the optimized velocity field u)

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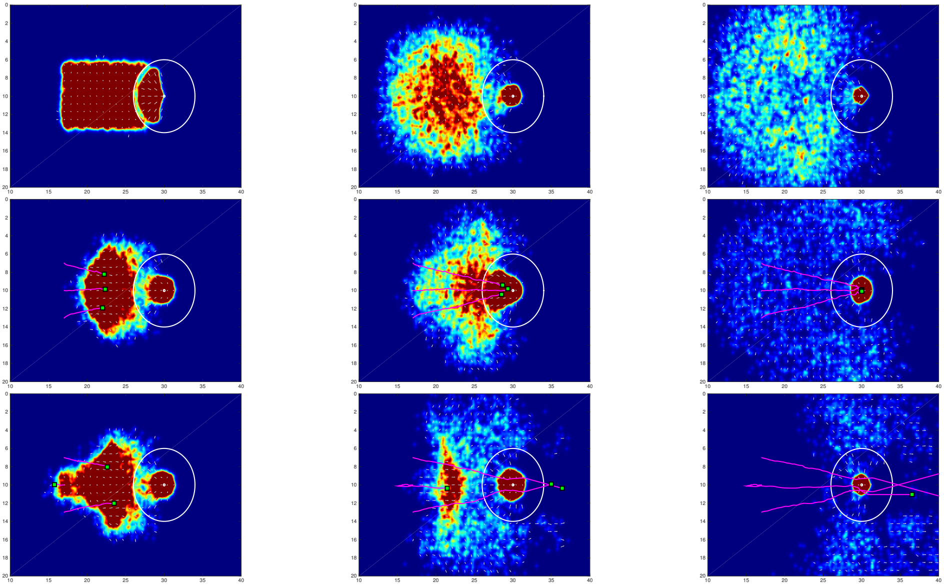

$41,2\%$

$71,3\%$

$85,2\%$

courtesy of the authors of [Albi,Bongini,Cristiani,Kalise 2016]

Given ν, we can build a weak solution of

$$\begin{cases} \partial_t \rho = \nabla \cdot \left(\rho(\partial^0 W * \rho + \nabla V * \nu)\right) & (0,T) \times \mathbb{R}^d \\ \rho(0,\cdot) = \rho_0 \in \mathscr{P}_2(\mathbb{R}^d) \end{cases} \tag{TE}$$

following a suitable generalisation of the JKO scheme that applies to time-depending energies

$$\mathcal{F}_{\nu(t)}[\mu] = \frac{1}{2}\int_{\mathbb{R}^d} W * \mu d\mu + \int_{\mathbb{R}^d} V * \nu(t) d\mu$$

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and works updating at each iteration the energy functional in the variational problem

$$\begin{cases} \rho^0 := \rho_0 \\ \rho^{i+1} \in \arg\min_{\mu \in \mathscr{P}_2(\mathbb{R}^d)} \left(\frac{1}{2\tau} W_2^2(\rho^i, \mu) + \mathcal{F}_{\nu(\tau(i+1))}[\mu]\right) \end{cases}$$

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- lsc of $W_2 \Longrightarrow$ $\rho(t) \in \mathscr{P}_2(\mathbb{R}^d)$ for all t and $W_2(\rho(s), \rho(t)) \leq L(M, LipW, LipV)|s - t|$

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$\Longrightarrow \quad \rho \in Lip_{L,2}(0,T;\mathscr{P}_2(\mathbb{R}^d))$ and is a solution of (TE)

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The control problem

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$$\inf_{\mathfrak{S}} \left(\mathcal{J}(\rho,\nu) + \chi_{\mathfrak{U}}(\rho,\nu)\right) \quad \text{where} \quad \mathfrak{U} = \{(\rho,\nu) \in \mathfrak{S} : \ \rho,\nu \ \text{satisfy (TE)}\} \tag{OC}$$

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Independent compactness:

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$$\text{then} \quad \mathcal{J}(\rho,\nu) + \chi_{\mathfrak{U}}(\rho,\nu) \le \liminf_k \mathcal{J}(\rho_k,\nu_k) + \chi_{\mathfrak{U}}(\rho_k,\nu_k) = \inf_{\mathfrak{S}}(\mathcal{J} + \chi_{\mathfrak{U}})$$

Open problems

- extend the analysis to the functionals $\int_0^T |\partial_t \int_\Omega d\nu(t,x)|dt$ and $\int_0^T |\nu'(t)|dt$
- consider more involved evolution equations (different transport terms,more singular kernels,different mobilities etcetc)
- study more performant numerical schemes (for example try the JKO)

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Thank you
for your kind attention!